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Multiple positive solutions to n-component coupled system of iterative systems on time scales

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ABSTRACT. This study explores advances in detecting positive solutions for second order n-component coupled system of iterative systems with two-point boundary conditions on time scales. In this system, each component interacts with both itself and the next component in a cyclic manner. We establish criteria for existence of at least one positive solution using Guo–Krasnosel'skii fixed point theorem and at least three positive solutions utilizing Ren-Ge-Ren fixed point theorem.

1. INTRODUCTION

A time scale is a non-empty closed subset of the real numbers that represents the set of points at which a dynamic system evolves. The idea behind time scales is to unify and extend continuous and discrete time models into one framework, allowing for a more versatile approach to modeling systems that can operate in both continuous and discrete time. It provides a powerful tool for modeling and analyzing dynamic systems that exhibit different types of time-dependent behavior. For more details, refer to [2, 3, 4, 13, 16].

A strong foundation for resolving dynamic equations in mixed discrete-continuous settings is offered by the iterative system of boundary value problems (BVPs) on time scales. Systems having recursive architecture or feedback loops are modeled using iterative dynamic equations. These systems often arise in optimization, numerical methods and simulations, where repeated steps are key to finding solutions and interpreting the solutions, see [6, 12, 17, 20, 22].

In an n-component system of iterative systems, where the last iteration of component leads to the first iteration of the next component, the system evolves through interdependencies between components. In such systems, each component's iteration is tightly tied to the iterations of its neighbors, resulting in a closed-loop interaction. The behavior of each component is determined not just by its own state, but also by the states of the other components. These interactive coupled systems are primarily used to model the dynamics of multiple layers in a neural network, where the output of one layer becomes the input for the next layer, to study the spread of diseases among multiple populations, where the infection rate of one population affects the infection rate of the next population and to model the dynamics of multiple species in an ecosystem, we refer to [1, 5, 8, 14, 19].

In 2006, Hao et. al, [11] established existence of positive solutions for the BVP on time scales

$$u^{\Delta\Delta}(t)+m(t)f(t,u(\sigma(t))=0,\ t\in[a,b]_{\mathbb{T}},$$

$$\mathsf{a}u(a)-\beta u^{\Delta}(a)=0,\mathsf{c}u(\sigma(b))+\mathsf{d}u^{\Delta}(\sigma(b))=0,$$

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Recently, Prasad et al. [18] in 2022, considered the following two-point iterative BVP on time scales

$$v_\ell^{\Delta\Delta}(t) + \mathsf{L}(t)\mathfrak{g}_\ell(v_{\ell+1}(t)) = 0,\ 1 \leqslant \ell \leqslant m,\ t \in (0, \mathfrak{T})_{\mathbb{T}},$$
$$v_{m+1}(t) = v_1(t),\ t \in (0, \mathfrak{T})_{\mathbb{T}},$$
$$v_\ell(0) = v_\ell^{\Delta}(0), v_\ell(\mathfrak{T}) = -v_\ell^{\Delta}(\mathfrak{T}),\ 1 \leqslant \ell \leqslant m,$$

where $m \in \mathbb{N}$, $\mathsf{L}(t) = \prod_{i=1}^{m} \mathsf{L}_i(t)$ and each $\mathsf{L}_i(t) \in \mathsf{L}_{\Delta}^{\mathfrak{p}_i}([0, \mathfrak{T}]_{\mathbb{T}})$ has m-singularities and established infinitely many positive solutions.

Based on the above studies, we wish to study the existence of positive solutions for second order n-component coupled system of iterative system with two-point boundary conditions on time scales

$$\left.\begin{aligned}
&(\mathsf{u}_j^{(i)})^{\Delta\Delta}(\mathsf{r}) + \mathsf{L}_j(\mathsf{r})\, \mathfrak{f}_j^{(i)}(\mathsf{u}_j^{(i+1)}(\mathsf{r})) = 0,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\ \mathsf{r} \in [0, w]_{\mathbb{T}},\\
&\mathsf{u}_j^{(m+1)}(\mathsf{r}) = \mathsf{u}_{j+1}^{(1)}(\mathsf{r}),\ \mathsf{u}_{n+1}^{(1)}(\mathsf{r}) = \mathsf{u}_1^{(1)}(\mathsf{r}),\\
&\mathsf{a}_j\mathsf{u}_j^{(i)}(0) - \mathsf{b}_j\, (\mathsf{u}_j^{(i)})^{\Delta}(0) = 0,\ \mathsf{c}_j\mathsf{u}_j^{(i)}(w) + \mathsf{d}_j\, (\mathsf{u}_j^{(i)})^{\Delta}(w) = 0,
\end{aligned}\right\} \tag{1.1}$$

where $\mathsf{n}, \mathsf{m} \in \mathbb{N}$, $\mathsf{u}_j^{(i)}$ represents i^{th} iteration of j^{th} component of system and $\mathsf{L}_j(\mathsf{r}) = \prod_{k=1}^{s} \mu_{jk}(\mathsf{r})$ and each $\mu_{jk}(\mathsf{r}) \in \mathsf{L}_{\Delta}^{\mathfrak{p}_k}([0, w]_{\mathbb{T}})$ ($\mathfrak{p}_k \geqslant 1$) has s–singularities in the interval $[0, w]_{\mathbb{T}}$. Throughout the study, we assume the following conditions hold:

- (L1) each $\mathfrak{f}_j^{(i)} : [0, \infty) \to [0, \infty)$ is continuous for $1 \leqslant j \leqslant n,\ 1 \leqslant i \leqslant m$,
- (L2) $\lim_{\mathsf{r} \to \mathsf{r}_k} \mu_{jk}(\mathsf{r}) = \infty$, where $0 < \mathsf{r}_n < \mathsf{r}_{n-1} < \cdots < \mathsf{r}_1 < w$ for $1 \leqslant j \leqslant n,\ 1 \leqslant k \leqslant s$,
- (L3) there exists $\mathsf{e}_{jk} > 0$ such that $\mu_{jk}(\mathsf{r}) > \mathsf{e}_{jk}$ for $1 \leqslant j \leqslant n,\ 1 \leqslant k \leqslant s$ for $\mathsf{r} \in [0, w]_{\mathbb{T}}$.

The rest of the paper is organized as follows: In section 2, we present some preliminaries, which are used in the paper. In Section 3, we construct the Green's function for the homogeneous BVP corresponding to (1.1) and establish bounds for the Green's function. Section 4 is devoted to establish the criteria for the existence of positive solutions to (1.1) using Guo–Krasnosel'skii fixed point theorem. In section 5, we establish the existence of at least three positive solutions using Ren-Ge-Ren fixed point theorem. Finally, last section provides examples to demonstrate our results.

2. Preliminaries

In this section, we present some basic definitions and lemmas that will be useful in our subsequent discussions.

Definition 2.1. [7] *A time scale $\mathbb{T}$ is a non-empty closed subset of the real numbers $\mathbb{R}$. $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$, are defined by*

$$\sigma(t) = inf\{\tau \in \mathbb{T} : \tau > t\},\ \rho(t) = sup\{\tau \in \mathbb{T} : \tau < t\}$$

respectively.

- *The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t, \sigma(t) = t, \sigma(t) > t$ respectively.*
- *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathsf{C}_{rd} = \mathsf{C}_{rd} = \mathsf{C}_{rd}(\mathbb{T}, \mathbb{R})$.*

- *A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of all ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathsf{C}_{\mathsf{ld}} = \mathsf{C}_{\mathsf{ld}} = \mathsf{C}_{\mathsf{ld}}(\mathbb{T}, \mathbb{R})$.*
- *By an interval time scale, we mean the intersection of a real interval with a given time scale, i.e., $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. Other intervals can be defined similarly.*

Definition 2.2. [10] *Let $E \subset \mathbb{T}$ be a Δ-measurable set and $p \in \bar{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ be such that $p \geq 1$ and $f : E \to \bar{\mathbb{R}}$ be a Δ- measurable function. We say that f belongs to $L^p_\Delta(E)$ provided that either*

$$\int_E |f|^p(s)\Delta s < \infty, \qquad if \quad p \in [1, \infty),$$

or there exists a constant $M \in \mathbb{R}$ such that

$$|f| \leq M, \quad \Delta - a.e. \;\; on \;\; E \;\; if \; p = \infty.$$

3. Kernel and bounds

Now, we express the solution of BVP as a solution of integral equation by determining the Green's function for the corresponding homogeneous BVP. We also establish certain properties of Green's function that are essential for subsequent discussions.

Lemma 3.1. *Let $\mathsf{H}_j(r) \in \mathsf{C}_{rd}([0, w]_{\mathbb{T}}, \mathbb{R}),\ 1 \leqslant j \leqslant n$. Then the BVP*

$$(u_j^{(1)})^{\Delta\Delta}(r) + H_j(r) = 0,\ r \in [0, w]_{\mathbb{T}}, \tag{3.2}$$

$$a_j u_j^{(1)}(0) - b_j (u_j^{(1)})^{\Delta}(0) = 0,\ c_j u_j^{(1)}(w) + d_j (u_j^{(1)})^{\Delta}(w) = 0, \tag{3.3}$$

has one and only one solution

$$u_j^{(1)}(r) = \int_0^w N_j(r, s) H_j(s)\Delta s, \tag{3.4}$$

where

$$N_j(r, s) = \frac{1}{A_j} \begin{cases} (a_j\sigma(s) + b_j)(c_j(w - r) + d_j),\ if\ \sigma(s) \leqslant r, \\ (a_j r + b_j)(c_j(w - \sigma(s)) + d_j),\ if\ r \leqslant s, \end{cases} \tag{3.5}$$

and

$$A_j = a_j d_j + c_j w a_j + b_j c_j \neq 0,\ j = 1, 2, ..., n.$$

Proof. An equivalent integral equation for (3.2) is

$$u_j^{(1)}(r) = -\int_0^r [r - \sigma(s)] H_j(s)\Delta s + \mathtt{f}_1 r + \mathtt{f}_2.$$

Applying the boundary conditions (3.3), we get

$$\mathtt{f}_1 = \frac{a_j}{A_j} \int_0^w [c_j(w - \sigma(s)) + d_j] H_j(s)\Delta s \text{ and } \mathtt{f}_2 = \frac{b_j \mathtt{f}_1}{a_j}.$$

Then,

$$\begin{aligned} u_j^{(1)}(r) &= -\int_0^r [r - \sigma(s)] H_j(s)\Delta s + \frac{(a_j r + b_j)}{A_j} \int_0^w [c_j(w - \sigma(s)) + d_j] H_j(s)\Delta s \\ &= \int_0^w N_j(r, s) H_j(s)\Delta s. \end{aligned}$$

□

Note that an mn-tuple $\big(u_1^{(1)}(r), u_1^{(2)}(r), ..., u_1^{(m)}(r), u_2^{(1)}(r), u_2^{(2)}(r), ..., u_2^{(m)}(r), ..., u_n^{(1)}(r), u_n^{(2)}(r),$ $..., u_n^{(m)}(r)\big)$ is a solution of (1.1) if and only if

$$u_j^{(i)}(r) = \int_0^w N_j(r,s)L_j(s)f_j^{(i)}(u_j^{(i+1)}(s))\Delta s,\ 1 \leqslant j \leqslant n,\ 1 \leqslant i \leqslant m,$$

$$u_j^{(m+1)}(r) = u_{j+1}^{(1)}(r),\ u_{n+1}^{(1)}(r) = u_1^{(1)}(r),$$

$$u_1^{(1)}(r) = \int_0^w N_1(r,s_1)L_1(s_1)f_1^{(1)}\bigg(\int_0^w N_1(s_1,s_2)L_1(s_2)f_1^{(2)}\bigg(\int_0^w N_1(s_2,s_3)L_1(s_3)f_1^{(3)}...$$

$$f_1^{(m-1)}\bigg(\int_0^w N_1(s_{m-1},s_m)L_1(s_m)f_1^{(m)}\bigg(\int_0^w N_2(s_m,s_{m+1})L_2(s_{m+1})f_2^{(1)}$$

$$\bigg(\int_0^w N_2(s_{m+2},s_{m+3})L_2(s_{m+3})f_2^{(2)}...f_2^{(m-1)}\bigg(\int_0^w N_2(s_{2m-1},s_{2m})L_2(s_{2m})f_2^{(m)}...$$

$$...$$

$$...$$

$$...$$

$$f_{n-1}^{(m)}\bigg(\int_0^w N_n(s_{mn-m},s_{mn-m+1})L_n(s_{mn-m+1})f_n^{(1)}\bigg(\int_0^w N_n(s_{mn-m+1},s_{mn-m+2})$$

$$L_n(s_{mn-m+2})f_n^{(2)}...f_n^{(m-1)}\bigg(\int_0^w N_n(s_{mn-1},s_{mn})L_n(s_{mn})f_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn}\bigg)...$$

$$\Delta s_{mn-m+2}\bigg)\Delta s_{mn-m+1}\bigg)...\Delta s_{2m}...\bigg)\Delta s_{m+2}\bigg)\Delta s_{m+1}\bigg)\Delta s_m\bigg)...\Delta s_3\bigg)\Delta s_2\bigg)\Delta s_1.$$

Lemma 3.2. *Let* $G_j(p_j) = \min\left\{\frac{a_j p_j + b_j}{a_j w + b_j}, \frac{c_j p_j + d_j}{c_j w + d_j}\right\} < 1$, *where* $p_j \in (0, \frac{w}{2})_{\mathbb{T}}$, $j = 1, 2, ..., n$. *Then* $N_j(r,s)$ *has the following properties:*

(i) $0 \leqslant N_j(r,s) \leqslant N_j(s,s)$ *for all* $r, s \in [0,w]_{\mathbb{T}}$,
(ii) $G_j(p_j)N_j(s,s) \leqslant N_j(r,s)$ *for all* $r \in [p_j, w-p_j]_{\mathbb{T}}$ *and* $s \in [0,w]_{\mathbb{T}}$.

