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Some comments on reverse derivations in rings

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ABSTRACT. In this note, we explore scenarios in which the concepts of generalized reverse derivation, multiplicative generalized reverse derivation, and multiplicative (generalized) reverse derivation lose significance. First, we extend a result of [Aboubakr A., González S., Generalized reverse derivations on semiprime rings. *Siberian Math. J.* **56** (2015), no. 2, 199–205]. Then, we demonstrate that in some previous studies, the assumptions imposed on these mappings to establish commutativity (or closely related) theorems are, in fact, unnecessary. Additionally, we provide insights into reverse $*$-derivations, where $*$ denotes the involution.

1. INTRODUCTION

Throughout this study, we denote by R an associative ring with center $Z(R)$. The commutator of any two elements $x, y \in R$ is defined as $[x, y] := xy - yx$. A ring R is called *prime* if for any $a, b \in R$, the condition $aRb = \{0\}$ implies $a = 0$ or $b = 0$. Similarly, R is termed *semiprime* if $aRa = \{0\}$ implies $a = 0$. Clearly, every prime ring is semiprime. A *derivation* is an additive mapping $d : R \to R$ that satisfies

$$d(xy) = d(x)y + xd(y), \quad \forall\, x, y \in R.$$

Moreover, d is called a *Jordan derivation* if it satisfies

$$d(x^2) = d(x)x + xd(x), \quad \forall\, x \in R.$$

The concept of a *reverse derivation* first appeared in the work of Herstein [6] while investigating Jordan derivations in prime rings; accordingly and additive mapping $d : R \to R$ that satisfies

$$d(yx) = d(x)y + xd(y), \quad \forall\, x, y \in R.$$

It is evident that every derivation and every reverse derivation is a Jordan derivation; however, the converse does not hold in general. Now let us recall from [5, Definition] that if d is a derivation of R, then an additive mapping $F : R \to R$ is called a *left generalized derivation* if

$$F(xy) = F(x)y + xd(y), \quad \forall\, x, y \in R,$$

and a *right generalized derivation* if

$$F(xy) = d(x)y + xF(y), \quad \forall\, x, y \in R.$$

Perhaps inspired by this, Aboubakr and González [1] extended the notion to reverse derivations. Specifically, given a reverse derivation d of R, an additive mapping $F : R \to R$ is called an *l-generalized reverse derivation* if

$$F(xy) = F(y)x + yd(x), \quad \forall\, x, y \in R,$$

and an *r-generalized reverse derivation* if

$$F(xy) = d(y)x + yF(x), \quad \forall\, x, y \in R.$$

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If we relax the assumption of additivity in [5, Definition], then the map is called *multiplicative l-generalized derivation* and *multiplicative r-generalized derivation*, respectively (cf. [8, Definition 2.11]). Following the same approach, one can similarly define *multiplicative l-generalized reverse derivation* and *multiplicative r-generalized reverse derivation*.

Moreover, let $d : R \to R$ be a mapping that is neither necessarily additive nor a derivation. A function $F : R \to R$ (not necessarily additive) is called a *multiplicative (generalized)-derivation* associated with d if it satisfies the functional equation

$$F(xy) = F(x)y + xd(y), \quad \text{for all } x, y \in R,$$

as introduced in [4]. Subsequent research has established that the mapping d must be a multiplicative derivation, provided that R is semiprime (cf. [3, Lemma 2]). In 2018, Tiwari et al. [11] introduced the notion of a multiplicative (generalized) reverse derivation associated with a mapping d. While the properties of d remain largely unexplored in this setting, the first author and Kumar provided insights into this matter for prime rings (see [10, Proposition 3.12]). Consequently, it appears more natural to study multiplicative (generalized) reverse derivations under the assumption that the associated mapping is a reverse derivation.

The primary objective of this study is to demonstrate that the concept of reverse derivation, along with its generalizations, holds limited mathematical significance, particularly in the framework of prime and semiprime rings. This is due to its intrinsic tendency to impose a commutative-like structure on these rings, rendering further investigation into its properties largely redundant.

