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New Results on Existence for $\psi-$Hilfer Fractional Delay Differential Problem

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ABSTRACT. In this paper, we examine the existence and uniqueness of a solution to a nonlinear fractional delay differential equation with $\psi-$Hilfer derivative. The contraction mapping principle will be used as a main tool for existence, and generalized Gronwall inequality for continuous dependence results. Several numerical examples are included to illustrate our findings.

1. INTRODUCTION

In recent decades, fractional calculus has gained significant importance due to its wide range of applications in various fields such as physics, mechanics, chemistry, engineering, and finance see [5, 6, 15, 21, 34]. While classical calculus has long been recognized as a powerful tool for modeling dynamic processes, many complex systems in nature are more accurately described by fractional differential equations (FDEs). Examples of such systems include the transport of chemical pollutants through rocks, the behavior of viscoelastic materials like polymers, air pollution diffusion, cellular diffusion, and signal transmission across networks in strong magnetic fields see [4, 17, 18, 25, 26, 28, 30]. In these cases, the systems exhibit complex microscopic behavior that classical derivative models cannot fully capture. As a result, in many physical, chemical set ups, FDEs provide a more appropriate framework than traditional differential problems see [1, 3, 8, 9, 13, 14, 12, 22, 23, 27].

An important category of FDEs is fractional delay differential equations (FDDEs). These equations include delay parameters, meaning the unknown function depends on its past history. FDDEs have been studied for numerous applications. For example, recent work [10, 16, 19, 20, 24, 33], are evident. In recent time, the FDDEs are studied by utilizing fixed point technique and Caratheodory properties see [2, 29, 11]. Motivated by their work, we consider the following FDDEs

$$\mathfrak{D}_{a^+}^{\sigma,\delta;\psi}\zeta(t) = f(t,\zeta(t),\zeta(t-\tau));\ 0<\sigma<1,\ 0\le\delta\le 1\le 0,\ 0<t<T, \tag{1.1}$$

$$\zeta(t) = \phi(t),\ -\tau\le t<0, \tag{1.2}$$

$$\mathfrak{I}_{a^+}^{1-\eta;\psi} = c,\ c\in\mathfrak{R}, \tag{1.3}$$

where $\mathfrak{D}_{a^+}^{\sigma,\delta;\psi}(\cdot)$ is the $\psi-$Hilfer fractional derivative of order $0<\sigma<1$, type $0\le\delta\le 1$, $\zeta(t-\tau)$ denotes the amount of ζ at a fixed time τ unit in the previous time in which the impact of ζ on the present rate of alteration of ζ is belated by the time τ.

The paper is organized as follows: In section 2, we enlist some basic definitions, preliminary facts and lemmas whch are useful in the subsequent sections. In section 3, we prove the equivalence of the $\psi-$Hilfer FDDEs with Volterra integral equation. We prove the existence of a unique solution to FDDEs (1.1)–(1.3), continuous dependence in 4. In

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section 5, two examples illustrating our main results will be provided. The concluding remarks are presented in the final section.

2. PRELIMINARIES

Let $[a,b] \subset \mathfrak{R}^+,\ (0<a<b<\infty)$ and $\mathfrak{C}[a,b],\ \mathfrak{AC}^n[a,b],\ \mathfrak{C}^n[a,b]$ are the space of continuous real function, n-times absolutely continuous function and n-times continuously differentiable functions on $[a,b]$ respectively. Let $\mathfrak{L}^p(a,b),\ (1 \le p \le \infty)$ defines space of Lebesgue measurable function on (a,b). We recall the following norm and weighted space of continuous function:

$$\|\zeta\|_{\mathfrak{L}^p[a,b]} = \left[\int_a^b |\zeta(t)|^p dt\right]^{\frac{1}{p}} < \infty, \quad \forall\ \zeta \in \mathfrak{L}^p(a,b),$$

$$\|\zeta\|_{\mathfrak{C}[a,b]} = \max\{|\zeta(t) : t \in [a,b]\}, \quad \forall\ \zeta \in [a,b],$$

$$\mathfrak{AC}^n[a,b] = \{\zeta : [a,b] \longrightarrow \mathfrak{R} | \zeta^{(n-1)} \in \mathfrak{AC}[a,b]\},$$

and

$$\mathfrak{C}_{\eta;\psi}[a,b] = \{\zeta : [a,b] \longrightarrow \mathfrak{R} | (\psi(t)-\psi(a))^{\eta}\zeta(t) \in \mathfrak{C}[a,b]\}, \quad 0 \le \eta < 1,$$

$$\mathfrak{C}^n_{\eta;\psi}[a,b] = \{\zeta : [a,b] \longrightarrow \mathfrak{R} | \zeta(t) \in \mathfrak{C}^{n-1}[a,b]; \zeta^n(t) \in \mathfrak{C}_{\eta;\psi}[a,b]\}, \quad 0 \le \eta < 1,\ n \in \mathfrak{N},$$

$$\mathfrak{C}^{\sigma;\delta}_{\eta;\psi}[a,b] = \{\zeta(t) \in \mathfrak{C}_{\eta;\psi}[a,b]; \mathfrak{D}^{\sigma;\delta}\zeta \in \mathfrak{C}_{\eta;\psi}[a,b]\}, \quad \eta = \sigma+\delta-\sigma\delta.$$

We note that, if $n=0$, $\mathfrak{C}^0_{\eta;\psi}[a,b] = \mathfrak{C}_{\eta;\psi}[a,b]$ with

$$\|\zeta\|_{\mathfrak{C}_{\eta;\psi}[a,b]} = \|(\psi(t)-\psi(a))^{\eta}\zeta(t)\|_{\mathfrak{C}_{[a,b]}} = \max\{(\psi(t)-\psi(a))^{\eta}\zeta(t) : t \in [a,b]\},$$

and $\|\zeta\|_{\mathfrak{C}^n_{\eta;\psi}[a,b]} = \sum_{k=0}^{n-1} \|\zeta^k\|_{\mathfrak{C}_{[a,b]}} + \|\zeta^n\|_{\mathfrak{C}_{\eta;\psi[a,b]}}$.

Assume during the analysis, unless otherwise indicated, $0 \le a < b \le T < \infty$.

Definition 2.1. [31] *The Mittag–Leffler function for two parameters is defined as:*

$$E_{\sigma,\delta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\sigma k+\delta)}, \tag{2.4}$$

where $\sigma, \delta \in \mathfrak{C}$, $Re(\sigma) > 0$ *and* $\Gamma(z), z > 0$, *is Gamma function:* $\Gamma(z) = \int_0^{\infty} e^{-t}t^{z-1}dt.$

Definition 2.2. [31] *The left sided* $\psi-$*Riemann-Liouville fractional integral of order* σ, $(n-1 < \sigma < n)$ *for an integrable function* $\zeta : [a,b] \longrightarrow \mathfrak{R}$ *with respect to another function* $\psi : [a,b] \longrightarrow \mathfrak{R}$, *that is an increasing differentiable function such that* $\psi'(t) \neq 0$, *for all* $t \in [a,b]$, $(-\infty \le a < b \le \infty)$ *are respectively defined as follows:*

$$\mathfrak{I}^{\sigma;\psi}_{a^+}\zeta(t) = \frac{1}{\Gamma(\sigma)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}\zeta(s)ds$$

and

$$\mathfrak{D}^{\sigma;\psi}_{a^+}\zeta(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathfrak{I}^{n-\sigma;\psi}_{a^+}\zeta(t).$$

$$= \frac{1}{\Gamma(n-\sigma)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}\zeta(s)ds.$$

Definition 2.3. [32] *The left-sided $\psi-$Caputo fractional derivative of order σ, $(n-1<\sigma<n)$, $n=[\sigma]+1$, function $\zeta\in\mathfrak{C}^{n}[a,b]$ with respect to another function ψ is defined by*

$$\begin{aligned}{}^{c}\mathfrak{D}^{\sigma;\psi}_{a^+} &= \mathfrak{I}^{n-\sigma;\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}\zeta(t)\\ &= \frac{1}{\Gamma(n-\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{n-\sigma-1}\zeta^{(n)}_{\psi}(s)ds,\end{aligned}$$

where $\zeta^{(n)}_{\psi}(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}\zeta(t).$

Remark 2.1. *The $\psi-$Caputo fractional derivative of function $\zeta\in\mathfrak{AC}^{n}[a,b]$ is:*

$${}^{c}\mathfrak{D}^{\sigma;\psi}_{a^+} = \mathfrak{D}^{\sigma;\psi}_{a^+}\left[\zeta(t)-\sum_{k=0}^{n-1}\frac{(\frac{1}{\psi'(t)}\frac{d}{dt})^{k}\zeta(a)}{k!}(\psi(t)-\psi(a))^{k}\right].$$