Proof. We can establish the inequality (i) by algebraic computations. For the inequality (ii), let $r \in [p_j, w-p_j]_{\mathbb{T}}$ and $r \leq s$, we obtain

$$\frac{N_j(r,s)}{N_j(s,s)} = \frac{a_j r + b_j}{a_j s + b_j} \geqslant \frac{a_j p_j + b_j}{a_j w + b_j} \geqslant G_j(p_j).$$

Let $r \in [p_j, w-p_j]_{\mathbb{T}}$ and $r \geq s$, we obtain

$$\frac{N_j(r,s)}{N_j(s,s)} = \frac{c_j(w-r) + d_j}{c_j(w-s) + d_j} \geqslant \frac{c_j p_j + d_j}{c_j w + d_j} \geqslant G_j(p_j).$$

This completes the proof. □

4. Existence of at least one positive solution

In this section, we establish the presence of positive solution for (1.1) by employing the Hölder's inequality and Guo–Krasnosel'skii fixed point theorem.

Let $\mathsf{B} = \mathsf{C}_{\mathsf{rd}}([0, w]_{\mathbb{T}}, \mathbb{R})$ be a Banach space with norm $\|\mathsf{u}_{\mathsf{j}}\| = \max_{\mathsf{r} \in [0,w]_{\mathbb{T}}} |\mathsf{u}_{\mathsf{j}}(\mathsf{r})|$. For $p_{\mathsf{j}} \in (0, \frac{w}{2})_{\mathbb{T}}$, letting $\mathsf{K} = \min\{\mathsf{G}_1(p_1), \mathsf{G}_2(p_2), ..., \mathsf{G}_{\mathsf{n}}(p_{\mathsf{n}})\}$, $\mathsf{z} = \max\{p_1, p_2, ..., p_{\mathsf{n}}\}$, define the cone $\mathsf{P} \subset \mathsf{B}$ as

$$\mathsf{P} = \left\{\mathsf{u}_{\mathsf{j}} \in \mathsf{b} : \text{each } \mathsf{u}_{\mathsf{j}}(\mathsf{r}) \geqslant 0 \text{ on } [0, w]_{\mathbb{T}}, \min_{\mathsf{r} \in [\mathsf{z}, w-\mathsf{z}]_{\mathbb{T}}} \mathsf{u}_{\mathsf{j}}(\mathsf{r}) \geqslant \mathsf{K}\|\mathsf{u}_{\mathsf{j}}\|_{\mathsf{B}}\right\}.$$

For any $\mathsf{u}_1^{(1)} \in \mathsf{P}$, define an operator $\mathsf{T} : \mathsf{P} \to \mathsf{B}$ by

$$\begin{aligned}
\mathsf{u}_1^{(1)}(\mathsf{r}) = \int_0^w & N_1(\mathsf{r}, \mathsf{s}_1)\mathsf{L}_1(\mathsf{s}_1)\mathfrak{f}_1^{(1)}\Bigg(\int_0^w N_1(\mathsf{s}_1, \mathsf{s}_2)\mathsf{L}_1(\mathsf{s}_2)\mathfrak{f}_1^{(2)}\Bigg(\int_0^w N_1(\mathsf{s}_2, \mathsf{s}_3)\mathsf{L}_1(\mathsf{s}_3)\mathfrak{f}_1^{(3)}...\\
&\mathfrak{f}_1^{(\mathsf{m}-1)}\Bigg(\int_0^w N_1(\mathsf{s}_{\mathsf{m}-1}, \mathsf{s}_{\mathsf{m}})\mathsf{L}_1(\mathsf{s}_{\mathsf{m}})\mathfrak{f}_1^{(\mathsf{m})}\Bigg(\int_0^w N_2(\mathsf{s}_{\mathsf{m}}, \mathsf{s}_{\mathsf{m}+1})\mathsf{L}_2(\mathsf{s}_{\mathsf{m}+1})\mathfrak{f}_2^{(1)}\\
&\Bigg(\int_0^w N_2(\mathsf{s}_{\mathsf{m}+2}, \mathsf{s}_{\mathsf{m}+3})\mathsf{L}_2(\mathsf{s}_{\mathsf{m}+3})\mathfrak{f}_2^{(2)}...\mathfrak{f}_2^{(\mathsf{m}-1)}\Bigg(\int_0^w N_2(\mathsf{s}_{2\mathsf{m}-1}, \mathsf{s}_{2\mathsf{m}})\mathsf{L}_2(\mathsf{s}_{2\mathsf{m}})\mathfrak{f}_2^{(\mathsf{m})}...\\
&...\\
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&...\\
&\mathfrak{f}_{\mathsf{n}-1}^{(\mathsf{m})}\Bigg(\int_0^w N_{\mathsf{n}}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}}, \mathsf{s}_{\mathsf{mn}-\mathsf{m}+1})\mathsf{L}_{\mathsf{n}}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1})\mathfrak{f}_{\mathsf{n}}^{(1)}\Bigg(\int_0^w N_{\mathsf{n}}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1}, \mathsf{s}_{\mathsf{mn}-\mathsf{m}+2})\\
&\mathsf{L}_{\mathsf{n}}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+2})\mathfrak{f}_{\mathsf{n}}^{(2)}...\mathfrak{f}_{\mathsf{n}}^{(\mathsf{m}-1)}\Bigg(\int_0^w N_{\mathsf{n}}(\mathsf{s}_{\mathsf{mn}-1}, \mathsf{s}_{\mathsf{mn}})\mathsf{L}_{\mathsf{n}}(\mathsf{s}_{\mathsf{mn}})\mathfrak{f}_{\mathsf{n}}^{(\mathsf{m})}(\mathsf{u}_1^{(1)}(\mathsf{s}_{\mathsf{mn}}))\Delta \mathsf{s}_{\mathsf{mn}}\Bigg)...\\
&\Delta \mathsf{s}_{\mathsf{mn}-\mathsf{m}+2}\Bigg)\Delta \mathsf{s}_{\mathsf{mn}-\mathsf{m}+1}\Bigg)...\Delta \mathsf{s}_{2\mathsf{m}}...\Bigg)\Delta \mathsf{s}_{\mathsf{m}+2}\Bigg)\Delta \mathsf{s}_{\mathsf{m}+1}\Bigg)\Delta \mathsf{s}_{\mathsf{m}}\Bigg)...\Delta \mathsf{s}_3\Bigg)\Delta \mathsf{s}_2\Bigg)\Delta \mathsf{s}_1.
\end{aligned}$$

Lemma 4.3. *Assume that* (L1)−(L2) *hold. Then for each* $p_{\mathsf{j}} \in (0, \frac{w}{2})_{\mathbb{T}}$, $\mathsf{T}(\mathsf{P}) \subset \mathsf{P}$ *and* $\mathsf{T} : \mathsf{P} \to \mathsf{P}$ *is completely continuous.*

Proof. From Lemma 3.2, we have $N_{\mathsf{j}}(\mathsf{r}, \mathsf{s}) \geqslant 0$, for all $\mathsf{r}, \mathsf{s} \in [0, w]_{\mathbb{T}}$ and so $\mathsf{T}\mathsf{u}_1^{(1)}(\mathsf{r}) \geqslant 0$. We can easily establish that T is completely continuous. For any $\mathsf{u}_1^{(1)} \in \mathsf{P}$, we have

$$\begin{aligned}
\|\mathsf{T}\mathsf{u}_1^{(1)}\| = \max_{\mathsf{r} \in [0,w]_{\mathbb{T}}} \int_0^w & N_1(\mathsf{r}, \mathsf{s}_1)L_1(\mathsf{s}_1)\mathfrak{f}_1^{(1)}\Bigg(\int_0^w N_1(\mathsf{s}_1, \mathsf{s}_2)\mathsf{L}_1(\mathsf{s}_2)\mathfrak{f}_1^{(2)}\Bigg(\int_0^w N_1(\mathsf{s}_2, \mathsf{s}_3)\mathsf{L}_1(\mathsf{s}_3)\mathfrak{f}_1^{(3)}...\\
&\mathfrak{f}_1^{(\mathsf{m}-1)}\Bigg(\int_0^w N_1(\mathsf{s}_{\mathsf{m}-1}, \mathsf{s}_{\mathsf{m}})\mathsf{L}_1(\mathsf{s}_{\mathsf{m}})\mathfrak{f}_1^{(\mathsf{m})}\Bigg(\int_0^w N_2(\mathsf{s}_{\mathsf{m}}, \mathsf{s}_{\mathsf{m}+1})\mathsf{L}_2(\mathsf{s}_{\mathsf{m}+1})\mathfrak{f}_2^{(1)}\\
&\Bigg(\int_0^w N_2(\mathsf{s}_{\mathsf{m}+2}, \mathsf{s}_{\mathsf{m}+3})\mathsf{L}_2(\mathsf{s}_{\mathsf{m}+3})\mathfrak{f}_2^{(2)}...\mathfrak{f}_2^{(\mathsf{m}-1)}\Bigg(\int_0^w N_2(\mathsf{s}_{2\mathsf{m}-1}, \mathsf{s}_{2\mathsf{m}})\mathsf{L}_2(\mathsf{s}_{2\mathsf{m}})\mathfrak{f}_2^{(\mathsf{m})}...\\
&...\\
&...\\
&...\\
&\mathfrak{f}_{\mathsf{n}-1}^{(\mathsf{m})}\Bigg(\int_0^w N_{\mathsf{n}}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}}, \mathsf{s}_{\mathsf{mn}-\mathsf{m}+1})\mathsf{L}_{\mathsf{n}}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1})\mathfrak{f}_{\mathsf{n}}^{(1)}\Bigg(\int_0^w N_{\mathsf{n}}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1}, \mathsf{s}_{\mathsf{mn}-\mathsf{m}+2})\\
&\mathsf{L}_{\mathsf{n}}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+2})\mathfrak{f}_{\mathsf{n}}^{(2)}...\mathfrak{f}_{\mathsf{n}}^{(\mathsf{m}-1)}\Bigg(\int_0^w N_{\mathsf{n}}(\mathsf{s}_{\mathsf{mn}-1}, \mathsf{s}_{\mathsf{mn}})\mathsf{L}_{\mathsf{n}}(\mathsf{s}_{\mathsf{mn}})\mathfrak{f}_{\mathsf{n}}^{(\mathsf{m})}(\mathsf{u}_1^{(1)}(\mathsf{s}_{\mathsf{mn}}))\Delta \mathsf{s}_{\mathsf{mn}}\Bigg)...\\
&\Delta \mathsf{s}_{\mathsf{mn}-\mathsf{m}+2}\Bigg)\Delta \mathsf{s}_{\mathsf{mn}-\mathsf{m}+1}\Bigg)...\Delta \mathsf{s}_{2\mathsf{m}}...\Bigg)\Delta \mathsf{s}_{\mathsf{m}+2}\Bigg)\Delta \mathsf{s}_{\mathsf{m}+1}\Bigg)\Delta \mathsf{s}_{\mathsf{m}}\Bigg)...\Delta \mathsf{s}_3\Bigg)\Delta \mathsf{s}_2\Bigg)\Delta \mathsf{s}_1.
\end{aligned}$$

