A study on Sasakian manifolds admitting $*$-η-Ricci-Bourguignon solitons

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ABSTRACT. The aim of the the present paper is to investigate some curvature properties of Sasakian manifold with respect to $*$-η-Ricci-Bourguignon soliton. We prove that a Sasakian manifold in view of $*$-η-Ricci-Bourguignon soliton has a cyclic Ricci tensor. We also study some curvature identities like $R(\xi, L).Ric_g = 0$, $Ric_g(\xi, M).R = 0$, $\bar{H}(\xi, N).Ric_g = 0$, where $\bar{H}$ is Pseudo-projective curvature tensor. An example is illustrated to support the result of the paper.

1. INTRODUCTION

In 1981, Jean-Pierre Bourguignon[4] introduced and studied Ricci-Bourguignon following the work of Aubin[1]. The Ricci-Bourguignon (RB) flow is defined as follows:

Definition 1.1. *Let (M^n, g) be a Riemannian manifold and $h(t)$ be a group of metrics on M^n. Then $h(t)$ is called a evolve of the Ricci-Bourguignon flow [10] if the following relation is true for $h(t)$,*

$$(1.1) \qquad \frac{\delta h}{\delta t} = -2(Ric_g - crh),$$

with Ric stands for Ricci tensor under the metric g, r stands for scalar curvature and c in $\mathbb{R}$ stands for an invariant.

The Ricci-Bourguignon flow will be converted to Ricci flow when c will be equal to zero in the relation (1.1). For different values of c we get the following tensors:
For $c = \frac{1}{2}$, the tensor will become Einstein, for $c = \frac{1}{n}$ the tensor will become traceless Ricci and for $c = \frac{1}{2(n-1)}$ it will reduce to Schouten. The clear concept of RB flow is also given [2] in a space which is termed as Anti-de Sitter along a sphere. Shubham Dwivedi [10] classified the Ricci-Bourguignon soliton in the following ways:

Definition 1.2. *The RB-soliton of a Riemannian manifold M of dimension $(2n + 1)$ is defined in the following way:*

$$(1.2) \qquad \mathcal{L}_F g + 2Ric_g = 2(\vartheta r - \Omega)g,$$

where $\mathcal{L}$ represents the Lie derivative, F is a vector field on M and the real Ω is an invariant.

A. Ghosh et. al. [11] developed the concept of $*$-Ricci soliton on Sasakian manifolds and (κ, μ)-contact manifolds. The $*$- Ricci tensor on a manifold M of dimension $(2n + 1)$ is given by the following relation:

$$2Ric_g^*(X, Y) = Tr_g Z \longrightarrow R(X, \phi Y)\phi Z,$$

where the vector fields $X, Y, Z \in \chi(M)$. Furthermore, the relation $g(Q^*X, Y) = Ric_g^*(X, Y)$, holds for $X, Y \in \chi(M)$ for $* - Ricci$ operator Q^* which covers the $*$-scalar curvature r^*. The manifold M is considered as a $*$-Ricci flat when the $*$-Ricci tensor disappears in the same way. Now the manifold M becomes $*$-η-Einstein when $Ric_g^*(E, F) = \Omega g(E, F) + \psi \eta(E)\eta(F)$ holds, $\forall E, F \in \chi(M)$. Also, the manifold M turns into $*$-Einstein if $Ric_g^*(E, F) = \Omega g(E, F)$, i.e., if $\psi = 0, \forall E, F \in \chi(M)$.

Definition 1.3. *The $*$-η-RB soliton of a Riemannian manifold M of dimension $(2n + 1)$, $n \geq 1$ is defined by the existence of a vector field F in M with the following condition:*

$$(1.3) \qquad \frac{1}{2}\mathcal{L}_F g + Ric_g^* + \psi \eta \otimes \eta = [\vartheta r^* - \Omega]g,$$

where the vector field $F \in \chi(M)$.