2. Main result

Let I be a nonempty subset of the ring R. Then the set $C_R(I) := \{x \in R \mid xu = ux \; \forall \, u \in I\}$ is known as the centralizer of I in R. In [1, Theorem 3.1], Aboubakr and González proved that, if R is a semiprime ring, I is a nonzero ideal of R, then the following assertions are equivalent:

(i) $F : I \to R$ is an l-generalized reverse derivation (resp. r-generalized reverse derivation) with associated reverse derivation d;

(ii) $d(I) \subseteq C_R(I)$, $F(I) \subseteq C_R(I)$ and F is a r-generalized derivation (resp. l-generalized reverse derivation) with respect to d on I.

The authors assumed the additivity of F in their proof; however, we will now demonstrate that this assumption is unnecessary. Consequently, the results remain valid for multiplicative generalized reverse derivations.

Theorem 2.1. *Let R be a semiprime ring and I be a nonzero ideal of R. Then the following assertions are equivalent:*

(i) *$F : I \to R$ is a multiplicative l-generalized reverse derivation with associated reverse derivation d;*

(ii) *$d(I) \subseteq C_R(I)$, $F(I) \subseteq C_R(I)$ and F is a multiplicative r-generalized derivation with respect to d on I.*

Proof. (i)$\to$ (ii)
Let us assume that $F : I \to R$ be a multiplicative l-generalized reverse derivation. Therefore, we see that

$$F(x(yz)) = F(yz)x + yzd(x) = F(z)yx + zd(y)x + yzd(x) \; \forall \, x, y, z \in I. \tag{2.1}$$

Also,

$$F((xy)z) = F(z)xy + zd(xy) = F(z)xy + zd(y)x + zyd(x) \; \forall \, x, y, z \in I. \tag{2.2}$$

Now jointly considering (2.1) and (2.2), we get

$$F(z)[x,y]+[z,y]d(x)=0\ \forall\ x,y,z\in I. \tag{2.3}$$

In particular, we have $[z,y]d(y)=0$ for all $y,z\in I$. Since R is a semiprime ring, it contains a family of prime ideals, i.e., $\Omega=\{P_\lambda\ :\ \lambda\in\Lambda\}$ with $\cap_\lambda P_\lambda=\{0\}$. Let $P_\alpha\in\Omega$ be a typical member of the family. Thus, by our situation, we have $[z,y]d(y)\in P_\alpha$ for all $y,z\in I$. Replacing z by wz, we get $[w,y]Id(y)\subseteq P_\alpha$ for all $w,y\in I$. It implies that for each $y\in I$, either $[w,y]\in P_\alpha$ for all $w,z\in I$ or $d(y)\in P_\alpha$. Applying Brauer's trick (i.e., union of additive subgroups of I is not a subgroup), we obtain that either $[w,y]\in P_\alpha$ for all $w,y\in I$ or $d(I)\subseteq P_\alpha$. The latter case yields that $d(I)\subseteq\cap_\lambda P_\lambda=\{0\}$, hence $d(I)=\{0\}$.

Now on the other hand, let us suppose that $[w,y]\in P_\alpha$ for all $w,y\in I$. Taking $d(t)w$ for w in the last expression, we get $[d(t),y]w\in P_\alpha$ for all $t,y,w\in I$. It forces that $[d(t),y]I[d(t),y]\subseteq P_\alpha$ for all $t,y\in I$. It forces that $[d(t),y]\in P_\alpha$ for all $t,y\in I$. Since P_α is an arbitrary member of Ω, we get $[d(t),y]=0$ for all $t,y\in I$, and hence $d(I)\subseteq C_R(I)$. Therefore, together both case imply $d(I)\subseteq C_R(I)$. In this view, let us replace y by yu in Eq. (2.3), we get

$$F(z)[x,y]u+F(z)y[x,u]+[z,y]d(x)u+y[z,u]d(x)=0\ \forall\ x,y,z,u\in I. \tag{2.4}$$

Using (2.3), we arrive at

$$[F(z),y][x,u]=0\ \forall\ x,y,z,u\in I. \tag{2.5}$$

Putting $xF(z)$ in place of x in (2.5), we see that $(I[F(z),y])^2=\{0\}$ for all $z,y\in I$. It forces $I[F(z),y]=\{0\}$ for all $z,y\in I$. By Herstein's Lemma, we get $[F(I),I]=\{0\}$. This means that $F(I)\subseteq C_R(I)$. Thus F is a multiplicative r-generalized derivation of R with associated derivation d.