$$\leqslant \int_0^w N_1(\mathsf{s}_1,\mathsf{s}_1)\mathsf{L}_1(\mathsf{s}_1)\mathfrak{f}_1^{(1)}\Bigg(\int_0^w N_1(\mathsf{s}_1,\mathsf{s}_2)\mathsf{L}_1(\mathsf{s}_2)\mathfrak{f}_1^{(2)}\Bigg(\int_0^w N_1(\mathsf{s}_2,\mathsf{s}_3)\mathsf{L}_1(\mathsf{s}_3)\mathfrak{f}_1^{(3)}\ldots$$
$$\mathfrak{f}_1^{(\mathsf{m}-1)}\Bigg(\int_0^w N_1(\mathsf{s}_{\mathsf{m}-1},\mathsf{s}_\mathsf{m})\mathsf{L}_1(\mathsf{s}_\mathsf{m})\mathfrak{f}_1^{(\mathsf{m})}\Bigg(\int_0^w N_2(\mathsf{s}_\mathsf{m},\mathsf{s}_{\mathsf{m}+1})\mathsf{L}_2(\mathsf{s}_{\mathsf{m}+1})\mathfrak{f}_2^{(1)}$$
$$\Bigg(\int_0^w N_2(\mathsf{s}_{\mathsf{m}+2},\mathsf{s}_{\mathsf{m}+3})\mathsf{L}_2(\mathsf{s}_{\mathsf{m}+3})\mathfrak{f}_2^{(2)}\ldots\mathfrak{f}_2^{(\mathsf{m}-1)}\Bigg(\int_0^w N_2(\mathsf{s}_{2\mathsf{m}-1},\mathsf{s}_{2\mathsf{m}})\mathsf{L}_2(\mathsf{s}_{2\mathsf{m}})\mathfrak{f}_2^{(\mathsf{m})}\ldots$$
$$\ldots$$
$$\ldots$$
$$\ldots$$
$$\mathfrak{f}_{\mathsf{n}-1}^{(\mathsf{m})}\Bigg(\int_0^w N_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}},\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1})\mathsf{L}_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1})\mathfrak{f}_\mathsf{n}^{(1)}\Bigg(\int_0^w N_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1},\mathsf{s}_{\mathsf{mn}-\mathsf{m}+2})$$
$$\mathsf{L}_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+2})\mathfrak{f}_\mathsf{n}^{(2)}\ldots\mathfrak{f}_\mathsf{n}^{(\mathsf{m}-1)}\Bigg(\int_0^w N_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-1},\mathsf{s}_{\mathsf{mn}})\mathsf{L}_\mathsf{n}(\mathsf{s}_{\mathsf{mn}})\mathfrak{f}_\mathsf{n}^{(\mathsf{m})}(\mathsf{u}_1^{(1)}(\mathsf{s}_{\mathsf{mn}}))\Delta\mathsf{s}_{\mathsf{mn}}\Bigg)\ldots$$
$$\Delta\mathsf{s}_{\mathsf{mn}-\mathsf{m}+2}\Bigg)\Delta\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1}\Bigg)\ldots\Delta\mathsf{s}_{2\mathsf{m}}\ldots\Bigg)\Delta\mathsf{s}_{\mathsf{m}+2}\Bigg)\Delta\mathsf{s}_{\mathsf{m}+1}\Bigg)\Delta\mathsf{s}_\mathsf{m}\Bigg)\ldots\Delta\mathsf{s}_3\Bigg)\Delta\mathsf{s}_2\Bigg)\Delta\mathsf{s}_1.$$
$$\min_{\mathsf{r}\in[\mathsf{z},w-\mathsf{z}]_\mathbb{T}}\mathsf{T}\mathsf{u}_1^{(1)}(\mathsf{r}) = \min_{\mathsf{r}\in[\mathsf{z},w-\mathsf{z}]_\mathbb{T}}\int_0^w N_1(\mathsf{r},\mathsf{s}_1)\mathsf{L}_1(\mathsf{s}_1)\mathfrak{f}_1^{(1)}\Bigg(\int_0^w N_1(\mathsf{s}_1,\mathsf{s}_2)\mathsf{L}_1(\mathsf{s}_2)\mathfrak{f}_1^{(2)}\ldots$$
$$\mathfrak{f}_1^{(\mathsf{m}-1)}\Bigg(\int_0^w N_1(\mathsf{s}_{\mathsf{m}-1},\mathsf{s}_\mathsf{m})\mathsf{L}_1(\mathsf{s}_\mathsf{m})\mathfrak{f}_1^{(\mathsf{m})}\Bigg(\int_0^w N_2(\mathsf{s}_\mathsf{m},\mathsf{s}_{\mathsf{m}+1})\mathsf{L}_2(\mathsf{s}_{\mathsf{m}+1})\mathfrak{f}_2^{(1)}$$
$$\Bigg(\int_0^w N_2(\mathsf{s}_{\mathsf{m}+2},\mathsf{s}_{\mathsf{m}+3})\mathsf{L}_2(\mathsf{s}_{\mathsf{m}+3})\mathfrak{f}_2^{(2)}\ldots\mathfrak{f}_2^{(\mathsf{m}-1)}\Bigg(\int_0^w N_2(\mathsf{s}_{2\mathsf{m}-1},\mathsf{s}_{2\mathsf{m}})\mathsf{L}_2(\mathsf{s}_{2\mathsf{m}})\mathfrak{f}_2^{(\mathsf{m})}\ldots$$
$$\ldots$$
$$\ldots$$
$$\ldots$$
$$\mathfrak{f}_{\mathsf{n}-1}^{(\mathsf{m})}\Bigg(\int_0^w N_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}},\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1})\mathsf{L}_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1})\mathfrak{f}_\mathsf{n}^{(1)}\Bigg(\int_0^w N_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1},\mathsf{s}_{\mathsf{mn}-\mathsf{m}+2})$$
$$\mathsf{L}_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+2})\mathfrak{f}_\mathsf{n}^{(2)}\ldots\mathfrak{f}_\mathsf{n}^{(\mathsf{m}-1)}\Bigg(\int_0^w N_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-1},\mathsf{s}_{\mathsf{mn}})\mathsf{L}_\mathsf{n}(\mathsf{s}_{\mathsf{mn}})\mathfrak{f}_\mathsf{n}^{(\mathsf{m})}(\mathsf{u}_1^{(1)}(\mathsf{s}_{\mathsf{mn}}))\Delta\mathsf{s}_{\mathsf{mn}}\Bigg)\ldots$$
$$\Delta\mathsf{s}_{\mathsf{mn}-\mathsf{m}+2}\Bigg)\Delta\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1}\Bigg)\ldots\Delta\mathsf{s}_{2\mathsf{m}}\ldots\Bigg)\Delta\mathsf{s}_{\mathsf{m}+2}\Bigg)\Delta\mathsf{s}_{\mathsf{m}+1}\Bigg)\Delta\mathsf{s}_\mathsf{m}\Bigg)\ldots\Delta\mathsf{s}_3\Bigg)\Delta\mathsf{s}_2\Bigg)\Delta\mathsf{s}_1.$$
$$\geqslant \mathcal{G}_1(p_1)\int_0^w N_1(\mathsf{s}_1,\mathsf{s}_1)\mathsf{L}_1(\mathsf{s}_1)\mathfrak{f}_1^{(1)}\Bigg(\int_0^w N_1(\mathsf{s}_1,\mathsf{s}_2)\mathsf{L}_1(\mathsf{s}_2)\mathfrak{f}_1^{(2)}\Bigg(\int_0^w N_1(\mathsf{s}_2,\mathsf{s}_3)\mathsf{L}_1(\mathsf{s}_3)$$
$$\mathfrak{f}_1^{(3)}\ldots\mathfrak{f}_1^{(\mathsf{m}-1)}\Bigg(\int_0^w N_1(\mathsf{s}_{\mathsf{m}-1},\mathsf{s}_\mathsf{m})\mathsf{L}_1(\mathsf{s}_\mathsf{m})\mathfrak{f}_1^{(\mathsf{m})}\Bigg(\int_0^w N_2(\mathsf{s}_\mathsf{m},\mathsf{s}_{\mathsf{m}+1})\mathsf{L}_2(\mathsf{s}_{\mathsf{m}+1})\mathfrak{f}_2^{(1)}$$
$$\Bigg(\int_0^w N_2(\mathsf{s}_{\mathsf{m}+2},\mathsf{s}_{\mathsf{m}+3})\mathsf{L}_2(\mathsf{s}_{\mathsf{m}+3})\mathfrak{f}_2^{(3)}\ldots\mathfrak{f}_2^{(\mathsf{m}-1)}\Bigg(\int_0^w N_2(\mathsf{s}_{2\mathsf{m}-1},\mathsf{s}_{2\mathsf{m}})\mathsf{L}_2(\mathsf{s}_{2\mathsf{m}})\mathfrak{f}_2^{(\mathsf{m})}\ldots$$
$$\ldots$$
$$\ldots$$
$$\ldots$$
$$\mathfrak{f}_{\mathsf{n}-1}^{(\mathsf{m})}\Bigg(\int_0^w N_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}},\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1})\mathsf{L}_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1})\mathfrak{f}_\mathsf{n}^{(1)}\Bigg(\int_0^w N_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1},\mathsf{s}_{\mathsf{mn}-\mathsf{m}+2})$$
$$\mathsf{L}_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-\mathsf{m}+2})\mathfrak{f}_\mathsf{n}^{(2)}\ldots\mathfrak{f}_\mathsf{n}^{(\mathsf{m}-1)}\Bigg(\int_0^w N_\mathsf{n}(\mathsf{s}_{\mathsf{mn}-1},\mathsf{s}_{\mathsf{mn}})\mathsf{L}_\mathsf{n}(\mathsf{s}_{\mathsf{mn}})\mathfrak{f}_\mathsf{n}^{(\mathsf{m})}(\mathsf{u}_1^{(1)}(\mathsf{s}_{\mathsf{mn}}))\Delta\mathsf{s}_{\mathsf{mn}}\Bigg)\ldots$$
$$\Delta\mathsf{s}_{\mathsf{mn}-\mathsf{m}+2}\Bigg)\Delta\mathsf{s}_{\mathsf{mn}-\mathsf{m}+1}\Bigg)\ldots\Delta\mathsf{s}_{2\mathsf{m}}\ldots\Bigg)\Delta\mathsf{s}_{\mathsf{m}+2}\Bigg)\Delta\mathsf{s}_{\mathsf{m}+1}\Bigg)\Delta\mathsf{s}_\mathsf{m}\Bigg)\ldots\Delta\mathsf{s}_3\Bigg)\Delta\mathsf{s}_2\Bigg)\Delta\mathsf{s}_1.$$

$\geqslant K\|Tu_1^{(1)}\|.$

Hence, $\min_{r\in[z,w-z]_{\mathbb{T}}} Tu_1^{(1)}(r) \geqslant K\|Tu_1^{(1)}\|_B$.

Therefore, $T(P) \subset P$ and using Arzela-Ascoli theorem, it can be seen that T is completely continuous. □

Theorem 4.1. [15] *Let $f \in L^{\mathfrak{p}}_{\Delta}(J)$ with $\mathfrak{p} > 1, g \in L^{\mathfrak{q}}_{\Delta}(J)$ with $\mathfrak{q} > 1$, and $\frac{1}{\mathfrak{p}} + \frac{1}{\mathfrak{q}} = 1$. Then $fg \in L^1_{\Delta}(J)$ and $||fg||_{L^1_{\Delta}} \leq ||f||_{L^{\mathfrak{p}}_{\Delta}} ||g||_{L^{\mathfrak{q}}_{\Delta}}$, where*

$$||f||_{L^{\mathfrak{p}}_{\Delta}} := \begin{cases} \left[\displaystyle\int_J |f|^{\mathfrak{p}}(s)\Delta s\right]^{\frac{1}{\mathfrak{q}}}, & if\ \mathfrak{p} \in \mathbb{R}, \\ inf\{M \in \mathbb{R}/|f| \leq M\ \Delta - a.e\ on\ J\}, & if\ \mathfrak{p} = \infty, \end{cases}$$

and $J = [a, b)$.

Theorem 4.2. ***(Hölder's inequality)*** [15] *Let $f \in L^{\mathfrak{p}_k}_{\Delta}[0,1]$ with $\mathfrak{p}_k > 1$, for $k = 1, 2, ..., s$ and $\displaystyle\sum_{k=1}^{s} \frac{1}{\mathfrak{p}_k} = 1$. Then $\displaystyle\prod_{k=1}^{s} f_k \in L^1_{\Delta}[0,1]$ and $\left\|\displaystyle\prod_{k=1}^{s} f_k\right\|_1 \leqslant \displaystyle\prod_{k=1}^{s} \|f_k\|_{\mathfrak{p}_k}$. Further, if $f \in L^1_{\Delta}[0,1]$ and $g \in L^{\infty}_{\Delta}[0,1]$, then $fg \in L^1_{\Delta}[0,1]$ and $\|fg\|_1 \leqslant \|f\|_1\|g\|_{\infty}$.*

Theorem 4.3. [9] *Let P be a cone in a Banach space B and R_1, R_2 are open sets with $0 \in R_1, \bar{R}_1 \subset R_2$. Let $T : P \cap (\bar{R}_2 \setminus R_1) \to P$ be completely continuous operator such that*

(i) $\|Tr\| \leqslant \|r\|, r \in P \cap \partial R_1$, and $\|Tr\| \geqslant \|r\|, r \in P \cap \partial R_2$, or

(ii) $\|Tr\| \geqslant \|r\|, r \in P \cap \partial R_1$, and $\|Tr\| \leqslant \|r\|, r \in P \cap \partial R_2$.

Then T has a fixed point in $P \cap (\bar{R}_2 \setminus R_1)$.

We consider the cases for $\mu_{jk}(r) \in L^{\mathfrak{p}_k}_{\Delta}[0, w]_{\mathbb{T}}$:

(i) $\displaystyle\sum_{k=1}^{s} \frac{1}{\mathfrak{p}_k} < 1$, (ii) $\displaystyle\sum_{k=1}^{s} \frac{1}{\mathfrak{p}_k} = 1$, (iii) $\displaystyle\sum_{k=1}^{s} \frac{1}{\mathfrak{p}_k} > 1$.

Firstly, we present a result to establish existence of positive solution when $\displaystyle\sum_{k=1}^{s} \frac{1}{\mathfrak{p}_k} < 1$.

Theorem 4.4. *Suppose that (L1) − (L3) hold. Let $\{p_{j\ell}\}_{\ell=1}^{\infty}$ be a sequence with $p_{j\ell} \in (r_{\ell+1}, r_\ell)$, $z_\ell = \max\{p_{1\ell}, p_{2\ell}, ..., p_{n\ell}\}$, $0 < z_1 < \frac{w}{2}$ for $j = 1, 2, ..., n$. Let $\{D_\ell\}_{\ell=1}^{\infty}$ and $\{E_\ell\}_{\ell=1}^{\infty}$ be such that*

$$D_{\ell+1} < K_\ell E_\ell < E_\ell < M_j\, E_\ell < D_\ell,\ \ell \in \mathbb{N},$$

where

$$M_j = \max\left\{\left[K_1 \prod_{k=1}^{s} e_{jk} \int_{z_1}^{w-z_1} N_j(s,s)\Delta s\right]^{-1},\ 1\right\}.$$

Assume that $\mathfrak{f}_j^{(i)}$ satisfies

(H1) $\mathfrak{f}_j^{(i)}(u_j) \leqslant \mathfrak{h}_j D_\ell\ \forall\ r \in [0, w]_{\mathbb{T}},\ 0 \leqslant u_j \leqslant D_\ell$, *where*

$$\mathfrak{h}_j < \left[\|N_j\|_{L^{\mathfrak{q}}_{\Delta}} \prod_{k=1}^{s} \|\mu_{jk}(r)\|_{L^{\mathfrak{p}_k}_{\Delta}}\right]^{-1},$$

(H2) $\mathfrak{f}_j^{(i)}(u_j) \geqslant M_j E_\ell\ \forall\ r \in [z_\ell, w - z_\ell]_{\mathbb{T}},\ K_\ell E_\ell \leqslant u_j \leqslant E_\ell$.