Definition 2.4. [31] *Let $n-1<\sigma<n$; $n\in\mathfrak{N}, -\infty\le a<b\le\infty$ and $\psi\in\mathfrak{C}^{n}([a,b],\mathfrak{R})$ a function such that $\psi(t)$ is increasing and $\psi'(t)\neq 0$, $\forall\, t\in[a,b]$. The left-sided $\psi-$Hilfer fractional derivative of function $\zeta\in\mathfrak{C}^{n}[a,b]$ of order σ and type $\delta\in[0,1]$ is defined as:*

$$\mathfrak{D}^{\sigma,\delta;\psi}_{a^+}\zeta(t)=\mathfrak{I}^{\delta(n-\sigma);\psi}_{a^+}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}\mathfrak{I}^{(1-\delta)(n-\sigma)}_{a^+}\zeta(t),\quad t>a.$$

Remark 2.2. *The $\psi-$Hilfer fractional derivative of order σ and type δ is also defined as:*

$$\mathfrak{D}^{\sigma,\delta;\psi}_{a^+}\zeta(t)=\mathfrak{I}^{\delta(n-\sigma);\psi}_{a^+}\mathfrak{D}^{\eta;\psi}_{a^+}\zeta(t),\ t>a,\ \eta=\sigma+n\delta-\sigma\delta, \tag{2.5}$$

where $\mathfrak{D}^{\eta;\psi}_{a^+}\zeta(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}\mathfrak{I}^{(1-\delta)(n-\sigma);\psi}_{a^+}\zeta(t).$

Remark 2.3. *In particular, the $\psi-$Hilfer fractional derivative of order $\sigma\in(0,1)$ and type $\delta\in[0,1]$ can also be defined as:*

$$\begin{aligned}\mathfrak{D}^{\sigma,\delta;\psi}_{a^+}\zeta(t) &= \frac{1}{\Gamma(\eta-\sigma)}\int_a^t(\psi(t)-\psi(s))^{\eta-\sigma-1}\mathfrak{D}^{\eta;\psi}_{a^+}\zeta(s)ds\\ &= \mathfrak{I}^{\eta-\sigma:\psi}_{a^+}\mathfrak{D}^{\eta;\psi}_{a^+}\zeta(t),\end{aligned}$$

where $\eta=\sigma+\delta-\sigma\delta$, $\mathfrak{I}^{\eta-\sigma:\psi}_{a^+}(\cdot)$ *defined by (2.5) and* $\mathfrak{D}^{\eta;\psi}_{a^+}\zeta(t)=\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}\mathfrak{I}^{(1-\eta);\psi}_{a^+}\zeta(t).$

Lemma 2.1. [31] *Let $\sigma>0$, $0\le\delta<1$ and $\zeta\in L^{1}[a,b]$. Then*

$$\mathfrak{I}^{\sigma;\psi}_{a^+}\mathfrak{I}^{\delta;\psi}_{a^+}\zeta(t)=\mathfrak{I}^{\sigma+\delta:\psi}_{a^+}\zeta(t),\ a.e.\ t\in[a,b].$$

In particular,

(i): If $\zeta\in\mathfrak{C}_{\eta;\psi}[a,b]$, then $\mathfrak{I}^{\sigma;\psi}_{a^+}\mathfrak{I}^{\delta;\psi}_{a^+}\zeta(t)=\mathfrak{I}^{\sigma+\delta:\psi}_{a^+}\zeta(t)$, $t\in(a,b]$.
(ii): If $\zeta\in\mathfrak{C}[a,b]$, then $\mathfrak{I}^{\sigma;\psi}_{a^+}\mathfrak{I}^{\delta;\psi}_{a^+}\zeta(t)=\mathfrak{I}^{\sigma+\delta:\psi}_{a^+}\zeta(t)$, $t\in[a,b]$.

Lemma 2.2. [31] *Let $\sigma>0$, $0\le\delta<1$. If $\zeta\in\mathfrak{C}_{\eta;\psi}[a,b]$, then*

$$\mathfrak{D}^{\sigma,\delta;\psi}_{a^+}\mathfrak{I}^{\sigma;\psi}_{a^+}\zeta(t)=\zeta(t),\ t\in(a,b].$$

If $\zeta\in\mathfrak{C}^{1}[a,b]$ then

$$\mathfrak{D}^{\sigma,\delta;\psi}_{a^+}\mathfrak{I}^{\sigma;\psi}_{a^+}\zeta(t)=\zeta(t),\ t\in[a,b].$$

Lemma 2.3. [31] *Let* $0 < \sigma < 1,\ 0 \leq \delta \leq 1,$ *and* $\eta = \sigma + \delta - \sigma\delta$*. If* $\zeta \in \mathfrak{C}^{\eta}_{1-\eta;\psi}[a,b]$ *then*

$$\mathfrak{I}^{\eta;\psi}_{a^+}\mathfrak{D}^{\eta,\delta;\psi}_{a^+}\zeta(t) = \mathfrak{I}^{\sigma;\psi}_{a^+}\mathfrak{D}^{\sigma,\delta;\psi}_{a^+}\zeta(t)$$

and

$$\mathfrak{D}^{\eta,\delta;\psi}_{a^+}\mathfrak{I}^{\sigma;\psi}_{a^+}\zeta(t) = \mathfrak{D}^{\delta(1-\sigma);\psi}_{a^+}\zeta(t).$$

Lemma 2.4. [31] *Let* $t > a$, $\sigma \geq 0$, *and* $\theta > 0$*. Then*

$$\mathfrak{I}^{\sigma;\psi}_{a^+}(\psi(t)-\psi(a))^{\theta-1} = \frac{\Gamma(\theta)}{\Gamma(\theta+\sigma)}(\psi(t)-\psi(a))^{\theta+\sigma-1}$$

and if $0 < \theta < 1,$ *we have*

$$\mathfrak{D}^{\sigma;\psi}_{a^+}(\psi(t)-\psi(a))^{\sigma-1} = 0.$$

Lemma 2.5. [31] *Let* $0 < \sigma < 1,\ 0 \leq \delta \leq 1,$ *and* $\eta = \sigma + \delta - \sigma\delta$*, and let* $\psi \in \mathfrak{C}^1([a,b],\mathfrak{R})$ *be an increasing function such that* $\psi'(t) \neq 0,\ \forall\ t \in [a,b]$*. Then*

(i) $\mathfrak{I}^{\sigma;\psi}_{a^+}$ *maps* $\mathfrak{C}[a,b]$ *into* $\mathfrak{C}[a,b]$.
(ii) $\mathfrak{I}^{\sigma;\psi}_{a^+}$ *is bounded from* $\mathfrak{C}_{1-\eta;\psi}[a,b]$ *into* $\mathfrak{C}_{1-\eta;\psi}[a,b]$.
(iii) If $\eta \leq \sigma$*, then* $\mathfrak{I}^{\sigma;\psi}_{a^+}$ *is bounded from* $\mathfrak{C}_{1-\eta;\psi}[a,b]$ *into* $\mathfrak{C}[a,b]$.

Lemma 2.6. [31] *Let* $\sigma > 0,\ ,0 \leq \eta < 1,$ *and* $\zeta \in \mathfrak{C}_{\eta;\psi}[a,b]$*. If* $\sigma > \eta$*, then* $\mathfrak{I}^{\sigma;\psi}_{a^+}\zeta \in \mathfrak{C}[a,b]$ *and*

$$\mathfrak{I}^{\sigma;\psi}_{a^+}\zeta(a) = \lim_{t\to a^+}\mathfrak{I}^{\sigma;\psi}_{a^+}\zeta(t) = 0.$$

Lemma 2.7. [31] *Let* $0 \leq \eta < 1,\ a < c < b,\ \zeta \in \mathfrak{C}_{\eta;\psi}[a,c],\ \zeta \in \mathfrak{C}[a,b]$ *and* ζ *is continuous at* c*. Then* $\zeta \in \mathfrak{C}_{\eta;\psi}[a,b]$.