We mention the projective curvature tensor H [19] as well as the Riemannian-Christoffel curvature tensor R in the following manner:

$$(1.4) \qquad H(U, V)W = -\frac{1}{n-1}[g(V, W)QU - g(U, W)QV] + R(U, V)W,$$

$$(1.5) \qquad R(U, V)W + \nabla_{[U,V]}W = \nabla_U \nabla_V W - \nabla_V \nabla_U W,$$

where the Ricci operator Q is given by $Ric_g(U, V) = g(QU, V), \forall U, V, W \in \chi(M)$. Also the author in the paper [15] studied the Pseudo-projective curvature tensor. If we consider Pseudo-projective curvature tensor by $\bar{H}$, then we can write

$$(1.6) \qquad \begin{aligned} \bar{H}(D, E)F &+ \frac{r}{n}(\frac{\alpha}{n-1} + \beta)[g(E, F)D - Eg(D, F)] \\ &= \alpha R(D, E)F + \beta[Ric_g(E, F)D - VRic_g(D, F)], \end{aligned}$$

where $D, E, F \in \chi(M)$ and the scalar curvature r is given by $r = tr(Ric_g)Ric_g$ and $\alpha, \beta \neq 0$ are constants.

The $*$-Ricci soliton and its generalization have been studied by many authors for metric manifolds of contact and paracontact structures (see [6], [7], [8], [9], [13], [14]). The authors in [5], proved the trivial condition of Schouten, traceless or Einstein Ricci soliton when they consider the RB solitons. More recent research on this soliton is available in [12], [17], [18]. The present paper is arranged as follows.

After introduction in section 1, we give some preliminaries in section 2. Section 3 deals with Sasakian manifold satisfyiing $*$- η-Ricci-Bourguignon soliton. In sections 4, 5 and 6 we investigate some curvature properties of Sasakian manifold satisfying $R(\xi, U).Ric_g = 0$, $Ric_g(\xi, U).R = 0$, $\bar{H}(\xi, U).Ric_g = 0$, respectively for the Pseudo-projective curvature tensor $\bar{H}$. In section 7 we give an example of a Sasakian manifold to support our results.

2. PRELIMINARIES

If a $(1, 1)$ type tensor field ϕ, a vector field ξ, an 1-form η and the Riemannian metric g are admitted by the smooth manifold M^{2n+1}, then we can write[3]:

$$(2.7) \qquad \phi^2(E) = -E + \eta(E)\xi, \phi\xi = 0, \eta(\xi) = 1, \eta \circ \phi = 0,$$

$$(2.8) \qquad g(\phi E, \phi F) = g(E, F) - \eta(E)\eta(F),$$

$$(2.9) \qquad g(\phi E, F) = -g(E, \phi F),$$

$$(2.10) \qquad \eta(E) = g(E, \xi),$$

$\forall$ vector fields $E, F \in \chi(M)$.

An almost contact manifold M of dimension $(2n+1)$ is a contact metric with the structure (ϕ, ξ, η, g) if [3] $d\eta(E,F) + g(\phi E, F) = 0$. Furthermore, the normality condition for an almost contact structure of M holds if

$$(2.11) \qquad [\phi, \phi](E,F) = -2d\eta(E,F)\xi,$$

where Nijenhuis tensor is represented by $[\phi, \phi]$ [3].

A Sasakian manifold is defined as a normal contact metric manifold.

The following relations are true in a Sasakian manifold of dimension $(2n+1)$[16]:

$$(2.12) \qquad \nabla_E \xi = -\phi E,$$

$$(2.13) \qquad R(E,F)\xi = \eta(F)E - \eta(E)F,$$

$$(2.14) \qquad R(\xi, E)F = \xi g(E,F) - E\eta(F),$$

$$(2.15) \qquad (\nabla_E \eta)F = g(E, \phi F),$$

where $E, F \in \chi(M)$, the Levi-Civita connection for g is denoted by ∇ and the Riemannian curvature tensor is denoted by R.

3. Sasakian manifold admitting $*$ - η-Ricci Bourguignon soliton

Theorem 3.1. *If M be an odd-dimensional (> 1) Sasakian manifold and g heeds $*$-η-Ricci-Bourguignon soliton, then the manifold has cyclic Ricci tensor.*

Proof. We assume that M is an odd-dimensional Sasakian manifold with $*$-η-Ricci-Bourguignon soliton. Then we can write :

$$(3.16) \qquad (\mathcal{L}_\xi g)(U,V) + 2Ric_g^*(U,V) + 2(\Omega - \vartheta r^*)g(U,V) + 2\psi\eta(U)\eta(V) = 0,$$

$\forall\, U, V \in \chi(M)$.

In Sasakian manifold, we have

$$(3.17) \qquad (\mathcal{L}_\xi g)(U,V) = 0,$$

where $\mathcal{L}$ stands for Lie derivative.