Then (1.1) *has positive solutions $\{(u_1^{(1)})^{[\ell]}, (u_1^{(2)})^{[\ell]}, \ldots, (u_1^{(m)})^{[\ell]}, (u_2^{(1)})^{[\ell]}, (u_2^{(2)})^{[\ell]}, \ldots, (u_2^{(m)})^{[\ell]}, \ldots, (u_n^{(1)})^{[\ell]}, (u_n^{(2)})^{[\ell]}, \ldots, (u_n^{(m)})^{[\ell]}\}_{\ell=1}^{\infty}$ such that $(u_j^{(i)})^{[\ell]}(r) \geqslant 0$ on $[0, w]_{\mathbb{T}}$, $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$, $\ell \in \mathbb{N}$.*

Proof. Consider the sequences $\{E_\ell\}_{\ell=1}^{\infty}$ and $\{F_\ell\}_{\ell=1}^{\infty}$ defined by

$$E_\ell = \{u_j \in B : \|u_j\| \leqslant D_\ell\},$$
$$F_\ell = \{u_j \in B : \|u_j\| \leqslant E_\ell\},$$

which are open subsets of B. Let $p_{j\ell}$ be as in the hypothesis and note that $r^* < r_{\ell+1} < p_{j\ell} < r_\ell < \frac{w}{2}$, $\forall\, \ell \in \mathbb{N}$.
Denote, $P_\ell = \{u_j \in E : u_j(r) \geqslant 0 \text{ on } [0,w]_{\mathbb{T}} \text{ and } \min_{r\in[z_\ell, w-z_\ell]_{\mathbb{T}}} u_j(r) \geqslant K_\ell \|u_j(r)\|\}$
and let $u_1^{(1)} \in P_\ell \cap \partial E_\ell$. Then $u_1^{(1)}(s_{mn}) \leqslant D_\ell = \|u_1^{(1)}\|$ for all $s_{mn} \in [0,w]_{\mathbb{T}}$.
By (H_1) and for $s_{mn} \in [0,w]_{\mathbb{T}}$, we have

$$\int_0^w N_n(s_{mn-1}, s_{mn}) L_n(s_{mn}) \mathfrak{f}_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn} \leqslant \int_0^w N_n(s_{mn}, s_{mn}) L_n(s_{mn}) \mathfrak{h}_n D_\ell \Delta s_{mn}$$
$$\leqslant \mathfrak{h}_n D_\ell \int_0^w N_n(s_{mn}, s_{mn}) L_n(s_{mn})\Delta s_{mn} = \mathfrak{h}_n D_\ell \int_0^w N_n(s_{mn}, s_{mn}) \prod_{k=1}^{s} \mu_{nk}(s_{mn})\Delta s_{mn}.$$

Since $\sum_{k=1}^{s} \frac{1}{\mathfrak{p}_k} < 1$, there exists $\mathfrak{q} > 1$ such that $\frac{1}{\mathfrak{q}} + \sum_{k=1}^{s} \frac{1}{\mathfrak{p}_k} = 1$. So,

$$\int_0^w N_n(s_{mn-1}, s_{mn}) L_n(s_{mn}) \mathfrak{f}_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn} \leqslant \mathfrak{h}_n D_\ell \|N_n\|_{L_\Delta^{\mathfrak{q}}} \left\| \prod_{k=1}^{s} \mu_{nk} \right\|_{L_\Delta^{\mathfrak{p}_k}} \leqslant D_\ell.$$

$$\int_0^w N_n(s_{mn-2}, s_{mn-1}) L_n(s_{mn-1}) \mathfrak{f}_n^{(m-1)}\left(\int_0^w N_n(s_{mn-1}, s_{mn}) L_n(s_{mn}) \mathfrak{f}_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn}\right)\Delta s_{mn-1}$$
$$\leqslant \int_0^w N_n(s_{mn-2}, s_{mn-1}) L_n(s_{mn-1}) \mathfrak{f}_n^{(mn-1)}(D_\ell)\Delta s_{mn-1} \leqslant D_\ell.$$

Continuing in this way, we get $(Tu_1^{(1)})(r) \leqslant D_\ell$.
Since $D_\ell = \|u_1^{(1)}\|$ for $u_1^{(1)} \in P_\ell \cap \partial E_\ell$, we get

$$\|T(u_1^{(1)})\| \leqslant \|(u_1^{(1)})\|. \tag{4.6}$$

Let $r \in [z_\ell, w - z_\ell]_{\mathbb{T}}$, then

$$E_\ell = \|u_j^{(1)}\| \geqslant u_j^{(1)}(r) \geqslant \min_{r\in[p_{j\ell}, w-p_{j\ell}]_{\mathbb{T}}} u_j^{(1)}(r) \geqslant \min_{r\in[z_\ell, w-z_\ell]_{\mathbb{T}}} u_j^{(1)}(r) \geqslant G_j(p_{j\ell})\|u_j^{(1)}(r)\| \geqslant K_\ell E_\ell.$$

By (H2) and for $s_{mn} \in [z_\ell, w - z_\ell]_{\mathbb{T}}$, we have

$$\int_0^w N_n(s_{mn-1}, s_{mn}) L_n(s_{mn}) \mathfrak{f}_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn}$$
$$\geqslant G_n(p_{n\ell}) \int_{z_\ell}^{w-z_\ell} N_n(s_{mn}, s_{mn}) L_n(s_{mn}) \mathfrak{f}_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn}$$
$$\geqslant G_n(p_{n\ell}) M_n\, E_\ell \int_{z_\ell}^{w-z_\ell} N_n(s_{mn}, s_{mn}) L_n(s_{mn})\Delta s_{mn}$$
$$\geqslant K_1 M_n\, E_\ell \prod_{k=1}^{s} e_{nk} \int_{z_1}^{w-z_1} N_n(s_{mn}, s_{mn})\Delta s_{mn}$$
$$\geqslant E_\ell.$$

Continuing in this way, we get $(T_1 u_1^{(1)})(r) \geqslant E_\ell$.
Since $E_\ell = \|u_1^{(1)}\|$ for $u_1^{(1)} \in P_\ell \cap \partial F_\ell$, we get

$$\|Tu_1^{(1)}\| \geqslant \|u_1^{(1)}\|. \tag{4.7}$$

It is evident that $0 \in \partial F_\ell \subset \partial \bar{F}_\ell \subset \partial E_\ell$. Using (4.6) and (4.7), it follows from theorem 4.3 that T has a fixed point $((u_j^{(i)})^{[\ell]}) \in P_\ell \cap (\bar{E}_\ell \setminus F_\ell)$ such that $(u_j^{(i)})^{[\ell]} \geqslant 0$ on $[0,w]_{\mathbb{T}}$ and $\ell \in \mathbb{N}$. Next setting $u_j^{(m+1)}(r) = u_{j+1}^{(1)}(r)$, $u_{n+1}^{(1)}(r) = u_1^{(1)}(r)$, for $j = 1,2,...,n$, $r \in [0,w]_{\mathbb{T}}$, we obtain positive solutions $\{(u_1^{(1)})^{[\ell]}, (u_1^{(2)})^{[\ell]}, \dots, (u_1^{(m)})^{[\ell]}, (u_2^{(1)})^{[\ell]}, (u_2^{(2)})^{[\ell]}, \dots, (u_2^{(m)})^{[\ell]}, \dots, (u_n^{(1)})^{[\ell]}, (u_n^{(2)})^{[\ell]}, \dots, (u_n^{(m)})^{[\ell]}\}_{\ell=1}^{\infty}$ for (1.1). Thus, the proof is complete. □

For $\sum_{k=1}^{s} \frac{1}{p_k} = 1$, we have the following theorem.

Theorem 4.5. *Suppose that* (L1) − (L3) *hold. Let* $\{p_{j\ell}\}_{\ell=1}^{\infty}$ *be a sequence with* $p_{j\ell} \in (r_{\ell+1}, r_\ell)$, $z_\ell = \max\{p_{1\ell}, p_{2\ell}, ..., p_{n\ell}\}$, $0 < z_1 < \frac{w}{2}$ *for* $j = 1,2,...,n$. *Let* $\{D_\ell\}_{\ell=1}^{\infty}$ *and* $\{E_\ell\}_{\ell=1}^{\infty}$ *be such that*

$$D_{\ell+1} < K_\ell E_\ell < E_\ell < M_j\, E_\ell < D_\ell, \ \ell \in \mathbb{N}.$$

Assume that $f_j^{(i)}$ *satisfies* (H2)

(H3) $f_j^{(i)}(u_j) \leqslant g_j D_\ell \ \forall\, r \in [0,w]_{\mathbb{T}}, \ 0 \leqslant u_j \leqslant D_\ell$, *where*

$$g_j < \left[\|N_j\|_{L_\Delta^\infty} \prod_{k=1}^{s} \|\mu_{jk}(r)\|_{L_\Delta^{p_k}} \right]^{-1}.$$

Then (1.1) *has positive solutions* $\{(u_1^{(1)})^{[\ell]}, (u_1^{(2)})^{[\ell]}, \dots, (u_1^{(m)})^{[\ell]}, (u_2^{(1)})^{[\ell]}, (u_2^{(2)})^{[\ell]}, \dots, (u_2^{(m)})^{[\ell]}, \dots, (u_n^{1})^{[\ell]}, (u_n^{(2)})^{[\ell]}, \dots, (u_n^{(m)})^{[\ell]}\}_{\ell=1}^{\infty}$ *such that* $(u_j^{(i)})^{[\ell]}(r) \geqslant 0$ *on* $[0,w]_{\mathbb{T}}$, $j = 1,2,...,n$, $i = 1,2,...,m$, $\ell \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 4.4. Therefore, we omit the details here. □

Lastly, the case $\sum_{k=1}^{s} \frac{1}{p_k} > 1$.

Theorem 4.6. *Suppose that* (L1) − (L3) *hold. Let* $\{p_{j\ell}\}_{\ell=1}^{\infty}$ *be a sequence with* $p_{j\ell} \in (r_{\ell+1}, r_\ell)$, $z_\ell = \max\{p_{1\ell}, p_{2\ell}, ..., p_{n\ell}\}$, $0 < z_1 < \frac{w}{2}$ *for* $j = 1,2,...,n$. *Let* $\{D_\ell\}_{\ell=1}^{\infty}$ *and* $\{E_\ell\}_{\ell=1}^{\infty}$ *be such that*

$$D_{\ell+1} < K_\ell E_\ell < E_\ell < M_j\, E_\ell < D_\ell, \ \ell \in \mathbb{N}.$$

Assume that $f_j^{(i)}$ *satisfies* (H2)

(H4) $f_j^{(i)}(u_j) \leqslant d_j D_\ell \ \forall\, r \in [0,w]_{\mathbb{T}}, \ 0 \leqslant u_j \leqslant D_\ell$, *where*

$$d_j < \left[\|N_j\|_{L_\Delta^\infty} \prod_{k=1}^{s} \|\mu_{jk}(r)\|_{L_\Delta^{1}} \right]^{-1}.$$

Then (1.1) *has positive solutions* $\{(u_1^{(1)})^{[\ell]}, (u_1^{(2)})^{[\ell]}, \dots, (u_1^{(m)})^{[\ell]}, (u_2^{(1)})^{[\ell]}, (u_2^{(2)})^{[\ell]}, \dots, (u_2^{(m)})^{[\ell]}, \dots, (u_n^{(1)})^{[\ell]}, (u_n^{(2)})^{[\ell]}, \dots, (u_n^{(m)})^{[\ell]}\}_{\ell=1}^{\infty}$ *such that* $(u_j^{(i)})^{[\ell]}(r) \geqslant 0$ *on* $[0,w]_{\mathbb{T}}$, $j = 1,2,...,n$, $i = 1,2,...,m$, $\ell \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 4.4. Therefore, we omit the details here. □

5. Existence of at least two positive solutions

Let γ be a nonnegative continuous functional on a cone P of the real Banach space B. Then for any two positive real numbers a', c', define the sets
$\mathsf{P}(\gamma, c') = \{\mathsf{u} \in \mathsf{P} : \gamma(\mathsf{u}) < c'\}$ and $\mathsf{P}_{a'} = \{\mathsf{u} \in \mathsf{P} : \|\mathsf{u}\| < a'\}$.
Define nonnegative, increasing, continuous functionals $\gamma_\ell, \beta_\ell, \alpha_\ell$ by
$\gamma_\ell(\mathsf{u}_1^{(1)}) = \min_{\mathsf{r}\in[0,w]_{\mathbb{T}}} (\mathsf{u}_1^{(1)}(\mathsf{r}))$, $\beta_\ell(\mathsf{u}_1^{(i)}) = \max_{\mathsf{r}\in[0,w]_{\mathbb{T}}} (\mathsf{u}_1^{(1)}(\mathsf{r}))$, $\alpha_\ell(\mathsf{u}_1^{(1)}(\mathsf{r})) = \max_{\mathsf{r}\in[0,w]_{\mathbb{T}}} (\mathsf{u}_1^{(1)}(\mathsf{r}))$.
It is obvious that for each $\mathsf{u}_1^{(1)} \in \mathsf{P}$, $\gamma_\ell(\mathsf{u}_1^{(1)}) \leqslant \beta_\ell(\mathsf{u}_1^{(1)}) = \alpha_\ell(\mathsf{u}_1^{(1)})$.
In addition by lemma (2.2) for each $\mathsf{u}_1^{(1)} \in \mathsf{P}$, $\gamma_\ell(\mathsf{u}_1^{(1)}) = \min_{\mathsf{r}\in[0,w]_{\mathbb{T}}} (\mathsf{u}_1^{(1)}(\mathsf{r})) \geqslant \mathsf{K}\|\mathsf{u}_1^{(1)}\|$.
Thus, $\|\mathsf{u}_1^{(1)}\| \leqslant \frac{1}{\mathsf{K}}\gamma_\ell(\mathsf{u}_1^{(1)})$ for $\mathsf{u}_1^{(1)} \in \mathsf{P}$.