Lemma 2.8. [31] *If* $\zeta \in \mathfrak{C}^n[a,b],\ n-1 < \sigma < n,\ 0 \leq \delta \leq 1,\ \eta = \sigma + \delta - \sigma\delta$*. Then for all* $t \in [a,b]$,

$$\mathfrak{I}^{\sigma;\psi}_{a^+}\mathfrak{D}^{\sigma,\delta;\psi}_{a^+}\zeta(t) = \zeta(t) - \sum_{k=1}^{n}\frac{(\psi(t)-\psi(a))^{\eta-k}}{\Gamma(\eta)}\zeta^{(n-k)}_{\psi}\mathfrak{I}^{(1-\delta)(1-\sigma);\psi}_{a^+}\zeta(a).$$

In particular, if $0 < \sigma < 1$*, we have*

$$\mathfrak{I}^{\sigma;\psi}_{a^+}\mathfrak{D}^{\sigma,\delta;\psi}_{a^+}\zeta(t) = \zeta(t) - \frac{(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}\mathfrak{I}^{(1-\delta)(1-\sigma);\psi}_{a^+}\zeta(a).$$

Additionlly , if $\zeta \in \mathfrak{C}_{1-\eta;\psi}[a,b]$ *and* $\mathfrak{I}^{1-\eta;\psi}_{a^+}\zeta \in \mathfrak{C}^1_{1-\eta;\psi}[a,b]$ *such that* $0 < \eta < 1$*. Then for all* $t \in (a,b]$,

$$\mathfrak{I}^{\eta;\psi}_{a^+}\mathfrak{D}^{\eta;\psi}_{a^+}\zeta(t) = \zeta(t) - \frac{[\psi(t)-\psi(a)]^{\eta-1}}{\Gamma(\eta)}\mathfrak{I}^{1-\eta;\psi}_{a^+}\zeta(a).$$

Lemma 2.9. [31] *Let* $\zeta \in \mathfrak{L}^1[a,h]$*. Then*

$$\lim_{s\to h^+}\int_a^h(\psi(s)-\psi(t))^{\sigma-1}\zeta(t)dt = \int_a^h(\psi(h)-\psi(t))^{\sigma-1}\zeta(t)dt = \Gamma(\sigma)\mathfrak{I}^{\sigma;\psi}_{a^+}\zeta(c),\ \sigma > 0.$$

Lemma 2.10. [32] *(Gronwall lemma) Let* $p, q,$ *be two integrable functions and* ζ *is continuous on* $[a,b]$*. Let* $\psi \in \mathfrak{C}[a,b]$ *be an increasing function such that* $\psi'(t) \neq 0, \forall\ t \in [a,b]$*. Assume that* p *and* q *are non-negative and non-decreasing. If*

$$p(t) \leq q(t) + \zeta(t)\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\sigma-1}p(s)ds,$$

then, for all $t \in [a,b]$, *we have*

$$p(t) \le q(t) + \int_a^t \sum_{k=1}^{\infty} \frac{(\zeta(t)\Gamma(\sigma))^k}{\Gamma(\sigma k)} \psi'(s)(\psi(t)-\psi(s))^{\sigma k-1} q(s)ds. \tag{2.6}$$

Also, if q *is a non-decreasing function on* $[a,b]$, *then*

$$p(t) \le q(t) E_\sigma(\zeta(t)\Gamma(\sigma)(\psi(t)-\psi(a))^\sigma).$$

Theorem 2.1. [21, 7] *(Banach fixed point theorem) Let* (X,d) *be a complete metric space and* $T : X \to X$ *is a strict contraction, i.e. a map satisfying*

$$d(T(x),T(y)) \le a d(x,y), \forall x,y \in X \tag{2.7}$$

where $0 < a < 1$. *Then the operator* T *has a unique fixed point* $p \in X$. *The Picard iteration* $x_n{}_{n=0}^{\infty}$ *defined by*

$$x_{n+1} = Tx_n, \quad n = 0,1,2,\cdots$$

converge to p, *for any* $x_0 \in X$.

To prove the main result, we need the following axioms:

(A1): $\zeta : (0,T] \times \mathfrak{R} \times \mathfrak{R} \longrightarrow \mathfrak{R}$ be a function such that $f(t,\zeta(t),\lambda(t-\tau)) \in \mathfrak{C}_{1-\eta;\psi}[a,b]$ for any $\zeta, \lambda \in \mathfrak{C}_{1-\eta;\psi}[0,T]$.

(A2): $f(t,\zeta(t),\lambda(t))$ satisfies the Lipschitz's condition with respect to ζ, λ and is bounded in a region $\mathfrak{G} \subset \mathfrak{R}$, $\forall\ t \in [0,T]$ such that

$$\begin{aligned}\|f(t,\zeta_1(t),\lambda_1(t-\tau)) - f(t,\zeta_2(t),\lambda_2(t-\tau))\|_{\mathfrak{C}_{1-\eta;\psi}[0,T]} \\ \le L(|\zeta_1(t)-\lambda_1(t)|_{\mathfrak{C}_{1-\eta;\psi}[0,T]} + |\zeta_1(t-\tau)-\lambda_2(t-\tau)|_{\mathfrak{C}_{1-\eta;\psi}[0,T]}).\end{aligned}$$

3. Main Result

In this section, we demonstrate the equivalent Volterra integral solution for FDDEs and establish its existence and uniqueness results.

Lemma 3.11. *Let* $0 < \sigma < 1$, $0 \le \delta \le 1$, $\eta = \sigma + \delta - \sigma\delta$ *and let* ζ *as in* $(\mathbf{A1})$. *If* $\zeta \in \mathfrak{C}^{\eta}_{1-\eta;\psi}[a,b]$ *then* ζ *satisfies FDDEs (1.1)-(1.3) if and only if* ζ *satisfies:*

$$\zeta(t) = \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} + \frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(a))^{\sigma-1} f(s,\zeta(s),\zeta(s-\tau))ds. \tag{3.8}$$

Proof. In the bigining, we prove the first part, followed by its converse. Let $\zeta \in \mathfrak{C}^{\eta}_{1-\eta;\psi}[a,b]$ be a solution of FDDEs (1.1)-(1.3). We demonstrate that ζ also satisfies (3.8). By definition of $\mathfrak{C}^{\eta}_{1-\eta;\psi}[a,b]$, Lemma 2.5 and Definition 2.4, we have $\mathfrak{I}^{1-\eta;\psi}_{a^+}\zeta \in \mathfrak{C}[a,b]$ and

$$\mathfrak{D}^{\eta;\psi}_{a^+}\zeta(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathfrak{I}^{(1-\eta);\psi}_{a^+}\zeta(t) \in \mathfrak{C}_{1-\eta;\psi}[a,b].$$

Since $\psi \in \mathfrak{C}^1[a,b]$, and by definition of $\mathfrak{C}^{\eta}_{\eta;\psi}[a,b]$, clearly $\mathfrak{I}^{1-\eta;\psi}_{a^+}\zeta \in \mathfrak{C}^1_{1-\eta;\psi}[a,b]$. Hence, by using Theorem 2.8 and initial condition (1.2), for $t \in (a,b]$ we have

$$\begin{aligned}\mathfrak{I}^{\sigma;\psi}_{a^+}\mathfrak{D}^{\eta;\psi}_{a^+}\zeta(t) &= \zeta(t) - \frac{(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}\mathfrak{I}^{1-\eta;\psi}_{a^+}\zeta(a) \\ &= \zeta(t) - \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1}. \qquad (3.9)\end{aligned}$$

From the fact that $\mathfrak{D}^{\eta;\psi}_{a^+}\zeta \in \mathfrak{C}_{1-\eta;\psi}[a,b]$ and Lemma 2.3, we have

$$\begin{aligned}\mathfrak{I}^{\eta;\psi}_{a^+}\mathfrak{D}^{\eta;\psi}_{a^+}\zeta(t) &= \mathfrak{I}^{\sigma;\psi}_{a^+}\mathfrak{D}^{\sigma,\delta;\psi}_{a^+}\zeta(t) \\ &= \mathfrak{I}^{\sigma;\psi}_{a^+}\mathfrak{D}^{\sigma,\delta;\psi}_{a^+} f(t,\zeta(t),\zeta(t-\tau)). \qquad (3.10)\end{aligned}$$

Comparing (3.9) and (3.10), we reach at the expected integral equation (3.8) as

$$\begin{aligned}\zeta(t) &= \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} + \frac{1}{\Gamma(\sigma)}\mathfrak{I}_{a^+}^{\sigma;\psi} f(s,\zeta(s),\zeta(s-\tau)) \\ &= \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} + \frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(a))^{\sigma-1} f(s,\zeta(s),\zeta(s-\tau))ds.\end{aligned}$$