Equation (3.17) reduces to

$$(3.18) \qquad Ric_g^*(U,V) + (\Omega - \vartheta r^*)g(U,V) + \psi\eta(U)\eta(V) = 0.$$

With the help of the Lemma (5.3) (see page 284 in ([19]) and using definition of $*$-Ricci tensor, we get

$$(3.19) \qquad Ric_g^*(U,V) = Ric_g(U,V) - (2n-1)g(U,V) - \eta(U)\eta(V) = 0,$$

$\forall$ vector fields $U, V \in \chi(M)$.

Replacing (3.19) in (3.18) we obtain

$$(3.20) \qquad Ric_g(U,V) = [2n - 1 - \Omega + \vartheta r^*]g(U,V) - (\psi - 1)\eta(U)\eta(V)$$

Putting $V = \xi$ in equation (3.20), we have

$$(3.21) \qquad Ric_g(U,\xi) = [2n - \Omega + \vartheta r^* - \psi]\eta(U).$$

Again for an odd-dimensional Sasakian manifold with Ricci tensor field we have

$$(3.22) \qquad Ric_g(U,\xi) = 2n\eta(U).$$

Then from (3.21) and (3.22) we have

$$(3.23) \qquad \psi = \vartheta r^* - \Omega.$$

We know that

(3.24) $$(\nabla_U Ric_g)(V, W) = U Ric_g(V, W) - Ric_g(\nabla_U V, W) - Ric_g(V, \nabla_U W),$$

$\forall U, V, W \in \chi(M)$.
Using (3.20) and (2.15) we get

(3.25) $$(\nabla_U Ric_g)(V, W) = (\psi - 1)[\eta(V)g(\phi U, W) + \eta(W)g(\phi U, V)].$$

Similarly, we get

(3.26) $$(\nabla_V Ric_g)(W, U) = (\psi - 1)[g(\phi V, U)\eta(W) + g(\phi V, W)\eta(U)].$$

and

(3.27) $$(\nabla_W Ric_g)(U, V) = (\psi - 1)[g(\phi W, V)\eta(U) + g(\phi W, U)\eta(V)].$$

Then combining (3.25), (3.26) and (3.27) and using (2.9) we get

(3.28) $$(\nabla_U Ric_g)(V, W) + (\nabla_V Ric_g)(W, U) + (\nabla_W Ric_g)(U, V) = 0,$$

$\forall U, V, W \in \chi(M)$.
This gives the end of the proof. $\qquad\square$

Theorem 3.2. *Let M be an odd-dimensional Sasakian manifold. If g heeds $*$-η-Ricci-Bourguignon soliton, then $\psi = 1$ and $*$-scalar curvature,$r^* = \frac{\Omega+1}{\vartheta}$.*

Proof. We suppose that $\nabla Ric_g = 0$. Then putting $W = \xi$ in (3.25) we get

(3.29) $$g(\phi U, V)(\psi - 1) = 0,$$

$\forall U, V \in \chi(M)$.

The above relation leads to $\psi = 1$. Then from (3.23) we get $r^* = \frac{\Omega+1}{\vartheta}$.
Thus we have the proof. $\qquad\square$

4. SASAKIAN MANIFOLD ADMITTING $*$- η-RICCI-BOURGUIGNON SOLITON WITH $R(\xi, U).Ric_g = 0$

Theorem 4.3. *Let M be an odd-dimensional Sasakian manifold with $R(\xi, U).Ric_g = 0$. If $*$-η-Ricci-Bourguignon soliton is heeded by the metric g, then $\psi = 1$,$*$-scalar curvature $r^* = \frac{\Omega+1}{\vartheta}$.*

Proof. Let us supposed that, $R(\xi, U).Ric_g = 0$. This implies that

(4.30) $$Ric_g(R(\xi, U)V, W) + Ric_g(V, R(\xi, U)W) = 0,$$

$\forall U, V, W \in \chi(M)$.
Using (3.20) we get from (4.30)

(4.31) $$(\psi - 1)[\eta(W)g(U, V) + \eta(V)g(U, W) - 2\eta(U)\eta(V)\eta(W)] = 0.$$

Putting $W = \xi$ in the above equation we get

(4.32) $$(\psi - 1)g(\phi U, \phi V) = 0.$$

This gives

(4.33) $$\psi = 1.$$

Now from (3.23) we get

(4.34) $$r^* = \frac{\Omega + 1}{\vartheta}.$$

Hence the theorem is proved. $\qquad\square$

In view of (4.33) and (4.34) we get from (3.20)

$$(4.35) \qquad Ric_g(U,V) = (2n-1)g(U,V).$$

Now we arise to give a clarification by the following statement :