Theorem 5.7. ***(Ren-Ge-Ren)*** [21] *Let* P *be a cone in a Banach space* B*. Let α, β, γ be three increasing, nonnegative and continuous functionals on P satisfying for some $c' > 0$ and $M > 0$ such that $\gamma(z) \leqslant \beta(z) \leqslant \alpha(z)$ and $\|z\| \leqslant M\gamma(z)$, for all $z \in \overline{P(\gamma, c')}$. Suppose there exists a completely continuous operator $N : \overline{\mathsf{P}(\gamma, c')} \to P$ and $0 < a' < b' < c'$ such that*
(i) $\gamma(z) > c'$, for all $z \in \partial\mathsf{P}(\gamma, c')$,
(ii) $\beta(z) < b'$, for all $z \in \partial\mathsf{P}(\beta, b')$,
(iii) $\mathsf{P}(\alpha, a') \neq \phi$ and $\alpha(z) < a'$, for all $z \in \partial\mathsf{P}(\alpha, a')$. Then, N has at least three fixed points ${}^1z$, ${}^2z$, ${}^3z \in \overline{\mathsf{P}(\gamma, c')}$ such that $\alpha({}^1z) < a' < \alpha({}^2z), \beta({}^2z) < b' < \beta({}^3z))$ and $\gamma({}^3z) < c'$.

Theorem 5.8. *Suppose that* (L1) − (L3) *hold. Let $\{p_{j\ell}\}_{\ell=1}^{\infty}$ be a sequence with $p_{j\ell} \in (\mathsf{r}_{\ell+1}, \mathsf{r}_\ell)$, $\mathsf{z}_\ell = \max\{p_{1\ell}, p_{2\ell}, ..., p_{n\ell}\}$, $0 < \mathsf{z}_1 < \frac{w}{2}$ for $\mathsf{j} = 1, 2, ..., \mathsf{n}$. Let $\{\mathsf{R}_\ell\}_{\ell=1}^{\infty}$, $\{\mathsf{Q}_\ell\}_{\ell=1}^{\infty}$ and $\{\mathsf{S}_\ell\}_{\ell=1}^{\infty}$ be three sequences such that*

$$\mathsf{R}_{\ell+1} < \mathsf{Q}_\ell < \mathsf{K}_\ell \mathsf{S}_\ell < \mathsf{S}_\ell < \mathsf{R}_\ell,\ \mathsf{M}_\mathsf{j}\, \mathsf{S}_\ell < \mathsf{R}_\ell,\ \ell \in \mathbb{N},$$

where

$$\mathsf{M}_\mathsf{j} = \max\left\{ \left[\mathsf{K}_1 \prod_{\mathsf{k}=1}^{s} \mathsf{e}_{\mathsf{jk}} \int_{\mathsf{z}_1}^{w-\mathsf{z}_1} N_\mathsf{j}(\mathsf{s}, \mathsf{s})\Delta \mathsf{s} \right]^{-1},\ 1 \right\}.$$

Assume that $\mathfrak{f}_\mathsf{j}^{(\mathsf{i})}$ satisfies the following

(F1) $\mathfrak{f}_\mathsf{j}^{(\mathsf{i})}(\mathsf{u}_\mathsf{j}) \leqslant \mathfrak{h}_\mathsf{j} \mathsf{R}_\ell\ \forall\ \mathsf{r} \in [0, w]_{\mathbb{T}},\ 0 \leqslant \mathsf{u}_\mathsf{j} \leqslant \frac{1}{\mathsf{K}_\ell}\mathsf{R}_\ell$, *where*

$$\mathfrak{h}_\mathsf{j} < \left[\|N_\mathsf{j}\|_{\mathsf{L}^{\mathsf{q}}_{\Delta}} \prod_{\mathsf{k}=1}^{s} \|\mu_{\mathsf{jk}}(\mathsf{r})\|_{\mathsf{L}^{\mathsf{p}_\mathsf{k}}_{\Delta}} \right]^{-1},$$

(F2) $\mathfrak{f}_\mathsf{j}^{(\mathsf{i})}(\mathsf{u}_\mathsf{j}) \geqslant \mathsf{M}_\mathsf{j} \mathsf{S}_\ell\ \forall\ \mathsf{r} \in [\mathsf{z}_\ell, w - \mathsf{z}_\ell]_{\mathbb{T}},\ \mathsf{K}_\ell \mathsf{S}_\ell \leqslant \mathsf{u}_\mathsf{j} \leqslant \mathsf{S}_\ell$,
(F3) $\mathfrak{f}_\mathsf{j}^{(\mathsf{i})}(\mathsf{u}_\mathsf{j}) \leqslant \mathfrak{h}_\mathsf{j} \mathsf{Q}_\ell\ \forall\ \mathsf{r} \in [0, w]_{\mathbb{T}},\ 0 \leqslant \mathsf{u}_\mathsf{j} \leqslant \frac{1}{\mathsf{K}_\ell}\mathsf{Q}_\ell$.

Then (1.1) *has at least three positive solutions* $\big\{({}^1\mathsf{u}_1^{(1)})^{[\ell]}, ({}^1\mathsf{u}_1^{(2)})^{[\ell]}, \ldots, ({}^1\mathsf{u}_1^{(\mathsf{m})})^{[\ell]}, ({}^1\mathsf{u}_2^{(1)})^{[\ell]}, ({}^1\mathsf{u}_2^{(2)})^{[\ell]}, \ldots, ({}^1\mathsf{u}_2^{(\mathsf{m})})^{[\ell]}, \ldots, ({}^1\mathsf{u}_\mathsf{n}^{(1)})^{[\ell]}, ({}^1\mathsf{u}_\mathsf{n}^{(2)})^{[\ell]}, \ldots, ({}^1\mathsf{u}_\mathsf{n}^{(\mathsf{m})})^{[\ell]}\big\}_{\ell=1}^{\infty}$, $\big\{({}^2\mathsf{u}_1^{(1)})^{[\ell]}, ({}^2\mathsf{u}_1^{(2)})^{[\ell]}, \ldots, ({}^2\mathsf{u}_1^{(\mathsf{m})})^{[\ell]}, ({}^2\mathsf{u}_2^{(1)})^{[\ell]}, ({}^2\mathsf{u}_2^{(2)})^{[\ell]}, \ldots, ({}^2\mathsf{u}_2^{(\mathsf{m})})^{[\ell]}, \ldots, ({}^2\mathsf{u}_\mathsf{n}^{(1)})^{[\ell]}, ({}^2\mathsf{u}_\mathsf{n}^{(2)})^{[\ell]}, \ldots, ({}^2\mathsf{u}_\mathsf{n}^{(\mathsf{m})})^{[\ell]}\big\}_{\ell=1}^{\infty}$ *and* $\big\{({}^3\mathsf{u}_1^{(1)})^{[\ell]}, ({}^3\mathsf{u}_1^{(2)})^{[\ell]}, \ldots, ({}^3\mathsf{u}_1^{(\mathsf{m})})^{[\ell]}, ({}^3\mathsf{u}_2^{(1)})^{[\ell]}, ({}^3\mathsf{u}_2^{(2)})^{[\ell]}, \ldots, ({}^3\mathsf{u}_2^{(\mathsf{m})})^{[\ell]}, \ldots, ({}^3\mathsf{u}_\mathsf{n}^{(1)})^{[\ell]}, ({}^3\mathsf{u}_\mathsf{n}^{(2)})^{[\ell]}, \ldots, ({}^3\mathsf{u}_\mathsf{n}^{(\mathsf{m})})^{[\ell]}\big\}_{\ell=1}^{\infty}$ *such that* $({}^1\mathsf{u}_\mathsf{j}^{(\mathsf{i})})^{[\ell]}(\mathsf{r}) \geqslant 0$, $({}^2\mathsf{u}_\mathsf{j}^{(\mathsf{i})})^{[\ell]}(\mathsf{r}) \geqslant 0$ *and* $({}^3\mathsf{u}_\mathsf{j}^{(\mathsf{i})})^{[\ell]}(\mathsf{r}) \geqslant 0$ *on* $[0, w]_{\mathbb{T}}$, *with* $\alpha_\ell({}^1\mathsf{u}_\mathsf{j}^{(\mathsf{i})})^{[\ell]} \leqslant \mathsf{Q}_\ell \leqslant \alpha_\ell({}^2\mathsf{u}_\mathsf{j}^{(\mathsf{i})})^{[\ell]}, \beta_\ell({}^2\mathsf{u}_\mathsf{j}^{(\mathsf{i})})^{[\ell]} \leqslant \mathsf{S}_\ell \leqslant \beta_\ell({}^3\mathsf{u}_\mathsf{j}^{(\mathsf{i})})^{[\ell]}$, $\gamma_\ell({}^3\mathsf{u}_\mathsf{j}^{(\mathsf{i})})^{[\ell]} \leqslant \mathsf{R}_\ell$, $\mathsf{j} = 1, 2, \ldots, \mathsf{n}, \mathsf{i} = 1, 2, \ldots, \mathsf{m}$, $\ell \in \mathbb{N}$.

Proof. Consider the completely continuous operator T and the cone P which was established in previous section. So it is easy to check that $\mathsf{T} : \overline{\mathsf{P}(\beta, \mathsf{R}_\ell)} \to \mathsf{P}$ for $\ell \in \mathbb{N}$.
In order to prove that all conditions of theorem (5.7) are satisfied, we choose

$u_1^{(1)} \in \partial P(\beta, R_\ell)$. Then $\beta(u_1^{(1)}) = \max_{r\in[0,w]_{\mathbb{T}}} (u_1^{(1)}(r)) = R_\ell,\ 0 \leqslant u_1^{(1)} \leqslant \dfrac{1}{K_\ell} R_\ell$ for $r \in [0, w]_{\mathbb{T}}$.
We have $\|u_1^{(1)}\| \leqslant \frac{1}{K_\ell}\beta(u_1^{(1)}) = \frac{1}{K_\ell}R_\ell$.
So we have, $0 \leqslant u_1^{(1)} \leqslant \frac{1}{K_\ell}R_\ell, r \in [0, w]_{\mathbb{T}}$, for $0 \leqslant s_{mn-1} < w$,

$$\int_0^w N_n(s_{mn-1}, s_{mn})L_n(s_{mn})\mathfrak{f}_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn} \leqslant \int_0^w N_n(s_{mn}, s_{mn})L_n(s_{mn})\mathfrak{h}_n R_\ell \Delta s_{mn}$$
$$\leqslant \mathfrak{h}_n R_\ell \int_0^w N_n(s_{mn}, s_{mn})L_n(s_{mn})\Delta s_{mn} = \mathfrak{h}_n R_\ell \int_0^w N_n(s_{mn}, s_{mn}) \prod_{k=1}^{s} \mu_{nk}(s_{mn})\Delta s_{mn}.$$

Since $\sum_{k=1}^{s} \dfrac{1}{\mathfrak{p}_k} < 1$, there exists $\mathfrak{q} > 1$ such that $\dfrac{1}{\mathfrak{q}} + \sum_{k=1}^{s} \dfrac{1}{\mathfrak{p}_k} = 1$. So,

$$\int_0^w N_n(s_{mn-1}, s_{mn})L_n(s_{mn})\mathfrak{f}_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn} \leqslant \mathfrak{h}_n R_\ell \|N_n\|_{L_\Delta^{\mathfrak{q}}} \left\| \prod_{k=1}^{s} \mu_{nk} \right\|_{L_\Delta^{\mathfrak{p}_k}} \leqslant R_\ell.$$

$$\int_0^w N_n(s_{mn-2}, s_{mn-1})L_n(s_{mn-1})\mathfrak{f}_n^{(m-1)}\left(\int_0^w N_n(s_{mn-1}, s_{mn})L_n(s_{mn})\mathfrak{f}_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn}\right)\Delta s_{mn-1}$$
$$\leqslant \int_0^w N_n(s_{mn-2}, s_{mn-1})L_n(s_{mn-1})\mathfrak{f}_n^{(m-1)}(R_\ell)\Delta s_{mn-1} \leqslant R_\ell.$$

Continuing in this way, we get $(Tu_1^{(1)})(r) \leqslant R_\ell$.
$\beta_\ell(Tu_1^{(1)})(r) = \max_{r\in[0,w]_{\mathbb{T}}} (Tu_1^{(1)})(r) \leqslant R_\ell$, Hence condition (a) is satisfied.
$u_1^{(1)} \in \partial P(\gamma, S_\ell)$. Then
$S_\ell = \gamma(u_1^{(1)}) = \min_{r\in[0,w]_{\mathbb{T}}} (u_1^{(1)}(r)) \leqslant \max_{r\in[0,w]_{\mathbb{T}}} (u_1^{(1)}(r)) = \|u_1^{(1)}\| \leqslant \dfrac{1}{K_\ell}\gamma(u_1^{(1)}) \leqslant \dfrac{1}{K_\ell}S_\ell$.
We have $\|u_1^{(1)}\| \leqslant \frac{1}{K_\ell}\gamma_\ell(u_1^{(1)}) \leqslant \frac{1}{K_\ell}b_\ell(u_1^{(1)}) = \frac{1}{K_\ell}S_\ell. \therefore S_\ell \leqslant u_1^{(1)}(r) \leqslant \frac{1}{K_\ell}S_\ell, r \in [0, w]_{\mathbb{T}}$.
By (F2) and for $s_{mn} \in [z_\ell, w - z_\ell]_{\mathbb{T}}$, we have

$$\int_0^w N_n(s_{mn-1}, s_{mn})L_n(s_{mn})\mathfrak{f}_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn}$$
$$\geqslant G_n(p_{n\ell}) \int_{z_\ell}^{w-z_\ell} N_n(s_{mn}, s_{mn})L_n(s_{mn})\mathfrak{f}_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn}$$
$$\geqslant G_n(p_{n\ell})M_n\, S_\ell \int_{z_\ell}^{w-z_\ell} N_n(s_{mn}, s_{mn})L_n(s_{mn})\Delta s_{mn}$$
$$\geqslant K_1 M_n\, S_\ell \prod_{k=1}^{s} e_{nk} \int_{z_1}^{w-z_1} N_n(s_{mn}, s_{mn})\Delta s_{mn} \geqslant S_\ell.$$