Conversely, let $\zeta \in \mathfrak{C}_{1-\eta;\psi}^{\eta}[a,b]$ satisfies (3.8). We prove that ζ also satisfies the FDDEs (1.1)-(1.3). Applying $\mathfrak{D}_{a^+}^{\eta;\psi}$ on both sides of (3.8) and in view of Lemma 2.4, one can write

$$\begin{aligned}\mathfrak{D}_{a^+}^{\eta;\psi}\zeta(t) &= \mathfrak{D}_{a^+}^{\eta;\psi}\left(\frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} + \frac{1}{\Gamma(\sigma)}\mathfrak{I}_{a^+}^{\sigma;\psi} f(t,\zeta(t),\zeta(t-\tau))\right) \\ &= \frac{c}{\Gamma(\eta)}\mathfrak{D}_{a^+}^{\eta;\psi}(\psi(t)-\psi(a))^{\eta-1} + \mathfrak{D}_{a^+}^{\eta;\psi}\mathfrak{I}_{a^+}^{\sigma;\psi} f(t,\zeta(t),\zeta(t-\tau)) \\ &= \mathfrak{D}_{a^+}^{\delta(1-\sigma);\psi} f(t,\zeta(t),\zeta(t-\tau)). \qquad (3.11)\end{aligned}$$

Since $D_{a^+}^{\eta;\psi}\zeta \in \mathfrak{C}_{1-\eta;\psi}[a,b]$, equation (3.10) implicate

$$\mathfrak{D}_{a^+}^{\eta;\psi}\zeta(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \mathfrak{I}_{a^+}^{(1-\eta);\psi} f(t,\zeta(t),\zeta(t-\tau)) \in \mathfrak{C}_{1-\eta;\psi}[a,b].$$

Since $f(t,\zeta(t),\zeta(t-\tau)) \in \mathfrak{C}_{1-\eta;\psi}[a,b]$ and by Lemma 2.5, $\mathfrak{I}_{a^+}^{(1-\delta)(1-\sigma);\psi} f(t,\zeta(t),\zeta(t-\tau)) \in \mathfrak{C}_{1-\eta;\psi}[a,b]$. Now applying $\mathfrak{I}_{a^+}^{\delta(1-\sigma);\psi}$ on both sides of (3.10):

$$\begin{aligned}\mathfrak{I}_{a^+}^{\delta(1-\sigma);\psi}\mathfrak{D}_{a^+}^{\eta;\psi}\zeta(t) &= \mathfrak{I}_{a^+}^{\delta(1-\sigma);\psi}\mathfrak{D}_{a^+}^{\delta(1-\sigma);\psi} f(t,\zeta(t),\zeta(t-\tau)) \\ &= f(t,\zeta(t),\zeta(t-\tau)) - \frac{\mathfrak{I}_{a^+}^{(1-\delta)(1-\sigma);\psi} f(t,\zeta(t),\zeta(t-\tau))}{\Gamma(\delta(1-\sigma))}(\psi(t)-\psi(a))^{\delta(1-\sigma)} \\ &= f(t,\zeta(t),\zeta(t-\tau)). \qquad (3.12)\end{aligned}$$

Contrasting (2.5) and (3.12), we have

$$\mathfrak{I}_{a^+}^{\delta(1-\sigma);\psi}\mathfrak{D}_{a^+}^{\eta;\psi}\zeta(t) = \mathfrak{D}_{a^+}^{\sigma,\delta;\psi}\zeta(t) = f(t,\zeta(t),\zeta(t-\tau)).$$

Now we show that $\zeta \in \mathfrak{C}_{1-\eta;\psi}^{\eta}[a,b]$ given by (3.8) also satisfies the initial condition (1.2). Applying $\mathfrak{I}_{a^+}^{1-\eta;\psi}$ to both sides of (3.8), using Lemma 2.1 and Lemma 2.4, we have

$$\begin{aligned}\mathfrak{I}_{a^+}^{1-\eta;\psi}\zeta(t) &= \mathfrak{I}_{a^+}^{1-\eta;\psi}\left[\frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} + \mathfrak{I}_{a^+}^{\sigma;\psi} f(t,\zeta(t),\zeta(t-\tau))\right] \\ &= c + \mathfrak{I}_{a^+}^{1-\eta;\psi}\mathfrak{I}_{a^+}^{\sigma;\psi} f(t,\zeta(t),\zeta(t-\tau)) \\ &= c + \mathfrak{I}_{a^+}^{1-\delta(1-\sigma);\psi} f(t,\zeta(t),\zeta(t-\tau)). \qquad (3.13)\end{aligned}$$

Now, $t \to a^+$ in (3.13), and by Lemma 2.6, we conclude that $\mathfrak{I}_{a^+}^{1-\eta;\psi}\zeta(t) = c$. □

Theorem 3.2. *Let* $0 < \sigma < 1,\ 0 \le \delta \le 1$, *and* $\eta = \sigma + \delta - \sigma\delta$. *Assume that* $(\mathbf{A1}) - (\mathbf{A2})$ *hold. Then there exists unique solution* ζ *for FDDEs* (1.1)-(1.3) *in* $\mathfrak{C}_{1-\eta;\psi}^{\eta}[a,b]$.

Proof. We begin to prove the existence and uniqueness in $\mathfrak{C}_{1-\eta}[0,T]$. The proof is constructed by examining two cases, the first one is when $t \in (0,\tau] = [a,b]$ and second one is $t \in (\tau,T]$. In each cases, we divided the interval into j subinterval as $(t_0,t_1],[t_1,t_2],...[t_{j-1},t_j]$ on which operator χ is contraction mapping on all subinterval. Consider $\chi : X \to X$ with $X = \{\zeta \in \mathfrak{C}_{1-\eta}[0,T]\}$ defined as:

(3.14)

$$\chi(\zeta(t)) = \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} + \frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(a))^{\sigma-1} f(s,\zeta(s),\zeta(s-\tau))ds.$$

Step I: Let $t \in (0,\tau]$, so $\zeta(t-\tau) = \phi(t-\tau) = \lambda(t-\tau)$. Choose k_1, k_2 such that $0 < k_1 < k_2 \leq \tau$, so $\mathfrak{C}_{1-\eta}[k_1,k_2]$ is complete metric space with

$$\begin{aligned} d(\zeta(t)-\lambda(t)) &= \|\zeta(t)-\lambda(t)\|_{\mathfrak{C}_{1-\eta}[k_1,k_2]} \\ &= \max_{t\in[k_1,k_2]} \left|(\psi(t)-\psi(a))^{\sigma-1}(\zeta(t)-\lambda(t))\right|. \end{aligned}$$

Select $t_1 \in (0,\tau]$ such that

$$\Lambda_1 = \frac{L\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t)-\psi(a))^{\eta+\sigma-1} < 1.$$

Note that $\dfrac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} \in \mathfrak{C}_{1-\eta;\psi}[0,t_1]$ and in light of Lemma 2.5, $\chi(\zeta(t)) \in \mathfrak{C}_{1-\eta;\psi}[0,t_1]$; χ maps $\mathfrak{C}_{1-\eta;\psi}[0,t_1]$ ino itself.
Now, we prove that χ has a fixed point in $\mathfrak{C}_{1-\eta;\psi}[0,t_1]$ which is the unique solution to FDDEs (1.1)-(1.3) on $(0,t_1]$. To this end, it is sufficient to prove that the operator χ is a contraction map. For any $\zeta(t),\ \lambda(t) \in \mathfrak{C}_{1-\eta;\psi}[0,t_1]$, we have

$$\begin{aligned} \|\chi(\zeta(t))-\chi(\lambda(t))\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]} &= \|\mathfrak{I}^{\sigma;\psi}_{a^+} f(t,\zeta(t),\zeta(t-\tau)) - \mathfrak{I}^{\sigma;\psi}_{a^+} f(t,\lambda(t),\lambda(t-\tau))\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]} \\ &\leq \|\mathfrak{I}^{\sigma;\psi}_{a^+}|f(t,\zeta(t),\zeta(t-\tau)) - f(t,\lambda(t),\lambda(t-\tau))|\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]}. \end{aligned}$$