Corollary 4.1. *An odd-dimensional Sasakian manifold M heeding $*$-η-Ricci-Bourguignon soliton with $R(\xi,U).Ric_g = 0$ is Einstein manifold.*

5. Sasakian manifold admitting $*$-η-Ricci-Bourguignon soliton with $Ric_g(\xi,U).R = 0$

Theorem 5.4. *Let M be an odd-dimensional Sasakian manifold with $Ric_g(\xi,U).R = 0$. If $*$-η-Ricci-Bourguignon soliton is heeded by the metric g, then $\psi = 1 - 4n$ and $r^* = \frac{\Omega-4n+1}{\vartheta}$.*

Proof. We suppose that, $Ric_g(\xi,U).R = 0$. Then this implies that

$$(5.36) \qquad \begin{aligned} &Ric_g(U, R(V,W)X)\xi - Ric_g(\xi, R(V,W)X)U + Ric_g(U,V)R(\xi,W)X \\ - \; &Ric_g(\xi,V)R(U,W)X + Ric_g(U,W)R(V,\xi)X - Ric_g(\xi,W)R(V,U)X \\ + \; &Ric_g(U,X)R(V,W)\xi - Ric_g(\xi,X)R(V,W)U = 0, \end{aligned}$$

$\forall\, U,V,W,X \in \chi(M).$

Taking inner product on both-side of (5.36) by ξ we get

$$(5.37)$$
$$\begin{aligned} &Ric_g(U, R(V,W)X) - \eta(U)Ric_g(\xi, R(V,W)X) + \eta(R(\xi,W)X)Ric_g(U,V) \\ - \; &\eta(R(U,W)X)Ric_g(\xi,V) + \eta(R(V,\xi)X)Ric_g(U,W) - \eta(R(V,U)X)Ric_g(\xi,W)) \\ + \; &\eta(R(V,W)\xi)Ric_g(U,X) - \eta(R(V,W)U)Ric_g(\xi,X) = 0. \end{aligned}$$

Using (3.20) and putting $W = X = \xi$, we obtain from (5.37)

$$(5.38) \qquad [4n - 1 - 2\Omega + 2\vartheta r^* - \psi]g(\phi U, \phi V) = 0.$$

Hence the equation (5.38) gives

$$(5.39) \qquad 4n - 1 - 2\Omega + 2\vartheta r^* - \psi = 0.$$

In view of (3.23) we obtain from (5.39)

$$(5.40) \qquad r^* = \frac{\Omega - 4n + 1}{\vartheta}.$$

Again from (3.23) we obtain

$$(5.41) \qquad \psi = 1 - 4n.$$

This complete the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Furthermore, we have from (5.39), (5.41) and (3.20)

$$(5.42) \qquad Ric_g(U,V) = -2ng(U,V) + 4n\eta(U)\eta(V)$$

Hence we arrive to give a clarification by the following statement:

Corollary 5.2. *An odd-dimensional Sasakian manifold M heeding $*$-η-Ricci-Bourguignon soliton with $Ric_g(\xi,U).R = 0$ is η-Einstein manifold.*

6. SASAKIAN MANIFOLD ADMITTING $*$-η-RICCI-BOURGUIGNON SOLITON WITH $\bar{H}(\xi, U).Ric_g = 0$

Theorem 6.5. *Let M be an odd-dimensional Sasakian manifold with $\bar{H}(\xi, U).Ric_g = 0$. If $*$-η-Ricci-Bourguignon soliton is heeded by the metric g, so either $\psi = 1, r^* = \frac{\Omega+1}{\vartheta}$ or $r = \frac{2n(2n+1)\alpha}{\alpha+2n\beta}$, where $\alpha, \beta \neq 0$ are constants.*

Proof. We consider that $\bar{H}(\xi, U).Ric_g = 0$. Then we have the following:

$$(6.43) \qquad Ric_g(\bar{H}(\xi, U)V, W) + Ric_g(V, \bar{H}(\xi, U)W) = 0,$$

where $U, V, W \in \chi(M)$ and $\bar{H}$ is the Pseudo-projective curvature tensor in M.