Continuing in this way, we get $(Tu_1^{(1)})(r) \geqslant S_\ell$.
$\gamma(Tu_1^{(1)})(r) = \min_{r\in[0,w]_{\mathbb{T}}} T(u_1^{(1)}(r)) \geqslant S_\ell$.
Hence condition (b) is satisfied. Finally we verify that (c) of theorem 5.7 is also satisfied.
Since $0 \in P, Q_\ell > 0$, it follows that $P(\alpha_\ell, Q_\ell) \neq \phi$.
Now let $u_1^{(1)} \in \partial P(\alpha_\ell, Q_\ell)$. Then $\alpha_\ell(u_1^{(1)}(r)) = \max_{r\in[0,w]_{\mathbb{T}}} (u_1^{(1)}(r)) = \|u_1^{(1)}\| = Q_\ell$.
For $\|u_1^{(1)}\| \leqslant \frac{1}{K_\ell}\gamma_\ell(u_1^{(1)}) \leqslant \frac{1}{K_\ell}\alpha_\ell(u_1^{(1)}) = \frac{1}{K_\ell}Q_\ell$, then we get $0 \leqslant u_1^{(1)} \leqslant \frac{1}{K_\ell}R_\ell,\ r \in [0, w]_{\mathbb{T}}$.
For $0 \leqslant s_{mn-1} < w$,

$$\int_0^w N_n(s_{mn-1},s_{mn})L_n(s_{mn})f_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn} \leqslant \int_0^w N_n(s_{mn},s_{mn})L_n(s_{mn})h_n Q_\ell \Delta s_{mn}$$

$$\leqslant h_n Q_\ell \int_0^w N_n(s_{mn},s_{mn})L_n(s_{mn})\Delta s_{mn} = h_n Q_\ell \int_0^w N_n(s_{mn},s_{mn})\prod_{k=1}^{s}\mu_{nk}(s_{mn})\Delta s_{mn}.$$

Since $\sum_{k=1}^{s}\frac{1}{p_k} < 1$, there exists $q > 1$ such that $\frac{1}{q}+\sum_{k=1}^{s}\frac{1}{p_k} = 1$. So,

$$\int_0^w N_n(s_{mn-1},s_{mn})L_n(s_{mn})f_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn} \leqslant h_n Q_\ell \|N_n\|_{L_\Delta^q}\left\|\prod_{k=1}^{s}\mu_{nk}\right\|_{L_\Delta^{p_k}} \leqslant Q_\ell.$$

$$\int_0^w N_n(s_{mn-2},s_{mn-1})L_n(s_{mn-1})f_n^{(m-1)}\left(\int_0^w N_n(s_{mn-1},s_{mn})L_n(s_{mn})f_n^{(m)}(u_1^{(1)}(s_{mn}))\Delta s_{mn}\right)\Delta s_{mn-1}$$

$$\leqslant \int_0^w N_n(s_{mn-2},s_{mn-1})L_n(s_{mn-1})f_n^{(m-1)}(Q_\ell)\Delta s_{mn-1} \leqslant Q_\ell.$$

Continuing in this way, we get $(Tu_1^{(1)})(r) \leqslant Q_\ell$.

$\alpha_\ell(Tu_1^{(1)})(r) = \max_{r\in[0,w]_{\mathbb{T}}}(Tu_1^{(1)})(r) \leqslant Q_\ell$, hence condition (c) is satisfied.

Thus, all the conditions of Theorem 5.7 are satisfied. Hence, there exists at least three fixed points of T which are positive solutions of (1.1) such that $\alpha_\ell({}^1u_j^{(i)})^{[\ell]} \leqslant Q_\ell \leqslant \alpha_\ell({}^2u_j^{(i)})^{[\ell]}, \beta_\ell({}^2u_j^{(i)})^{[\ell]} \leqslant S_\ell \leqslant \beta_\ell({}^3u_j^{(i)})^{[\ell]}$, $\gamma_\ell({}^3u_j^{(i)})^{[\ell]} \leqslant R_\ell$, $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$, $\ell \in \mathbb{N}$. □

Theorem 5.9. *Suppose that* (L1) − (L3) *hold. Let* $\{p_{j\ell}\}_{\ell=1}^{\infty}$ *be a sequence with* $p_{j\ell} \in (r_{\ell+1}, r_\ell)$, $z_\ell = \max\{p_{1\ell}, p_{2\ell}, ..., p_{n\ell}\}$, $0 < z_1 < \frac{w}{2}$ *for* $j = 1, 2, ..., n$. *Let* $\{R_\ell\}_{\ell=1}^{\infty}$, $\{Q_\ell\}_{\ell=1}^{\infty}$ *and* $\{S_\ell\}_{\ell=1}^{\infty}$ *be three sequences such that*

$$R_{\ell+1} < Q_\ell < K_\ell S_\ell < S_\ell < R_\ell,\ M_j\, S_\ell < R_\ell,\ \ell \in \mathbb{N},$$

where

$$M_j = \max\left\{\left[K_1\prod_{k=1}^{s}e_{jk}\int_{z_1}^{w-z_1}N_j(s,s)\Delta s\right]^{-1},\ 1\right\}.$$

Assume that $f_j^{(i)}$ *satisfies*

(F4) $f_j^{(i)}(u_j) \leqslant m_j R_\ell\ \forall\, r \in [0,w]_{\mathbb{T}},\ 0 \leqslant u_j \leqslant \frac{1}{K_\ell}R_\ell$, *where*

$$m_j < \left[\|N_j\|_{L_\Delta^\infty}\prod_{k=1}^{s}\|\mu_{jk}(r)\|_{L_\Delta^{p_k}}\right]^{-1},$$

(F5) $f_j^{(i)}(u_j) \geqslant M_j S_\ell\ \forall\, r \in [z_\ell, w - z_\ell]_{\mathbb{T}},\ K_\ell S_\ell \leqslant u_j \leqslant S_\ell$,

(F6) $f_j^{(i)}(u_j) \leqslant m_j Q_\ell\ \forall\, r \in [0,w]_{\mathbb{T}},\ 0 \leqslant u_j \leqslant \frac{1}{K_\ell}Q_\ell$.

Then (1.1) *has at least three positive solutions* $\{({}^1u_1^{(1)})^{[\ell]}, ({}^1u_1^{(2)})^{[\ell]}, \ldots, ({}^1u_1^{(m)})^{[\ell]}, ({}^1u_2^{(1)})^{[\ell]}, ({}^1u_2^{(2)})^{[\ell]}, \ldots, ({}^1u_2^{(m)})^{[\ell]}, \ldots, ({}^1u_n^{(1)})^{[\ell]}, ({}^1u_n^{(2)})^{[\ell]}, \ldots, ({}^1u_n^{(m)})^{[\ell]}\}_{\ell=1}^{\infty}$, $\{({}^2u_1^{(1)})^{[\ell]}, ({}^2u_1^{(2)})^{[\ell]}, \ldots, ({}^2u_1^{(m)})^{[\ell]}, ({}^2u_2^{(1)})^{[\ell]}, ({}^2u_2^{(2)})^{[\ell]}, \ldots, ({}^2u_2^{(m)})^{[\ell]}, \ldots, ({}^2u_n^{(1)})^{[\ell]}, ({}^2u_n^{(2)})^{[\ell]}, \ldots, ({}^2u_n^{(m)})^{[\ell]}\}_{\ell=1}^{\infty}$ *and* $\{({}^3u_1^{(1)})^{[\ell]}, ({}^3u_1^{(2)})^{[\ell]}, \ldots, ({}^3u_1^{(m)})^{[\ell]}, ({}^3u_2^{(1)})^{[\ell]}, ({}^3u_2^{(2)})^{[\ell]}, \ldots, ({}^3u_2^{(m)})^{[\ell]}, \ldots, ({}^3u_n^{(1)})^{[\ell]}, ({}^3u_n^{(2)})^{[\ell]}, \ldots, ({}^3u_n^{(m)})^{[\ell]}\}_{\ell=1}^{\infty}$ *such that* $({}^1u_j^{(i)})^{[\ell]}(r) \geqslant 0$, $({}^2u_j^{(i)})^{[\ell]}(r) \geqslant 0$ *and* $({}^3u_j^{(i)})^{[\ell]}(r) \geqslant 0$ *on* $[0,w]_{\mathbb{T}}$, *with* $\alpha_\ell({}^1u_j^{(i)})^{[\ell]} \leqslant Q_\ell \leqslant \alpha_\ell({}^2u_j^{(i)})^{[\ell]}, \beta_\ell({}^2u_j^{(i)})^{[\ell]} \leqslant S_\ell \leqslant \beta_\ell({}^3u_j^{(i)})^{[\ell]}$, $\gamma_\ell({}^3u_j^{(i)})^{[\ell]} \leqslant R_\ell$, $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$, $\ell \in \mathbb{N}$.

Lastly, the case $\sum_{k=1}^{s} \frac{1}{\mathfrak{p}_k} > 1$.

Theorem 5.10. *Suppose that* $(L1) - (L3)$ *hold. Let* $\{p_{j\ell}\}_{\ell=1}^{\infty}$ *be a sequence with* $p_{j\ell} \in (r_{\ell+1}, r_\ell)$, $z_\ell = \max\{p_{1\ell}, p_{2\ell}, ..., p_{n\ell}\}$, $0 < z_1 < \frac{w}{2}$ *for* $j = 1, 2, ..., n$. *Let* $\{R_\ell\}_{\ell=1}^{\infty}$, $\{Q_\ell\}_{\ell=1}^{\infty}$ *and* $\{S_\ell\}_{\ell=1}^{\infty}$ *be three sequences such that*

$$R_{\ell+1} < Q_\ell < K_\ell S_\ell < S_\ell < R_\ell,\ M_j\, S_\ell < R_\ell,\ \ell \in \mathbb{N},$$

where

$$M_j = \max\left\{ \left[K_1 \prod_{k=1}^{s} e_{jk} \int_{z_1}^{w-z_1} N_j(s,s)\Delta s \right]^{-1},\ 1 \right\}.$$

Assume that $\mathfrak{f}_j^{(i)}$ *satisfies*

(F7) $\mathfrak{f}_j^{(i)}(u_j) \leqslant \mathtt{n}_j R_\ell\ \forall\, r \in [0,w]_{\mathbb{T}},\ 0 \leqslant u_j \leqslant \frac{1}{K_\ell} R_\ell$, *where*

$$\mathtt{n}_j < \left[\|N_j\|_{L^\infty_\Delta} \prod_{k=1}^{s} \|\mu_{jk}(r)\|_{L^1_\Delta} \right]^{-1},$$

(F8) $\mathfrak{f}_j^{(i)}(u_j) \geqslant M_j S_\ell\ \forall\, r \in [z_\ell, w - z_\ell]_{\mathbb{T}},\ K_\ell S_\ell \leqslant u_j \leqslant S_\ell$,

(F9) $\mathfrak{f}_j^{(i)}(u_j) \leqslant \mathtt{n}_j Q_\ell\ \forall\, r \in [0,w]_{\mathbb{T}},\ 0 \leqslant u_j \leqslant \frac{1}{K_\ell} Q_\ell$.

Then (1.1) *has at least three positive solutions* $\big\{({}^1u_1^{(1)})^{[\ell]}, ({}^1u_1^{(2)})^{[\ell]}, \ldots, ({}^1u_1^{(m)})^{[\ell]}, ({}^1u_2^{(1)})^{[\ell]}, ({}^1u_2^{(2)})^{[\ell]}, \ldots, ({}^1u_2^{(m)})^{[\ell]}, \ldots, ({}^1u_n^{(1)})^{[\ell]}, ({}^1u_n^{(2)})^{[\ell]}, \ldots, ({}^1u_n^{(m)})^{[\ell]}\big\}_{\ell=1}^{\infty}$, $\big\{({}^2u_1^{(1)})^{[\ell]}, ({}^2u_1^{(2)})^{[\ell]}, \ldots, ({}^2u_1^{(m)})^{[\ell]}, ({}^2u_2^{(1)})^{[\ell]}, ({}^2u_2^{(2)})^{[\ell]}, \ldots, ({}^2u_2^{(m)})^{[\ell]}, \ldots, ({}^2u_n^{(1)})^{[\ell]}, ({}^2u_n^{(2)})^{[\ell]}, \ldots, ({}^2u_n^{(m)})^{[\ell]}\big\}_{\ell=1}^{\infty}$ *and* $\big\{({}^3u_1^{(1)})^{[\ell]}, ({}^3u_1^{(2)})^{[\ell]}, \ldots, ({}^3u_1^{(m)})^{[\ell]}, ({}^3u_2^{(1)})^{[\ell]}, ({}^3u_2^{(2)})^{[\ell]}, \ldots, ({}^3u_2^{(m)})^{[\ell]}, \ldots, ({}^3u_n^{(1)})^{[\ell]}, ({}^3u_n^{(2)})^{[\ell]}, \ldots, ({}^3u_n^{(m)})^{[\ell]}\big\}_{\ell=1}^{\infty}$ *such that* $({}^1u_j^{(i)})^{[\ell]}(r) \geqslant 0$, $({}^2u_j^{(i)})^{[\ell]}(r) \geqslant 0$ *and* $({}^3u_j^{(i)})^{[\ell]}(r) \geqslant 0$ *on* $[0,w]_{\mathbb{T}}$, *with* $\alpha_\ell({}^1u_j^{(i)})^{[\ell]} \leqslant Q_\ell \leqslant \alpha_\ell({}^2u_j^{(i)})^{[\ell]}, \beta_\ell({}^2u_j^{(i)})^{[\ell]} \leqslant S_\ell \leqslant \beta_\ell({}^3u_j^{(i)})^{[\ell]}, \gamma_\ell({}^3u_j^{(i)})^{[\ell]} \leqslant R_\ell$, $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$, $\ell \in \mathbb{N}$.

6. Numerical Examples

In this section, we consider certain problems to verify our results.