Note that

$$\begin{aligned} &\|f(t,\zeta(t),\zeta(t-\tau))-f(t,\lambda(t),\lambda(t-\tau))\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]} \\ &\qquad= \max_{t\in[0,t_1]} (\psi(t)-\psi(a))^{1-\eta}|f(t,\zeta(t),\zeta(t-\tau)) - f(t,\lambda(t),\lambda(t-\tau))| \end{aligned}$$

so,

$$\begin{aligned} &\|f(t,\zeta(t),\zeta(t-\tau))-f(t,\lambda(t),\lambda(t-\tau))\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]} \\ &\qquad\geq (\psi(t)-\psi(a))^{1-\eta}|f(t,\zeta(t),\zeta(t-\tau)) - f(t,\lambda(t),\lambda(t-\tau))|. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|f(t,\zeta(t),\zeta(t-\tau))-f(t,\lambda(t),\lambda(t-\tau))\|t_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]} \\ &\qquad\leq (\psi(t)-\psi(a))^{\eta-1}|f(t,\zeta(t),\zeta(t-\tau)) - f(t,\lambda(t),\lambda(t-\tau))| \end{aligned}$$

$$\begin{aligned} &\|\mathfrak{I}^{\sigma;\psi}_{a^+}(f(t,\zeta(t),\zeta(t-\tau))-f(t,\lambda(t),\lambda(t-\tau)))\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]} \\ &\qquad\leq \|\mathfrak{I}^{\sigma;\psi}_{a^+}|f(t,\zeta(t),\zeta(t-\tau)) - f(t,\lambda(t),\lambda(t-\tau))|\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]} \end{aligned}$$

consequently,

$$\begin{aligned} \|\chi(\zeta(t))-\chi(\lambda(t))\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]} &\leq \frac{\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t)-\psi(a))^{\eta+\sigma-1} \\ &\quad\times L\Big(\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]} + \|\zeta(t-\tau)-\lambda(t-\tau)\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]}\Big) \\ &\leq \frac{\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t)-\psi(a))^{\eta+\sigma-1} \\ &\quad\times L\Big(\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]} + \|\phi(t-\tau)-\phi(t-\tau)\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]}\Big) \\ &\leq \frac{\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t)-\psi(a))^{\eta+\sigma-1} L\Big(\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]}\Big) \\ &= \Lambda_1\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]}. \end{aligned}$$

Since $\Lambda_1 < 1$, by contraction mapping theorem, we can deduce that a single fixed point exists, that is the solution $\zeta_0(t) \in \mathfrak{C}_{1-\eta;\psi}[0, t_1]$ on $(0, t_1]$. If $t_1 \neq \tau$, we consider $[t_1, \tau]$ and note that

$$\begin{aligned}
\chi(\zeta(t)) =& \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} + \frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(a))^{\sigma-1} f(s,\zeta(s),\zeta(s-\tau))ds \\
=& \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} + \frac{1}{\Gamma(\sigma)}\int_a^{t_1} \psi'(s)(\psi(t)-\psi(a))^{\sigma-1} f(s,\zeta(s),\zeta(s-\tau))ds \\
&+ \frac{1}{\Gamma(\sigma)}\int_{t_1}^t \psi'(s)(\psi(t)-\psi(a))^{\sigma-1} f(s,\zeta(s),\zeta(s-\tau)) \\
=& \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} + \frac{1}{\Gamma(\sigma)}\int_a^{t_1} \psi'(s)(\psi(t)-\psi(a))^{\sigma-1} \\
&\times f(s,\zeta(s),\zeta(s-\tau))ds + \mathfrak{I}_{a^+}^{\sigma;\psi} f(t,\zeta(t),\zeta(t-\tau)).
\end{aligned}$$

Select $\Lambda_2 < 1$ such that $\Lambda_2 = \dfrac{L\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t_2)-\psi(t_1))^{\eta+\sigma-1} < 1$. Let $\zeta(t),\ \lambda(t) \in \mathfrak{C}[t_1, t_2]$, for some $t_1 < t_2 < \tau$, we get

$$\begin{aligned}
\|\chi(\zeta(t)) - \chi(\lambda(t))\|_{\mathfrak{C}[t_1,t_2]} &= \|\mathfrak{I}_{t_1^+}^{\sigma;\psi} f(t,\zeta(t),\zeta(t-\tau)) - \mathfrak{I}_{t_1^+}^{\sigma;\psi} f(t,\lambda(t),\lambda(t-\tau))\|_{\mathfrak{C}[t_1,t_2]} \\
&\le \|\mathfrak{I}_{t_1^+}^{\sigma;\psi} |f(t,\zeta(t),\zeta(t-\tau)) - f(t,\lambda(t),\lambda(t-\tau))|\|_{\mathfrak{C}[t_1,t_2]} \\
&\le \frac{\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t_2)-\psi(t_1))^{\eta+\sigma-1} \\
&\times \|f(t,\zeta(t),\zeta(t-\tau)) - f(t,\lambda(t),\lambda(t-\tau))\|_{\mathfrak{C}[t_1,t_2]} \\
&= \frac{L\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t_2)-\psi(t_1))^{\eta+\sigma-1}\Big(\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}[t_1,t_2]}\Big) \\
&= \Lambda_2\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}[t_1,t_2]}.
\end{aligned}$$

Since, $\Lambda_2 < 1$, χ is contraction on $[t_1, t_2]$ and there exists a unique solution $\zeta_1(t)$ for $t \in [t_1, t_2]$, by Lemma 2.9, we can see $\zeta_0(t) = \zeta_1(t)$. So

$$\zeta(t) = \begin{cases} \zeta_0(t) & 0 < t \le t_1 \\ \zeta_1(t) & t_1 < t \le t_2. \end{cases}$$

By lemma 2.7 $\zeta \in \mathfrak{C}_{1-\eta}[0, t_2]$, therefore $\zeta(t)$ is unique solution of FDDEs (1.1)-(1.3) on $[0, t_2]$.

If $t_2 \neq \tau$, we iterate the above step more $(j-2)$ times, and we get a unique solution $\zeta_i(t)$ for $[t_i, t_{i+1}]$ with $i = 2, 3, \cdots j$ where $0 = t_0 < t_1 < \cdots < t_j = \tau$ such that $\Lambda_{i+1} = \dfrac{L\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t_{i+1})-\psi(t_i))^{\eta+\sigma-1} < 1$.

Hence, we have unique solution $\zeta(t) \in \mathfrak{C}_{1-\eta;\psi}[0, \tau]$ such that

$$\zeta(t) = \begin{cases} \zeta_0(t), & 0 < t \le t_1, \\ \zeta_1(t), & t_1 < t \le t_2, \\ \vdots & \\ \zeta_n(t), & t_{j-1} < t \le t_n = \tau. \end{cases}$$

Now, since $\zeta(t) \in \mathfrak{C}_{1-\eta;\psi}[0, \tau]$ is unique solution and satisfies (3.8). By conversing Lemma 3.11, we proved $\zeta(t) \in \mathfrak{C}^{\eta}_{1-\eta;\psi}[0, \tau]$.

Step II: If $t \in (\tau, T]$ and $\zeta(t),\ \lambda(t) \in \mathfrak{C}_{1-\eta}[\tau, T]$. Divide $[0, T]$ into $[0,\tau] \cup \cdots \cup [(i_0 - 1)\tau, i_0\tau] \cup [i_0\tau, T]$, where $i_0 \in j$ such that $0 \leq T - i_0\tau \leq T$. Suppose that FDDEs (1.1)-(1.3) possesses a unique solution say $\zeta_i^*(t)$ on $[\tau, i\tau]$, where $1 \leq i \leq i_0$. So, we want to prove that unique solution $\zeta_{i+1}^*(t)$ exists on $[i\tau, (i+1)\tau]$. Suppose that χ is contraction map for $t \in [\tau, i\tau]$ where $\|\chi\zeta(t) - \chi\lambda(t)\|_{\mathfrak{C}[\tau,i\tau]} \leq \Lambda_i \|\zeta(t) - \lambda(t)\|_{\mathfrak{C}[\tau,i\tau]}$.
Let $t \in [i\tau, (i+1)\tau]$. Then $\zeta(t-\tau) = \lambda(t-\tau) = \phi_i(t-\tau)$ and $k_1,\ k_2$ such that $\tau < k_1 < k_2 \leq i\tau$. So $\mathfrak{C}_{1-\eta;\psi}[k_1, k_2]$ is complete metric space with

$$\begin{aligned} d(\zeta(t) - \lambda(t)) &= \|\zeta(t) - \lambda(t)\|_{\mathfrak{C}_{1-\eta}[k_1,k_2]} \\ &= \max_{t\in[k_1,k_2]} \left|(\psi(t) - \psi(a))^{\sigma-1}(\zeta(t) - \lambda(t))\right|. \end{aligned}$$

Select $t_1 \in (i\tau, (i+1)\tau]$ such that

$$\Lambda_{1i} = \frac{L\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t) - \psi(a))^{\eta+\sigma-1} < 1.$$

Note that $\dfrac{c}{\Gamma(\eta)}(\psi(t) - \psi(a))^{\eta-1} \in \mathfrak{C}_{1-\eta;\psi}[i\tau, t_1]$ so by Lemma 2.5 $\chi(\zeta(t)) \in \mathfrak{C}_{1-\eta;\psi}[i\tau, t_1]$; χ maps $\mathfrak{C}_{1-\eta;\psi}[i\tau, t_1]$ into itself.
Now for each $\zeta(t),\ \lambda(t) \in \mathfrak{C}_{1-\eta;\psi}[i\tau, t_1]$ we have

$$\begin{aligned} &\|\chi(\zeta(t)) - \chi(\lambda(t))\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]} \\ &\quad= \|\mathfrak{I}_{a^+}^{\sigma;\psi} f(t, \zeta(t), \zeta(t-\tau)) - \mathfrak{I}_{a^+}^{\sigma;\psi} f(t, \lambda(t), \lambda(t-\tau))\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]} \\ &\quad\leq \|\mathfrak{I}_{a^+}^{\sigma;\psi} |f(t, \zeta(t), \zeta(t-\tau)) - f(t, \lambda(t), \lambda(t-\tau))|\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]} \end{aligned}$$