Now from (1.6) we can write

$$(6.44) \qquad \begin{aligned} \bar{H}(\xi, U)V &= \alpha R(\xi, U)V + \beta\{Ric_g(U, V)\xi - Ric_g(\xi, V)U\} \\ &- \tfrac{r}{2n+1}(\tfrac{\alpha}{2n} + \beta)\{g(U, V)\xi - g(\xi, V)U\}, \end{aligned}$$

where $\alpha, \beta \neq 0$ are constants.
In view of (2.14), (3.20) and (3.21) we obtain from (6.44)

$$(6.45) \qquad \begin{aligned} \bar{H}(\xi, U)V &= \{\alpha - \tfrac{r}{2n+1}(\tfrac{\alpha}{2n} + \beta)\}\{g(U, V)\xi - \eta(V)U\} \\ &+ \beta\{2n - 1 - \Omega + \vartheta r^*\}\{g(U, V)\xi - \eta(V)U\} \\ &+ \beta(\psi - 1)\{\eta(V)U - \xi\eta(U)\eta(V)\}. \end{aligned}$$

Using (3.20) in the equation (6.45) we obtain

$$(6.46) \qquad \begin{aligned} &Ric_g(\bar{H}(\xi, U)V, W) \\ =\ & \{2n - 1 - \Omega + \vartheta r^*\}\{\alpha - \tfrac{r}{2n+1}(\tfrac{\alpha}{2n} + \beta)\}\{\eta(W)g(U, V) - g(U, W)\eta(V)\} \\ +\ & \beta[2n - 1 - \Omega + \vartheta r^*]^2\{g(U, V)\eta(W) - g(U, W)\eta(V)\} \\ -\ & \beta(\psi - 1)\{2n - 1 - \Omega + \vartheta r^*\}[g(U, V)\eta(W) - g(U, W)\eta(V)] \\ -\ & (\psi - 1)\{\alpha - \tfrac{r}{2n+1}(\tfrac{\alpha}{2n} + \beta)\}\{\eta(W)g(U, V) - \eta(U)\eta(V)\eta(W)\}. \end{aligned}$$

By interchanging V and W in equation (6.46) we obtain

$$(6.47) \qquad \begin{aligned} &Ric_g(\bar{H}(\xi, U)W, V) \\ =\ & \{2n - 1 - \Omega + \vartheta r^*\}\{\alpha - \tfrac{r}{2n+1}(\tfrac{\alpha}{2n} + \beta)\}\{\eta(V)g(U, W) - g(U, V)\eta(W)\} \\ +\ & \beta\{2n - 1 - \Omega + \vartheta r^*\}^2\{\eta(W)g(U, V) - g(U, V)\eta(W)\} \\ -\ & \beta(\psi - 1)\{2n - 1 - \Omega + \vartheta r^*\}\{\eta(V)g(U, W) - g(U, V)\eta(W)\} \\ -\ & (\psi - 1)\{\alpha - \tfrac{r}{2n+1}(\tfrac{\alpha}{2n} + \beta)\}\{\eta(V)g(U, W) - \eta(U)\eta(W)\eta(V)\}. \end{aligned}$$

Using (6.46) and (6.47) in (6.43) we have
(6.48)
$$\{(\psi - 1)\alpha - \tfrac{r}{2n+1}(\tfrac{\alpha}{2n} + \beta)\}\{\eta(W)g(U, V) + g(U, W)\eta(V) - 2\eta(U)\eta(V)\eta(W)\} = 0.$$

Now putting $W = \xi$ in (6.48) we get

$$(6.49) \qquad (\psi - 1)\{\alpha - \frac{r}{2n+1}(\frac{\alpha}{2n} + \beta)\}g(\phi U, \phi V) = 0.$$

So from (6.49) we have either, $\psi = 1$ or $r = \frac{2n(2n+1)\alpha}{\alpha+2n\beta}$.
Now if $\psi = 1$, then from (3.23) we obtain

$$(6.50) \qquad\qquad\qquad\qquad r^* = \frac{\Omega + 1}{\vartheta}.$$

This complete the proof. $\square$

Now if $\psi = 1$ and $r^* = \frac{\Omega+1}{\vartheta}$, then from (3.20) we get

$$(6.51) \qquad Ric_g(U, V) = (2n - 1)g(U, V).$$

Hence we arrive to give a clarification by the following statement:

Corollary 6.3. *An odd-dimensional Sasakian manifold M admitting $*$-η-Ricci-Bourguignon soliton with $\bar{H}(\xi, U).Ric_g = 0$ is Einstein manifold when $r \neq \frac{2(2n+1)n\alpha}{\alpha + 2n\beta}$, where $\alpha, \beta \neq 0$ are constants.*

7. EXAMPLE

Let us first give an example of a five dimensional Sasakian manifold. Let us consider the five dimensional manifold $M = \{(u, v, w, p, q) \in \mathbb{R}^5 : q \neq 0\}$, where (u, v, w, p, q) are the standard co-ordinates in $\mathbb{R}^5$.