Example 6.1. *Consider the BVP*

$$\left.\begin{aligned}
&(u_1^{(1)})^{\Delta\Delta}(r) + L_1(r)\, \mathfrak{f}_1^{(1)}(u_2^{(1)}(r)) = 0,\ i = 1, j = 1, 2\ r \in [0,1]_{\mathbb{T}},\\
&(u_2^{(1)})^{\Delta\Delta}(r) + L_2(r)\, \mathfrak{f}_2^{(1)}(u_1^{(1)}(r)) = 0,\\
&u_1^{(2)}(r) = u_2^{(1)}(r), u_3^{(1)}(r) = u_1^{(1)}(r),\\
&u_1^{(1)}(0) - (u_1^{(1)})^{\Delta}(0) = 0,\ u_1^{(1)}(1) + (u_1^{(1)})^{\Delta}(1) = 0,\\
&u_2^{(1)}(0) - (u_2^{(1)})^{\Delta}(0) = 0,\ u_2^{(1)}(1) + (u_2^{(1)})^{\Delta}(1) = 0,
\end{aligned}\right\} \tag{6.8}$$

where $\mathbb{T} = [0, \frac{1}{5}] \cup \{\frac{1}{4}, \frac{2}{5}, \frac{3}{5}\} \cup [\frac{3}{4}, 1]$, $L_1(r) = \mu_{11}(r)\mu_{12}(r)$, $L_2(r) = \mu_{21}(r)\mu_{22}(r)$, $\mu_{11}(r) = \dfrac{1}{|r - \frac{1}{8}|^{\frac{1}{2}}}$, $\mu_{12}(r) = \dfrac{1}{|r - \frac{1}{6}|^{\frac{1}{2}}}$, $\mu_{21}(r) = \dfrac{1}{|r - \frac{4}{5}|^{\frac{1}{2}}}$, $\mu_{22}(r) = \dfrac{1}{|r - \frac{7}{8}|^{\frac{1}{2}}}$,

$$\mathfrak{f}_1^{(1)}(\mathsf{u}_1)=\begin{cases}0.05\times10^{-4},\ \mathsf{u}_1\in(10^{-4},\infty),\\ \dfrac{17\times10^{-(4\ell+3)}-0.05\times10^{-(4\ell)}}{10^{-(4\ell+3)}-10^{-(4\ell)}}(\mathsf{u}_1-10^{-(4\ell)})+0.05\times10^{-(8\ell)},\\ \mathsf{u}_1\in\left[10^{-(4\ell+3)},10^{-(4\ell)}\right],\\ 17\times10^{-(4\ell+3)},\ \mathsf{u}_1\in\left(\dfrac{1}{5}\times10^{-(4\ell+3)},10^{-(4\ell+3)}\right),\\ \dfrac{17\times10^{-(4\ell+3)}-0.05\times10^{-(8\ell)}}{0.05\times10^{-(4\ell+3)}-10^{-(4\ell+4)}}(\mathsf{u}_1-10^{-(4\ell+4)})+0.05\times10^{-(8\ell)},\\ \mathsf{u}_1\in(10^{-(4\ell+4)},\dfrac{1}{5}\times10^{-(4\ell+3)}],\\ 0,\ \mathsf{u}_1=0,\end{cases}$$

$$\mathfrak{f}_2^{(1)}(\mathsf{u}_2)=\begin{cases}0.05\times10^{-4},\ \mathsf{u}_2\in(10^{-4},\infty),\\ \dfrac{16\times10^{-(4\ell+3)}-0.05\times10^{-(4\ell)}}{10^{-(4\ell+3)}-10^{-(4\ell)}}(\mathsf{u}_2-10^{-(4\ell)})+0.05\times10^{-(8\ell)},\\ \mathsf{u}_2\in[10^{-(4\ell+3)},10^{-(4\ell)}],\\ 16\times10^{-(4\ell+3)},\ \mathsf{u}_2\in\left(\dfrac{1}{5}\times10^{-(4\ell+3)},10^{-(4\ell+3)}\right),\\ \dfrac{16\times10^{-(4\ell+3)}-0.05\times10^{-(8\ell)}}{0.05\times10^{-(4\ell+3)}-10^{-(4\ell+4)}}(\mathsf{u}_2-10^{-(4\ell+4)})+0.05\times10^{-(8\ell)},\\ \mathsf{u}_2\in(10^{-(4\ell+4)},\dfrac{1}{5}\times10^{-(4\ell+3)}],\\ 0,\ \mathsf{u}_2=0.\end{cases}$$

Let $\mathsf{r}_\ell=\frac{31}{64}-\sum_{\mathsf{k}=1}^{\ell}\frac{1}{4(\mathsf{k}+1)^4},\ p_{\mathsf{j}\ell}=\frac{1}{2}(\mathsf{r}_\ell+\mathsf{r}_{\ell+1}),\ \mathsf{j}=1,2,\ \ell=1,2,3,\dots,$
then $p_{\mathsf{j}1}=\frac{15}{32}-\frac{1}{648}<\frac{15}{32}$ *and* $\mathsf{r}_{\ell+1}<p_{\mathsf{j}\ell}<\mathsf{r}_\ell,\ p_{\mathsf{j}\ell}>\frac{1}{5},\mathsf{z}_1=\frac{15}{32}-\frac{1}{648}.$
It is clear that $\mathsf{r}_1=\frac{15}{32}<\frac{1}{2},\ \mathsf{r}_\ell-\mathsf{r}_{\ell+1}=\dfrac{1}{4(\ell+2)^4}.$
Since $\sum_{\mathsf{k}=1}^{\infty}\frac{1}{\mathsf{k}^4}=\frac{\pi^4}{90}$ *and* $\sum_{\mathsf{k}=1}^{\infty}\frac{1}{\mathsf{k}^2}=\frac{\pi^2}{6}$*, it follows that*

$$\mathsf{r}^*=\lim_{\ell\to\infty}\mathsf{r}_\ell=\frac{31}{64}-\sum_{\mathsf{k}=1}^{\infty}\frac{1}{4(\mathsf{k}+1)^4}=\frac{47}{64}-\frac{\pi^4}{360}>\frac{1}{5}.$$

$\mathsf{G}_\mathsf{j}(p_{\mathsf{j}1})=0.7336033951,\ \mathsf{e}_{11}=\mathsf{e}_{12}=(\frac{5}{3})^{\frac{1}{2}}$*, and* $\mathsf{e}_{21}=\mathsf{e}_{22}=(\frac{5}{2})^{\frac{1}{3}}$*,*

$$\int_{\mathsf{z}_1}^{w-\mathsf{z}_1}N_\mathsf{j}(\mathsf{s},\mathsf{s})d\mathsf{s}=\int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}}\frac{(2-\mathsf{s})(1+\mathsf{s})}{3}d\mathsf{s}=0.04918197801.$$

Thus, we obtain

$$\mathsf{M}_1=25.557267815288,\ \mathsf{M}_2=26.212302578035.$$

It follows that

$$\prod_{\mathsf{k}=1}^{2}\|\mu_{1\mathsf{k}}\|_{\mathsf{L}^{\mathsf{p}_\mathsf{k}}}=\int_0^{\frac{1}{4}}\mu_{11}(\mathsf{r})\mu_{12}(\mathsf{r})\Delta\mathsf{r}+\int_{\frac{3}{4}}^{1}\mu_{11}(\mathsf{r})\mu_{12}(\mathsf{r})\Delta\mathsf{r}+\left[\sigma\left(\frac{1}{4}\right)-\frac{1}{4}\right]\mu_{11}\left(\frac{1}{4}\right)\mu_{12}\left(\frac{1}{4}\right)+$$

$$\left[\sigma\left(\frac{2}{5}\right)-\frac{2}{5}\right]\mu_{11}\left(\frac{2}{5}\right)\mu_{12}\left(\frac{2}{5}\right)+\left[\sigma\left(\frac{3}{5}\right)-\frac{3}{5}\right]\mu_{11}\left(\frac{3}{5}\right)\mu_{12}\left(\frac{3}{5}\right)+\left[\sigma\left(\frac{1}{5}\right)-\frac{1}{5}\right]\mu_{11}\left(\frac{1}{5}\right)\mu_{12}\left(\frac{1}{5}\right),$$

$$\prod_{k=1}^{2}\|\mu_{1k}\|_{L^{p_k}} \approx 12.004 \prod_{k=1}^{2}\|\mu_{2k}\|_{L^{p_k}} \approx 8.57 \textit{ and } \|N_j\|_\infty = \frac{2}{3}.$$

We have $g_1 < 0.1249560140438, g_2 < 0.1751029108175$, *taking* $g_1 = \frac{1}{10}, g_2 = \frac{3}{20}$.
In addition, if we take $D_\ell = 10^{-4\ell},\ E_\ell = 10^{-(4\ell+3)}$, *then*

$$D_{\ell+1} = 10^{-(4\ell+4)} < \frac{1}{5} \times 10^{-(4\ell+3)} < K_\ell E_\ell < E_\ell = 10^{-(4\ell+3)} < D_\ell = 10^{-4\ell},$$

$$M_1\ E_\ell = 25.557267815288 \times 10^{-(4\ell+3)} < \frac{1}{10} \times 10^{-4\ell} = g_1 D_\ell,\ \ell = 1, 2, 3, \ldots,$$

$$D_{\ell+1} = 10^{-(4\ell+4)} < \frac{1}{5} \times 10^{-(4\ell+3)} < K_\ell E_\ell < E_\ell = 10^{-(4\ell+3)} < D_\ell = 10^{-4\ell},$$

$$M_2\ E_\ell = 26.212302578035 \times 10^{-(4\ell+3)} < \frac{3}{20} \times 10^{-4\ell} = g_2 D_\ell,\ \ell = 1, 2, 3, \ldots,$$

and $f_j^{(i)}$ *satisfies the following growth conditions:*

$f_1^{(1)}(u_1) \leqslant g_1\ D_\ell = \frac{1}{10} \times 10^{-4\ell},\ u_1 \in [0, 10^{-4\ell}]$,

$f_1^{(1)}(u_1) \geqslant M_1\ E_\ell = 25.557267815288 \times 10^{-(4\ell+3)},\ u_1 \in \left[\frac{1}{5} \times 10^{-(4\ell+3)}, 10^{-(4\ell+3)}\right]$,

$f_2^{(1)}(u_2) \leqslant g_2\ D_\ell = \frac{3}{20} \times 10^{-4\ell},\ u_2 \in [0, 10^{-4\ell}]$,

$f_2^{(1)}(u_2) \geqslant M_2\ E_\ell = 26.212302578035. \times 10^{-(4\ell+3)},\ u_2 \in \left[\frac{1}{5} \times 10^{-(4\ell+3)}, 10^{-(4\ell+3)}\right]$.

Hence, all conditions in Theorem 4.5 are satisfied. Therefore, by Theorem 4.5 the system of iterative BVP (6.8) has at least one positive solution $\left\{(u_1^{(1)})^{[\ell]}, (u_2^{(1)})^{[\ell]}\right\}_{\ell=1}^{\infty}$ *such that* $(u_j^{(i)})^{[\ell]}(r) \geqslant 0$ *on* $[0,1]_{\mathbb{T}},\ j = 1, 2, i = 1$ *and* $\ell \in \mathbb{N}$.

Example 6.2. *Consider the BVP*

$$\left.\begin{aligned}
&(u_1^{(1)})^{\Delta\Delta}(r) + L_1(r)\, f_1^{(1)}(u_2^{(1)}(r)) = 0,\ i = 1, j = 1, 2\ r \in [0,1]_{\mathbb{T}},\\
&(u_2^{(1)})^{\Delta\Delta}(r) + L_2(r)\, f_2^{(1)}(u_1^{(1)}(r)) = 0,\\
&u_1^{(2)}(r) = u_2^{(1)}(r), u_3^{(1)}(r) = u_1^{(1)}(r),\\
&u_1^{(1)}(0) - (u_1^{(1)})^{\Delta}(0) = 0,\ u_1^{(1)}(1) + (u_1^{(1)})^{\Delta}(1) = 0,\\
&u_2^{(1)}(0) - (u_2^{(1)})^{\Delta}(0) = 0,\ u_2^{(1)}(1) + (u_2^{(1)})^{\Delta}(1) = 0,
\end{aligned}\right\} \tag{6.9}$$

where $\mathbb{T} = [0, \frac{1}{4}] \cup \{\frac{1}{3}, \frac{2}{5}, \frac{1}{2}\} \cup [\frac{3}{4}, 1],\ L_1(r) = \mu_{11}(r)\mu_{12}(r),\ L_2(r) = \mu_{21}(r)\mu_{22}(r)$, $\mu_{11}(r) = \dfrac{1}{|r - \frac{1}{5}|^{\frac{1}{2}}}, \mu_{12}(r) = \dfrac{1}{|r - \frac{4}{5}|^{\frac{1}{2}}}, \mu_{21}(r) = \dfrac{1}{|r - \frac{1}{6}|^{\frac{1}{2}}}, \mu_{22}(r) = \dfrac{1}{|r - \frac{5}{6}|^{\frac{1}{2}}}$,

$$\mathfrak{f}_1^{(1)}(\mathsf{u}_1)=\begin{cases}0.05\times 10^{-4},\ \mathsf{u}_1\in(10^{-4},\infty),\\ \dfrac{25\times 10^{-(4\ell+3)}-0.05\times 10^{-(4\ell)}}{10^{-(4\ell+3)}-10^{-(4\ell)}}(\mathsf{u}_1-10^{-(4\ell)})+0.05\times 10^{-(8\ell)},\\ \ \mathsf{u}_1\in\left[10^{-(4\ell+3)},10^{-(4\ell)}\right],\\ 25\times 10^{-(4\ell+3)},\ \mathsf{u}_1\in\left(\dfrac{1}{5}\times 10^{-(4\ell+3)},10^{-(4\ell+3)}\right),\\ \dfrac{25\times 10^{-(4\ell+3)}-0.05\times 10^{-(8\ell)}}{0.05\times 10^{-(4\ell+3)}-10^{-(4\ell+4)}}(\mathsf{u}_1-10^{-(4\ell+4)})+0.05\times 10^{-(8\ell)},\\ \ \mathsf{u}_1\in(10^{-(4\ell+4)},\dfrac{1}{5}\times 10^{-(4\ell+3)}],\\ 0,\ \mathsf{u}_1=0,\end{cases}$$

$$\mathfrak{f}_2^{(1)}(\mathsf{u}_2)=\begin{cases}0.05\times 10^{-4},\ \mathsf{u}_2\in(10^{-4},\infty),\\ \dfrac{32\times 10^{-(4\ell+3)}-0.05\times 10^{-(4\ell)}}{10^{-(4\ell+3)}-10^{-(4\ell)}}(\mathsf{u}_2-10^{-(4\ell)})+0.05\times 10^{-(8\ell)},\\ \ \mathsf{u}_2\in[10^{-(4\ell+3)},10^{-(4\ell)}],\\ 32\times 10^{-(4\ell+3)},\ \mathsf{u}_2\in\left(\dfrac{1}{5}\times 10^{-(4\ell+3)},10^{-(4\ell+3)}\right),\\ \dfrac{32\times 10^{-(4\ell+3)}-0.05\times 10^{-(8\ell)}}{0.05\times 10^{-(4\ell+3)}-10^{-(4\ell+4)}}(\mathsf{u}_2-10^{-(4\ell+4)})+0.05\times 10^{-(8\ell)},\\ \ \mathsf{u}_2\in(10^{-(4\ell+4)},\dfrac{1}{5}\times 10^{-(4\ell+3)}],\\ 0,\ \mathsf{u}_2=0.\end{cases}$$