Note that

$$\begin{aligned} &\|f(t, \zeta(t), \zeta(t-\tau)) - f(t, \lambda(t), \lambda(t-\tau))\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]} \\ &= \max_{t\in[0,t_1]} (\psi(t) - \psi(a))^{1-\eta} |f(t, \zeta(t), \zeta(t-\tau)) - f(t, \lambda(t), \lambda(t-\tau))|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]} \end{aligned}$$

however,

$$\begin{aligned} &\|f(t, \zeta(t), \zeta(t-\tau)) - f(t, \lambda(t), \lambda(t-\tau))\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]} \\ &\qquad\geq (\psi(t) - \psi(a))^{1-\eta} |f(t, \zeta(t), \zeta(t-\tau)) - f(t, \lambda(t), \lambda(t-\tau))|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]} \end{aligned}$$

therefore,

$$\begin{aligned} &\|f(t, \zeta(t), \zeta(t-\tau)) - f(t, \lambda(t), \lambda(t-\tau))\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]} \\ &\qquad\leq (\psi(t) - \psi(a))^{\eta-1} |f(t, \zeta(t), \zeta(t-\tau)) - f(t, \lambda(t), \lambda(t-\tau))|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]} \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\|\mathfrak{I}_{a^+}^{\sigma;\psi}(f(t, \zeta(t), \zeta(t-\tau)) - f(t, \lambda(t), \lambda(t-\tau)))\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]} \\ &\qquad\leq \|\mathfrak{I}_{a^+}^{\sigma;\psi} |f(t, \zeta(t), \zeta(t-\tau)) - f(t, \lambda(t), \lambda(t-\tau))|\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]} \end{aligned} \tag{3.16}$$

which gives

$$\begin{aligned}\|\chi\zeta(t)-\chi\lambda(t)\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]} &\leq \frac{\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t)-\psi(a))^{\eta+\sigma-1}\\ &\times L\bigg(\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}_{1-\eta;\psi}[0,t_1]}+\|\zeta(t-\tau)-\lambda(t-\tau)\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]}\bigg)\\ &\leq \frac{\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t)-\psi(a))^{\eta+\sigma-1}\\ &\times L\bigg(\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]}+\|\phi(t-\tau)-\phi(t-\tau)\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]}\bigg)\\ &\leq \frac{\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t)-\psi(a))^{\eta+\sigma-1}L\bigg(\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]}\bigg)\\ &= \Lambda_{1i}\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]}.\end{aligned}$$

Since $\Lambda_{1i}<1$, by contraction mapping theorem we can deduce a single fix point which is a solution $\zeta_0^*(t)\in\mathfrak{C}_{1-\eta;\psi}[i\tau,t_1]$.
If $t_1\neq(i+1)\tau$, then set $[t_1,(i+1)\tau$. Furthermore

$$\begin{aligned}\chi(\zeta(t)) &= \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1}+\frac{1}{\Gamma(\sigma)}\int_a^t\psi'(s)(\psi(t)-\psi(a))^{\sigma-1}f(s,\zeta(s),\zeta(s-\tau))ds\\ &= \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1}+\frac{1}{\Gamma(\sigma)}\int_a^{t_1}\psi'(s)(\psi(t)-\psi(a))^{\sigma-1}f(s,\zeta(s),\zeta(s-\tau))ds\\ &+\frac{1}{\Gamma(\sigma)}\int_{t_1}^t\psi'(s)(\psi(t)-\psi(a))^{\sigma-1}f(s,\zeta(s),\zeta(s-\tau))\\ &= \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1}+\frac{1}{\Gamma(\sigma)}\int_a^{t_1}\psi'(s)(\psi(t)-\psi(a))^{\sigma-1}\\ &\times f(s,\zeta(s),\zeta(s-\tau))ds+\mathfrak{I}_{a^+}^{\sigma;\psi}f(t,\zeta(t),\zeta(t-\tau)).\end{aligned}$$

Choose $\Lambda_{2i}<1$ such that $\Lambda_2=\dfrac{L\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t_1)-\psi(t_2))^{\eta+\sigma-1}<1$. Let $\zeta(t),\ \lambda(t)\in\mathfrak{C}[t_1,t_2]$, for some $t_1<t_2<(i+1)\tau$, we get

$$\begin{aligned}&\|\chi(\zeta(t))-\chi(\lambda(t))\|_{\mathfrak{C}[t_1,t_2]}\\ &=\|\mathfrak{I}_{t_1^+}^{\sigma;\psi}f(t,\zeta(t),\zeta(t-\tau))-\mathfrak{I}_{t_1^+}^{\sigma;\psi}f(t,\lambda(t),\lambda(t-\tau))\|_{\mathfrak{C}[t_1,t_2]}\\ &\leq\|\mathfrak{I}_{t_1^+}^{\sigma;\psi}|f(t,\zeta(t),\zeta(t-\tau))-f(t,\lambda(t),\lambda(t-\tau))|\|_{\mathfrak{C}[t_1,t_2]}\\ &\leq\frac{\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t_2)-\psi(t_1))^{\eta+\sigma-1}\|f(t,\zeta(t),\zeta(t-\tau))-f(t,\lambda(t),\lambda(t-\tau))\|_{\mathfrak{C}[t_1,t_2]}\\ &\leq\frac{L\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t_2)-\psi(t_1))^{\eta+\sigma-1}\bigg(\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}[t_1,t_2]}+\|\zeta(t-\tau)-\lambda(t-\tau)\|_{\mathfrak{C}[t_1,t_2]}\bigg)\\ &\leq\frac{L\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t_2)-\psi(t_1))^{\eta+\sigma-1}\bigg(\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}[t_1,t_2]}+\|\phi(t-\tau)-\phi(t-\tau)\|_{\mathfrak{C}[t_1,t_2]}\bigg)\\ &=\frac{L\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t_2)-\psi(t_1))^{\eta+\sigma-1}\bigg(\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}[t_1,t_2]}\bigg)\\ &=\Lambda_{2i}\|\zeta(t)-\lambda(t)\|_{\mathfrak{C}[t_1,t_2]}.\end{aligned}$$

Since, $\Lambda_{2i} < 1$, χ is contraction on $[t_1, t_2]$ and there exists a unique solution $\zeta_1(t)$ for $t \in [t_1, t_2]$, by Lemma 2.9, we can see $\zeta_0(t) = \zeta_1(t)$.

$$\zeta(t) = \begin{cases} \zeta_0(t), & 0 < t \le t_1 \\ \zeta_1(t), & t_1 < t \le t_2. \end{cases}$$

By Lemma 2.7 $u \in \mathfrak{C}_{1-\eta}[0, t_2]$, therefore $\zeta(t)$ is unique solution of FDDEs (1.1)-(1.3) on $[0, t_2]$.

If $t_2 \neq (i+1)\tau$, we iterate the above step more $(j-2)$ times, after that we get the unique solution $\lambda_i(t)$ for $[t_i, t_{i+1}]$ with $i = 2, 3, \cdots j$, where $i\tau = t_0 < t_1 < \cdots < t_j = (i+1)\tau$ such that

$$\Lambda_{i+1} = \frac{L\Gamma(\eta)}{\Gamma(\eta+\sigma)}(\psi(t_{i+1}) - \psi(t_i))^{\eta+\sigma-1} < 1.$$

Hence, we have unique solution $\zeta(t) \in \mathfrak{C}_{1-\eta;\psi}[i\tau, (i+1)\tau]$ such that

$$\zeta(t) = \begin{cases} \zeta_0(t), & i\tau < t \le t_1, \\ \zeta_1(t), & t_1 < t \le t_2, \\ \vdots & \\ \zeta_n(t), & t_{j-1} < t \le t_j = (i+1)\tau. \end{cases}$$

Now, since $\zeta(t) \in \mathfrak{C}_{1-\eta;\psi}[i\tau, (i+1)\tau]$ is a unique solution and satisfies (3.8). By converse part of Lemma 3.11 we proved $\zeta(t) \in \mathfrak{C}^{\eta}_{1-\eta;\psi}[i\tau, (i+1)\tau]$. □