(ii)→ (i) This part is trivial. □

Following the same terminology, we can observe the following result:

Theorem 2.2. *Let R be a semiprime ring and I be a nonzero ideal of R. Then the following assertions are equivalent:*

(i) *$F:I\to R$ is a multiplicative r-generalized reverse derivation with associated reverse derivation d;*

(ii) *$d(I)\subseteq C_R(I)$, $F(I)\subseteq C_R(I)$ and F is a multiplicative l-generalized derivation with respect to d on I.*

Corollary 2.1. *Let R be a semiprime ring. If $F:R\to R$ is a multiplicative l-generalized reverse derivation with associated reverse derivation d of R. Then $d(R)\subseteq Z(R)$ and $F(R)\subseteq Z(R)$. Moreover, R contains a nonzero central ideal.*

Corollary 2.2. *Let R be a prime ring. If $F:R\to R$ is a multiplicative l-generalized reverse derivation with associated reverse derivation d of R. Then either $F=0$ or R is commutative.*

The following example shows that Corollary 2.2 can not hold true for arbitrary rings.

Example 2.1. *Let $\mathbb{F}$ be a field, and let e_{ij} denote the standard matrix units. Consider the ring*

$$R=\{x(e_{12}+e_{34})+y(e_{13}-e_{24})+z(e_{14})\mid x,y,z\in\mathbb{F}\}.$$

It is straightforward to verify that R is not a prime ring. Define mappings $F,d:R\to R$ by

$$F(a)=x(e_{12}+e_{34})+y(-e_{13}+e_{24})+zy(e_{14}),$$

$$d(a)=x(e_{12}+e_{34})+y(-e_{13}+e_{24})+0(e_{14}),$$

for all $a\in R$. Clearly, F is a nonzero multiplicative l-generalized reverse derivation associated with the reverse derivation d. However, R is not commutative, illustrating that the existence of such a mapping does not necessarily imply commutativity.

In 2019, the author and Kumar [10] demonstrated that if a prime ring R admits a multiplicative generalized reverse derivation (i.e., a multiplicative l-generalized reverse derivation) F, associated with a reverse derivation d, and satisfies the following annihilator conditions on a nonzero ideal of R: (i) $a(F(xy) \pm xy) = 0$, (ii) $a(F(x)F(y) \pm xy) = 0$, (iii) $a(F(xy) \pm F(y)F(x)) = 0$, (iv) $a(F(x)F(y) \pm yx) = 0$, (v) $a(F(xy) \pm yx) = 0$, then R must be commutative.

Moreover, in the same year, Huang [7] established certain central-valued conditions involving generalized reverse derivations (i.e., l-generalized reverse derivations) that also lead to the commutativity of R.

It is important to observe that Corollary 2.2 renders these studies redundant, as the commutativity of R follows trivially.

Theorem 2.3. *Let R be a semiprime ring and I a nonzero ideal of R. Suppose R admits a multiplicative (generalized) reverse derivation $F : R \to R$ with an associated mapping $d : R \to R$. Then, for all $x \in I$, the commutator identity*

$$[d(x), x] = 0$$

holds.

Proof. Let F be a multiplicative (generalized) reverse derivation of R associated with a mapping d. Then, for all $x, y, z \in I$, we have

$$F(x(yz)) = F(yz)x + yzd(x) = F(z)yx + zd(y)x + yzd(x). \tag{2.6}$$

Also,

$$F((xy)z) = F(z)xy + zd(xy). \tag{2.7}$$

From equations (2.6) and (2.7), it follows that

$$0 = F(z)[x, y] + z\big(d(xy) - d(y)x - yd(x)\big) + [z, y]d(x),$$

for all $x, y, z \in I$.