Let $\mathsf{r}_\ell=\frac{31}{64}-\sum_{\mathsf{k}=1}^{\ell}\dfrac{1}{4(\mathsf{k}+1)^4},\ p_{\mathsf{j}\ell}=\dfrac{1}{2}(\mathsf{r}_\ell+\mathsf{r}_{\ell+1}),\ \mathsf{j}=1,2,\ \ell=1,2,3,\dots,$
then $p_{\mathsf{j}1}=\frac{15}{32}-\frac{1}{648}<\frac{15}{32}$ *and* $\mathsf{r}_{\ell+1}<p_{\mathsf{j}\ell}<\mathsf{r}_\ell,\ p_{\mathsf{j}\ell}>\frac{1}{5},\mathsf{z}_1=\frac{15}{32}-\frac{1}{648}$.
It is clear that $\mathsf{r}_1=\frac{15}{32}<\frac{1}{2},\ \mathsf{r}_\ell-\mathsf{r}_{\ell+1}=\dfrac{1}{4(\ell+2)^4}$.
Since $\sum_{\mathsf{k}=1}^{\infty}\dfrac{1}{\mathsf{k}^4}=\dfrac{\pi^4}{90}$ *and* $\sum_{\mathsf{k}=1}^{\infty}\dfrac{1}{\mathsf{k}^2}=\dfrac{\pi^2}{6}$, *it follows that*

$$\mathsf{r}^*=\lim_{\ell\to\infty}\mathsf{r}_\ell=\frac{31}{64}-\sum_{\mathsf{k}=1}^{\infty}\frac{1}{4(\mathsf{k}+1)^4}=\frac{47}{64}-\frac{\pi^4}{360}>\frac{1}{5}.$$

$\mathsf{G}_\mathsf{j}(p_{\mathsf{j}1})=0.7336033951,\ \mathsf{e}_{11}=\mathsf{e}_{12}=(\frac{5}{3})^{\frac{1}{2}}$, *and* $\mathsf{e}_{21}=\mathsf{e}_{22}=(\frac{5}{2})^{\frac{1}{3}}$,

$$\int_{\mathsf{z}_1}^{w-\mathsf{z}_1}N_\mathsf{j}(\mathsf{s},\mathsf{s})d\mathsf{s}=\int_{\frac{15}{32}-\frac{1}{648}}^{1-\frac{15}{32}+\frac{1}{648}}\frac{(2-\mathsf{s})(1+\mathsf{s})}{3}d\mathsf{s}=0.04918197801.$$

Thus, we obtain

$$\mathsf{M}_1=24.393296158,\ \mathsf{M}_2=31.392175118.$$

It follows that

$$\prod_{\mathsf{k}=1}^{2}\|\mu_{1\mathsf{k}}\|_{\mathsf{L}^{\mathsf{p}_\mathsf{k}}}=\int_0^{\frac{1}{4}}\mu_{11}(\mathsf{r})\mu_{12}(\mathsf{r})\Delta\mathsf{r}+\int_{\frac{3}{4}}^{1}\mu_{11}(\mathsf{r})\mu_{12}(\mathsf{r})\Delta\mathsf{r}+\left[\sigma\left(\frac{1}{4}\right)-\frac{1}{4}\right]\mu_{11}\left(\frac{1}{4}\right)\mu_{12}\left(\frac{1}{4}\right)+$$

$$\left[\sigma\left(\frac{1}{3}\right)-\frac{1}{3}\right]\mu_{11}\left(\frac{1}{3}\right)\mu_{12}\left(\frac{1}{3}\right)+\left[\sigma\left(\frac{2}{5}\right)-\frac{2}{5}\right]\mu_{11}\left(\frac{2}{5}\right)\mu_{12}\left(\frac{2}{5}\right)+\left[\sigma\left(\frac{1}{2}\right)-\frac{1}{2}\right]\mu_{11}\left(\frac{1}{2}\right)\mu_{12}\left(\frac{1}{2}\right),$$

$$\prod_{k=1}^{2}\|\mu_{1k}\|_{L^{p_k}} \approx 5.68 \prod_{k=1}^{2}\|\mu_{2k}\|_{L^{p_k}} \approx 5.04 \text{ and } \|N_j\|_\infty = \frac{2}{3}.$$

We have $m_1 < 0.2641395704, m_2 < 0.2976190476$, *taking* $m_1 = \frac{1}{5}, m_2 = \frac{1}{4}$.
In addition, if we take $R_\ell = 10^{-4\ell},\ Q_\ell = 10^{-(4\ell+3)}, S_\ell = 10^{-4(\ell+2)}$ *then*

$$R_{\ell+1} < Q_\ell < K_\ell S_\ell < S_\ell < R_\ell,\ M_j\,S_\ell < R_\ell,$$

$$R_{\ell+1} = 10^{-(4\ell+4)} < Q_\ell = 10^{-(4\ell+3)} < K_\ell S_\ell < S_\ell = 10^{-(4\ell+2)} < R_\ell = 10^{-4\ell},$$
$$M_1\,S_\ell = 24.393296158 \times 10^{-(4\ell+2)} < \frac{1}{5} \times 10^{-4\,\ell} = m_1 R_\ell,\ \ell = 1,2,3,\ldots,$$
$$M_2\,S_\ell = 31.392175118 \times 10^{-(4\ell+2)} < \frac{1}{4} \times 10^{-4\ell} = m_2 R_\ell,\ \ell = 1,2,3,\ldots,$$

and $f_j^{(i)}$ *satisfies the following growth conditions:*

$$f_1^{(1)}(u_1) \leqslant m_1\,R_\ell = \tfrac{1}{5} \times 10^{-4\ell},\ u_1 \in [0, 10^{-4\ell}],$$

$$f_1^{(1)}(u_1) \geqslant M_1\,S_\ell = 24.393296158 \times 10^{-(4\ell+2)},\ u_1 \in \left[\tfrac{1}{5} \times 10^{-(4\ell+2)}, 10^{-(4\ell+2)}\right],$$

$$f_1^{(1)}(u_1) \leqslant m_1\,Q_\ell = \tfrac{1}{5} \times 10^{-4(\ell+3)},\ u_1 \in [0, 10^{-4(\ell+3)}],$$

$$f_2^{(1)}(u_2) \leqslant m_2\,R_\ell = \tfrac{1}{4} \times 10^{-4\ell},\ u_2 \in [0, 10^{-4\ell}],$$

$$f_2^{(1)}(u_2) \geqslant M_2\,S_\ell = 31.392175118 \times 10^{-(4\ell+2)},\ u_2 \in \left[\tfrac{1}{5} \times 10^{-(4\ell+2)}, 10^{-(4\ell+2)}\right],$$

$$f_2^{(1)}(u_2) \leqslant m_2\,Q_\ell = \tfrac{1}{4} \times 10^{-4(\ell+3)},\ u_2 \in [0, 10^{-4(\ell+3)}].$$

Hence, all conditions in Theorem 5.9 are satisfied. Therefore, by Theorem 5.9 the system of iterative BVP (6.9) has at least three positive solutions $\left\{(u_1^{(1)})^{[\ell]}, (u_2^{(1)})^{[\ell]}\right\}_{\ell=1}^{\infty}$ *such that* $(u_j^{(i)})^{[\ell]}(r) \geqslant 0$ *on* $[0,1]_{\mathbb{T}},\ j = 1,2, i = 1$ *and* $\ell \in \mathbb{N}$.

Conclusion

This paper aims to extend and generalize the existing results in the literature, see [6, 18, 19]. For instance, if $j = 1$, then the system reduces to an iterative system of two-point boundary value problem on time scales with one component. And also, if $i = 1$, then the system becomes a system of dynamical equations with n components. The present paper establishes the existence of positive solutions for an n-component coupled system of iterative system associated with two-point boundary conditions on time scales. The approach is based on the application of the Guo–Krasnosel'skii fixed point theorem and Hölder's inequality. Further, by applying the Ren-Ge-Ren fixed point theorem, we also establish the multiple positive solutions of the problem. Moreover, the present work can be further extended to multi-point boundary value problems and integral-type boundary value problems by applying various new fixed point theorems.

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REFERENCES

[1] Ammi, M. R. S.; Malinowska, A. B. Existence of solutions for a coupled quasilinear system on time scales. *Electron. J. Diff. Equ.* **2011** (2011), no. 131, 1–11.
[2] Agarwal, R. P.; Bohner, M. Basic calculus on time scales and some of its applications. *Results. Math.* **35** (1999), no. 1, 3–22.
[3] Agarwal, R. P.; Bohner, M. O'Regan, D.; Peterson, A. Dynamic equations on time scales: a survey. *J. Comput. Appl. Math.* **141** (2002), no. 1-2, 1–26.
[4] Agarwal, R. P.; Otero-Espinar, V.; Perera, K.; Vivero, D. R. Basic properties of Sobolev's spaces on time scales. *Adv. Difference. Equ.* **2006** (2006), 1–14.
[5] Agarwal, R. P; O'Regan, D. A coupled system of boundary value problems. *Appl. Anal.* **69** (1998), no. 3-4, 381–385.
[6] Benchohra, M.; Berhoun, F.; Hamani, S.; Henderson, J.; Ntouyas, S. K.; Ouahab, A.; Purnaras, I. K. Eigenvalues for iterative systems of nonlinear boundary value problems on time scales. *Nonlinear Dyn. Syst. Theory* **9** (2009), no. 1, 11–22.
[7] Bohner, M.; Peterson, A. Dynamic equations on time scales: An introduction with applications, Boston: Birkh¨auser Boston, Inc., 2001.
[8] Cao, Z.; Lv, Y.; Jiang, D.; Zu, L. Positive solutions to singular semipositone boundary value problems of second order coupled differential systems. *J. Appl. Math. Comput.* **46** (2014), no. 1, 1–16.
[9] Guo, D.; Lakshmikantham, V. *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 2014.
[10] Guseinov, G. S. Integration on time scales, *J. Math. Anal. Appl.* **285** (2003), no. 1, 107–127.
[11] Hao, Z. C.; Liang, J.; Xiao, T. J. Existence results for time scale boundary value problem. *J. Comput. Appl. Math.* **197** (2006), no. 1, 156–168.
[12] Khuddush, M.; Prasad, K. R. Nonlinear two-point iterative functional boundary value problems on time scales. *J. Appl. Math. Comput.* **68** (2022), no. 6, 4241–4251.
[13] Liu, X. L.; Wang, W. X.; Wu, J. Delay dynamic equations on time scales. *Appl. Anal.* **89** (2010), no. 8, 1241–1249.
[14] Lü, H.; Yu, H.; Liu, Y. Positive solutions for singular boundary value problems of a coupled system of differential equations. *J. Math. Anal.* **302** (2005), no. 1, 14–29.
[15] Özkan, U. M.; Sarikaya, M. Z.; Yildirim, H. Extensions of certain integral inequalities on time scales. *Appl. Math. Lett.* **21** (2008), no. 10, 993–1000.
[16] Peterson, A. C.; Tisdell, C. C. Boundedness and uniqueness of solutions to dynamic equations on time scales. *J. Differ. Equ. Appl.* **10** (2004), no. 13-15, 1295–1306.
[17] Prasad, K. R.; Khuddush, M.; Vidyasagar, K. V. Denumerably many positive solutions for iterative system of boundary value problems with n-singularities on time-scales. *Kragujev. J. Math.* **47** (2023), no. 3, 369–385.
[18] Prasad, K. R.; Khuddush, M.; Vidyasagar, K. V. Infinitely many positive solutions for an iterative system of singular BVP on time scales. *Cubo* **24** (2022), no. 1, 21–35.
[19] Prasad, K. R.; Sravani, B.; Sreedhar. N. Countably many positive solutions for system of iterative system of boundary value problems on time scales. *Russ. Math.* **68** (2024), no. 10, 52–65.
[20] Rao, A. K.; Rao, S. N. Positive solutions for iterative systems of nonlinear boundary value problems on time scales. *Asian-Eur. J. Math.* **4** (2011), no. 1, 95–107.
[21] Ren, J. L.; Ge, W. G.; Ren, B. X. Existence of three positive solutions for quasi-linear boundary value problem. *Acta Math. Appl. Sin.* **21** (2005), no. 3, 353–358.
[22] Zheng, F.; Wang, X.; Cheng, X.; Du, B. Infinitely many positive solutions to nonlinear first-order iterative systems of singular bvps on time scales. Symmetry **15** (2023), no. 8, 1–9.

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