4. Continuous Dependence

Theorem 4.3. *Let $\psi \in \mathfrak{C}([a,b], \mathfrak{R})$ a function such that it is increasing and $\psi'(t) \neq 0$, for all $t \in [a,b]$. Also, let $f \in (a,b] \times \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$ is continuation and satisfies Lipschitz condition (**A2**) on $\mathfrak{R}$ and $\sigma > 0$, $\varepsilon > 0$ such that $0 < \sigma - \epsilon < \sigma \le 1$ with $0 \le \delta \le 1$. For any $a \le t \le b$, assume that ζ is the solution of the FDDEs (1.1)-(1.3) and λ^* is the solution of the following problem:*

$$\mathfrak{D}^{\sigma-\varepsilon,\delta;\psi}_{a^+}\lambda^*(t) = f(t, \lambda^*(t), \lambda^*(t-\tau)), \;\; 0 < \sigma < 1, 0 \le \delta \le 1, t > a, \tag{4.17}$$

$$\lambda^*(t) = \phi(t), \;\; -\tau \le t < 0 \tag{4.18}$$

$$\mathfrak{I}^{1-\eta-\varepsilon(\delta-1);\psi}_{a^+}\lambda^*(a) = c^*. \qquad \eta = \sigma + \delta - \sigma\delta. \tag{4.19}$$

Then, for $a < t \le b$,

$$|\lambda^*(t) - \lambda(t)| \le B(t) + \int_a^t \left[\sum_{k=1}^{\infty} \left(\frac{L\Gamma(\sigma-\varepsilon)}{\Gamma(\sigma)}\right)^k \frac{\psi'(s)(\psi(t)-\psi(s))^{k(\sigma-\varepsilon)-1}}{\Gamma(k(\sigma-\varepsilon))} B(s)\right] ds,$$

where

$$\begin{aligned} B(t) = & \left|\frac{c^*(\psi(t)-\psi(a))^{\eta+\varepsilon(\delta-1)-1}}{\Gamma(\eta+\varepsilon(\delta-1))} - \frac{c(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}\right| \\ & + \|f\| \left|\frac{(\psi(t)-\psi(a))^{\sigma-\varepsilon}}{\Gamma(\sigma-\varepsilon+1)} - \frac{(\psi(t)-\psi(a))^{\sigma-\varepsilon}}{\Gamma(\sigma-\varepsilon)\Gamma(\sigma)}\right| \\ & + \|f\| \left|\frac{(\psi(t)-\psi(a))^{\sigma-\varepsilon}}{\Gamma(\sigma-\varepsilon)\Gamma(\sigma)} - \frac{(\psi(t)-\psi(a))^{\sigma}}{\Gamma(\sigma+1)}\right|, \end{aligned} \tag{4.20}$$

$\|f\| = \max_{t \in [a,b]} |f(t, \lambda(t), \lambda(t-\tau))|$.

Proof. The FDDEs (1.1)-(1.3) and (4.17)-(4.19), have integral equations which are given by

$$\begin{aligned}\lambda(t) &= \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1}\\ &+\frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}f(s,\lambda(s),)ds,\ t>a,\end{aligned}$$

and

$$\begin{aligned}\lambda^*(t) &= \frac{c^*}{\Gamma(\eta+\varepsilon(\delta-1))}(\psi(t)-\psi(a))^{\eta+\varepsilon(\delta-1)-1}\\ &+\frac{1}{\Gamma(\sigma-\varepsilon)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-\varepsilon-1}f(s,\lambda^*(s),\lambda^*(s-\tau))ds,\ t>a,\end{aligned}$$

respectively. It follows that

$$\begin{aligned}|\lambda^*(t)-\lambda(t)| \le &\left|\frac{c^*(\psi(t)-\psi(a))^{\eta+\varepsilon(\delta-1)-1}}{\Gamma(\eta+\varepsilon(\delta-1))}-\frac{c(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}\right|\\ &+\left|\frac{1}{\Gamma(\sigma-\varepsilon)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-\varepsilon-1}f(s,\lambda^*(s),\lambda^*(s-\tau))ds\right.\\ &\left.-\frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}f(s,\lambda(s),\lambda(s-\tau))ds\right|\\ \le&\left|\frac{c^*(\psi(t)-\psi(a))^{\eta+\varepsilon(\delta-1)-1}}{\Gamma(\eta+\varepsilon(\delta-1))}-\frac{c(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}\right|\\ &+\left|\int_a^t \psi'(s)\left[\frac{(\psi(t)-\psi(s))^{\sigma-\varepsilon-1}}{\Gamma(\sigma-\varepsilon)}-\frac{(\psi(t)-\psi(s))^{\sigma-\varepsilon-1}}{\Gamma(\sigma)}\right]f(s,\lambda^*(s),\lambda^*(s-\tau))ds\right.\\ &+\frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-\varepsilon-1}\left[f(s,\lambda^*(s),\lambda^*(s-\tau))-f(s,\lambda(s),\lambda(s-\tau))\right]ds\\ &\left.+\int_a^t \psi'(s)\left[\frac{(\psi(t)-\psi(s))^{\sigma-\varepsilon-1}}{\Gamma(\sigma)}-\frac{(\psi(t)-\psi(s))^{\sigma-1}}{\Gamma(\sigma)}\right]f(s,\lambda(s),\lambda(s-\tau))ds\right|.\end{aligned}$$

Since

$$\begin{aligned}|\lambda^*(t)-\lambda(t)| =&|f(t,\lambda^*(t),\lambda^*(t-\tau))-f(t,\lambda(t),\lambda(-\tau))|\\ &\le L(|\lambda^*(t)-\lambda(t)|+|\lambda^*(t-\tau)-\lambda(-\tau)|\\ &\le L|\lambda^*(t)-\lambda(t)|.\end{aligned}$$

Then

$$\begin{aligned}|\lambda^*(t)-\lambda(t)| &\le \left|\frac{c^*(\psi(t)-\psi(a))^{\eta+\varepsilon(\delta-1)-1}}{\Gamma(\eta+\varepsilon(\delta-1))}-\frac{c(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}\right|\\ &+\|f\|\left|\frac{(\psi(t)-\psi(a))^{\sigma-\varepsilon}}{\Gamma(\sigma-\varepsilon+1)}-\frac{(\psi(t)-\psi(a))^{\sigma-\varepsilon}}{\Gamma(\sigma)\Gamma(\sigma-\varepsilon)}\right|\\ &+L\frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-\varepsilon-1}\left|\lambda^*(s)-\lambda(s)\right|ds\\ &+\|f\|\left[\frac{(\psi(t)-\psi(s))^{\sigma-\varepsilon}}{\Gamma(\sigma)\Gamma(\sigma-\varepsilon)}-\frac{(\psi(t)-\psi(s))^{\sigma}}{\Gamma(\sigma+1)}\right]\\ &= B(t)+L\frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-\varepsilon-1}\left|\lambda^*(s)-\lambda(s)\right|ds,\end{aligned}$$

where $B(t)$ is defined as in (4.20). By applying Lemma 2.10, we conclude that

$$|\lambda^*(t)-\lambda(t)| \le B(t) + \int_a^t \left[\sum_{k=1}^{\infty}\left(\frac{L\Gamma(\sigma-\varepsilon)}{\Gamma(\sigma)}\right)^k \frac{\psi'(s)(\psi(t)-\psi(s))^{k(\sigma-\varepsilon)-1}}{\Gamma(k(\sigma-\varepsilon))} B(s)\right] ds.$$

□

Next, we consider the following fractional differential equation

$$\mathfrak{D}_{a^+}^{\sigma,\delta;\psi}\lambda(t) = f(t,\lambda(t),\lambda(t-\tau)),\;\; 0<\sigma<1, 0\le\delta\le 1, t>a. \tag{4.21}$$

with condition

$$\lambda(t) = \phi(t),\;\; -\tau\le t<0 \tag{4.22}$$

$$\mathfrak{I}_{a^+}^{1-\eta;\psi}\lambda(a) = c+\rho, \qquad \eta=\sigma+\delta-\sigma\delta. \tag{4.23}$$

Theorem 4.4. *Assume that hypotheses of Theorem 3.2 hold. Let ζ and λ^* are solutions of the FDDEs* (1.1)-(1.3) *and* (4.21)-(4.23) *respectively. Then*

$$|\lambda(t)-\lambda^*(t)| \le |\rho|\,(\psi(t)-\psi(a))^{\eta-1}E_{\sigma,\eta}\left[L(\psi(t)-\psi(a))^{\sigma}\right],\; t\in[a,b].$$