Writing z by uz in the above equation, it gives

$$0 = (F(z)u - uF(z))[x, y] + zd(u)[x, y] + [u, y]zd(x), \tag{2.8}$$

for all $x, y, z, u \in I$.

Replacing y by yx in (2.8) and simplifying, we obtain

$$[u, y][x, zd(x)] + y[u, x]zd(x) = 0, \tag{2.9}$$

for all $x, y, z, u \in I$.

Now, substituting y with $zd(x)y$ in (2.9), we get

$$[u, zd(x)]y[x, zd(x)] = 0,$$

for all $x, y, z, u \in I$. In particular, this implies

$$[x, zd(x)]I[x, zd(x)] = \{0\},$$

for all $x, z \in I$, which leads to $[x, zd(x)] = 0$ for all $x, z \in I$.

Finally, replacing z with $d(x)z$ in the last equation, we obtain

$$[x, d(x)]zd(x) = 0,$$

for all $x, z \in I$. This directly implies that

$$[d(x), x] = 0,$$

for all $x \in I$, completing the proof. □

Remark 2.1. *The preceding theorem yields a direct conclusion that stands independent of the conditions imposed in Theorem 2.5, Corollary 2.6, Theorem 2.7, and Theorems 2.10-2.13 of* [11]. *Consequently, the identities established in these results by the authors are rendered superfluous.*

Recall that an anti-automorphism of R of order 2 is called an *involution*. An additive mapping $\delta : R \to R$ is called a *reverse $*$-derivation* [2] if it satisfies the condition

$$\delta(xy) = \delta(y)x^* + y\delta(x), \quad \text{for all } x, y \in R.$$

This definition can be seen as a modification of standard derivations, adjusting the order in which elements are processed.

Moreover, it is important to note that if d is a derivation and δ is a reverse derivation, then for any fixed $n \in 2\mathbb{Z}^+$, we have

$$d^n(x) = \delta^n(x), \quad \text{for all } x \in R.$$

However, such a direct connection does not necessarily hold between $*$-derivations and reverse $*$-derivations.

A result due to Samman [9] states that a mapping d on a semiprime ring R is a reverse derivation if and only if it is a derivation that maps R into its center $Z(R)$. This naturally raises the question: *What can we say about reverse $*$-derivations in this context?*

To gain intuition, consider the field of complex numbers $\mathbb{C}$, which is a semiprime ring. The involution $*$ in this case is defined as the standard complex conjugation on $\mathbb{C}$. Now, define a mapping $\delta : \mathbb{C} \to \mathbb{C}$ by

$$\delta(a + ib) = b, \quad \text{for } a, b \in \mathbb{R}.$$

We observe that δ is a reverse $*$-derivation and satisfies $\delta(\mathbb{C}) \subseteq Z(\mathbb{C})$. This observation motivates the following result.

Theorem 2.4. *Let R be a semiprime ring with involution $*$. If $\delta : R \to R$ is a reverse $*$-derivation (not necessarily additive) of R, then δ maps R into the center $Z(R)$ of R.*

Proof. For any $x, y, z \in R$, using the associativity of R, we have

$$\delta((xy)z) = \delta(z)y^*x^* + z\delta(y)x^* + zy\delta(x),$$
$$\delta(x(yz)) = \delta(z)y^*x^* + z\delta(y)x^* + yz\delta(x).$$

Comparing both expressions, we obtain

$$[y, z]\delta(x) = 0 \quad \text{for all } x, y, z \in R.$$

Replacing y by ry, we get

$$[r, z]y\delta(x) = 0 \quad \forall\, x, y, z, r \in R.$$

This implies that

$$\delta(x)[r, z]R\delta(x)[r, z] = \{0\} \quad \text{for all } x, r, z \in R.$$

Consequently, we deduce that

$$\delta(x)[r, z] = 0 \quad \text{for all } x, r, z \in R.$$

In view of [9, Lemma], we conclude that $\delta(x) \in Z(R)$ for all $x \in R$. □

Corollary 2.3. *Let R be a prime ring with involution $*$. If $\delta : R \to R$ is a reverse $*$-derivation (not necessarily additive) of R, then either $\delta = 0$ or R is commutative.*

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