We consider linearly independent vector fields on M given by

$$f_1 = 2(q\frac{\partial}{\partial w} - \frac{\partial}{\partial u}), \quad f_2 = 2\frac{\partial}{\partial q}, \quad f_3 = -2\frac{\partial}{\partial w}, \quad f_4 = 2(v\frac{\partial}{\partial w} - \frac{\partial}{\partial p}), \quad f_5 = -2\frac{\partial}{\partial v}.$$

Now metric g is given by

$$g(f_s, f_t) = 0, \quad s \neq t; s, t = 1, 2, 3, 4, 5.$$

Also we have

$$g(f_1, f_1) = g(f_2, f_2) = g(f_3, f_3) = g(f_4, f_4) = g(f_5, f_5) = 1.$$

Now 1-form η is written with $\eta(U) = g(U, f_3)$, $\forall$ vector field $U \in M$. We consider the $(1, 1)$-tensor field ϕ in the following way:

$$\phi f_1 = f_2, \quad \phi f_2 = f_1, \quad \phi f_3 = 0, \quad \phi f_4 = f_5, \quad \phi f_5 = -f_4.$$

Now we can write

$$\eta(f_3) = 1, \quad \phi^2(W) = -W + \eta(W)f_3, \quad g(\phi W, \phi X) = g(W, X) - \eta(W)\eta(X)$$

where W, X are the vector fields on M.

So for $f_3 = \xi$, the structure (ϕ, ξ, η, g) forms a Sasakian formation on M.

We consider the Levi-Civita connection ∇. So we can proceed as follows:

$$[f_1, f_2] = 2f_3, \quad [f_4, f_5] = 2f_3.$$

For the remaining cases $[f_s, f_t] = 0$. The Koszul formula is given by

$$2g(\nabla_U V, W) = Ug(V, W) + Vg(W, U) - Wg(U, V) - g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]),$$

where $U, V, W \in \chi(M)$.

By Koszul's formula, we obtain the following:

$$\nabla_{f_3} f_1 = 0, \quad \nabla_{f_1} f_2 = f_3, \quad \nabla_{f_1} f_3 = f_2, \quad \nabla_{f_1} f_4 = 0, \quad \nabla_{f_1} f_5 = 0,$$

$$\nabla_{f_1} f_1 = 0, \quad \nabla_{f_1} f_2 = f_3, \quad \nabla_{f_1} f_3 = f_2, \quad \nabla_{f_1} f_4 = 0, \quad \nabla_{f_1} f_5 = 0,$$

$$\nabla_{f_2} f_1 = f_3, \quad \nabla_{f_2} f_2 = 0, \quad \nabla_{f_2} f_3 = f_1, \quad \nabla_{f_2} f_1 = 0, \quad \nabla_{f_2} f_1 = 0,$$

$$\nabla_{f_3} f_1 = f_2, \quad \nabla_{f_3} f_2 = f_1, \quad \nabla_{f_3} f_3 = 0, \quad \nabla_{f_3} f_4 = 0, \quad \nabla_{f_3} f_5 = f_4,$$

$$\nabla_{f_4} f_1 = 0, \quad \nabla_{f_4} f_2 = 0, \quad \nabla_{f_4} f_3 = f_5, \quad \nabla_{f_4} f_4 = 0, \quad \nabla_{f_4} f_5 = f_3,$$

$$\nabla_{f_5} f_1 = \nabla_{f_5} f_2 = \nabla_{f_5} f_3 = \nabla_{f_5} f_4 = \nabla_{f_5} f_5 = 0.$$

So this implies that the structure (M, ϕ, ξ, η, g) forms a Sasakian manifold. Obviously, the results can be easily established.

8. CONCLUSION

The article deals with the study of Sasakian manifold with $*$-η-Ricci-Bourguignon soliton. Also we apply some geometrical identities on Sasakian manifold. The outcomes of the work play a significant role to determine the scalar curvature and $*$-scalar curvature in geometrical analysis. Also few achievements of research in this way exist in others framework like quasi-Sasakian manifold etc. Working on this research we have noticed that if the relation between Ricci tensor and $*$-Ricci tensor is established on different type of manifolds, then many significant results may be disclosed in near future.

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