Proof. In view of Theorem 3.2, we have $\lambda(t) = \lim_{k\to\infty}\lambda_k(t)$ with

$$\lambda_0(t) = \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} \tag{4.24}$$

and

$$\begin{aligned}\lambda_k(t) &= \lambda_0(t) + \mathfrak{I}_{a^+}^{\sigma;\psi}F_{\lambda_{k-1}}(t)\\ &= \frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} + \frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}f(s,\lambda_{k-1}(s),\lambda_{k-1}(s))ds.\end{aligned} \tag{4.25}$$

Clearly, we can write $\lambda^*(t) = \lim_{k\to\infty}\lambda_k^*(t)$ with

$$\lambda_0^*(t) = \frac{(c+\rho)}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} \tag{4.26}$$

and

$$\begin{aligned}\lambda_k^*(t) &= \lambda_0^*(t) + \mathfrak{I}_{a^+}^{\sigma;\psi}f_{\lambda_{k-1}^*}(t)\\ &= \frac{(c+\rho)}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} + \frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}f(s,\lambda_{k-1}^*(s),\lambda_{k-1}^*(s))ds.\end{aligned} \tag{4.27}$$

By (4.24) and (4.26) we get

$$|\lambda_0(t)-\lambda_0^*(t)| = \left|\frac{c}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1} - \frac{(c+\rho)}{\Gamma(\eta)}(\psi(t)-\psi(a))^{\eta-1}\right| \le |\rho|\,\frac{(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}. \tag{4.28}$$

Using relations (4.24)-(4.27), the Lipschitz condition ($\mathbf{A2}$) and the inequality (4.28), we get

$$\begin{aligned}|\lambda_1(t)-\lambda_1^*(t)| \leq& |\rho|\frac{(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}+\frac{1}{\Gamma(\sigma)}\\ &\times\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}|f(s,\lambda_0(s),\lambda_0(s-\tau))-f(s,\lambda_0^*(s),\lambda_0^*(s-\tau)|ds\\ &\leq |\rho|\frac{(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}+\frac{L}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}|\lambda_k(t)-\lambda_0^*(t)|ds\\ &\leq |\rho|\frac{(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}+\frac{L|\rho|}{\Gamma(\eta)}\frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}(\psi(s)-\psi(a))^{\eta-1}ds\\ =&|\rho|\frac{(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}+\frac{L|\rho|}{\Gamma(\eta+\sigma)}(\psi(t)-\psi(a))^{\eta+\sigma-1}.\end{aligned}$$

Hence,

$$|\lambda_1(t)-\lambda_1^*(t)| \leq |\rho|\,(\psi(t)-\psi(a))^{\eta-1}\sum_{i=0}^{1}(L)^i\frac{(\psi(t)-\psi(a))^{\sigma i}}{\Gamma(\eta+\sigma k)}. \tag{4.29}$$

On the other hand, we have

$$\begin{aligned}|\lambda_2(t)-\lambda_2^*(t)| \leq& |\rho|\frac{(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}+\frac{1}{\Gamma(\sigma)}\\ &\times\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}|f(s,\lambda_1(s),\lambda_1(s-\tau))-f(s,\lambda_1^*(s),\lambda_1^*(s-\tau)|ds\\ \leq&|\rho|\frac{(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}+\frac{1}{\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}L|\lambda_1(t)-\lambda_1^*(t)|ds\\ \leq&|\rho|\frac{(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}+\frac{L|\rho|}{\Gamma(\eta)\Gamma(\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}(\psi(s)-\psi(a))^{\eta-1}ds\\ &+\frac{L^2|\rho|}{\Gamma(\sigma)\Gamma(\eta+\sigma)}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\sigma-1}(\psi(s)-\psi(a))^{\eta+\sigma-1}ds\\ \leq&|\rho|\frac{(\psi(t)-\psi(a))^{\eta-1}}{\Gamma(\eta)}+\frac{L|\rho|}{\Gamma(\eta+\sigma)}(\psi(t)-\psi(a))^{\eta+\sigma-1}\\ &+\frac{L^2|\rho|}{\Gamma(\eta+2\sigma)}(\psi(t)-\psi(a))^{\eta+2\sigma-1}\\ =&|\rho|(\psi(t)-\psi(a))^{\eta-1}\sum_{i=0}^{2}L^i\frac{(\psi(t)-\psi(a))^{\sigma i}}{\Gamma(\eta+\sigma i)}.\end{aligned}$$

Using the mathematical induction, we get

$$|\lambda_k(t)-\lambda_k^*(t)| \leq |\rho|\,(\psi(t)-\psi(a))^{\eta-1}\sum_{i=0}^{k}L^i\frac{(\psi(t)-\psi(a))^{\sigma i}}{\Gamma(\eta+\sigma i)}. \tag{4.30}$$

Taking the limit $k\to\infty$ in inequation (4.30), we obtain

$$|\lambda_k(t)-\lambda_k^*(t)| \leq |\rho|\,(\psi(t)-\psi(a))^{\eta-1}E_{\sigma,\eta}\left(L(\psi(t)-\psi(a))^{\sigma}\right).$$

□

5. Example

Example 5.1. *Consider the following nonlinear FDDEs:*

$$\mathfrak{D}_{a^+}^{\sigma,\delta;\psi} y(t) = -ky + f(t - y(t)), \tag{5.31}$$

for $f(t, y(t)) = \sin(t) + y^2(t)$, $t \geq \tau$ *and* $y(t) = g(t),\ t \in [0, \tau]$. *The integral solution can be written as*

$$y(t) = g(t) + \frac{1}{\Gamma(\sigma)} \int_0^t \psi'(s)(t - \psi(s))^{\sigma-1}(-ky(s-\tau) + \sin(s) + y^2(s))ds. \tag{5.32}$$

For $\psi(t) = \sin(t)$ *with* $\sigma = 0.5,\ 0.65,\ 0.7,\ 0.85,\ 0.95$, *and* $\delta = 1$, *all the conditions of Theorem 3.2 satisfied. Hence there exists a fixed point solution* $y(t)$ *given by* (5.32).

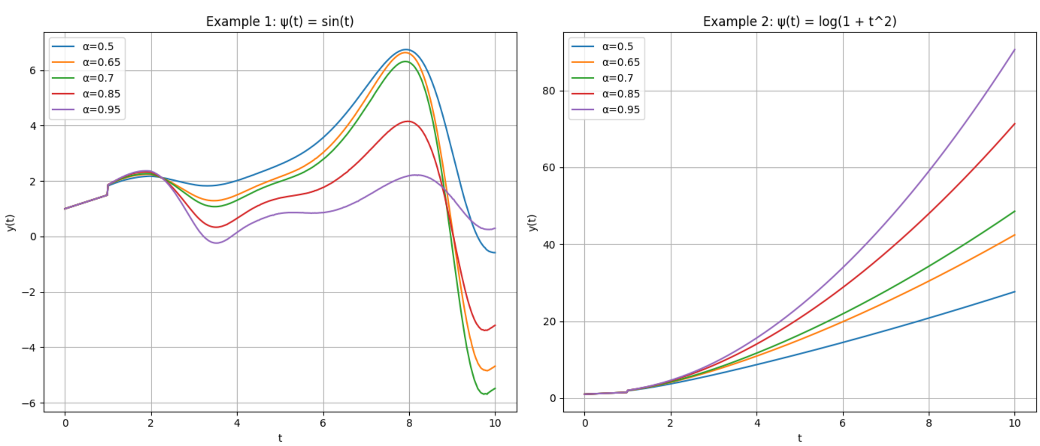
Example 5.2. *Consider the following nonlinear FDDEs:*

$$\mathfrak{D}_{a^+}^{\sigma,\delta;\psi} y(t) = h(t, y(t - \tau)), \tag{5.33}$$

for $h(t, y(t-\tau)) = e^{-t}\cos(y(t-\tau)) + t^2,\ t \geq \tau, y(t) = g(t),\ t \in [0, \tau]$. *The equivalent integral solution, by Lemma 2.1, is:*

$$y(t) = g(t) + \frac{1}{\Gamma(\sigma)} \int_0^t \psi'(s)(t - \psi(s))^{\sigma-1}(e^{-s}\cos(y(s-\tau)) + s^2)ds. \tag{5.34}$$

For $\psi(t) = \log(1 + t^2)$ *with* $\sigma = 0.5,\ 0.65,\ 0.7,\ 0.85,\ 0.95$, *and* $\delta = 1$.



6. Conclusion and future scope

We established the existence and uniqueness of solutions to nonlinear fractional delay differential equations (FDDEs) by applying the contraction mapping principle. Further, the result of continuous dependence of solution on order of differentiation using a generalized Gronwall inequality. Main results are justified with illustrative examples. This work may proven to be foundational work for nonlocal delay boundary value problems, some physico-chemical applications such as thermal shock problem with practical initial and boundary conditions in near future.

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