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Existence, Uniqueness and Stability Analysis of Solutions for a ψ-Caputo Fractional Spatial Heterogeneous Viral Infection Model

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ABSTRACT. This paper investigates a ψ-Caputo fractional-order spatial heterogeneous viral infection model. Utilizing fixed point theorems, specifically Banach's contraction principle, we establish the existence and uniqueness of mild solutions for the proposed model. Furthermore, we analyze the stability properties of its solutions, proving that they are Ulam-Hyers (UH) and Ulam-Hyers-Rassias (UHR) stable under appropriate conditions. The theoretical results are supported by numerical examples that demonstrate the practical applicability of the stability criteria. This work extends the analysis of fractional differential equations by incorporating a generalized fractional derivative with respect to a function ψ providing a framework for studying complex spatial dynamical systems in virology.

1. INTRODUCTION

The present paper is concerned with the analysis of an abstract fractional spatial heterogeneous viral infection model. The existence of solutions and their stability in the sense of Ulam–Hyers (UH) and Ulam–Hyers–Rassias (UHR) type criteria are the primary focus of this investigation. Fractional calculus serves as a fundamental tool in this analysis, extending classical concepts of differentiation and integration to non-integer orders. The application of fractional operators has proven to be a powerful modeling framework across diverse scientific disciplines, including physics, mechanics, signal and image processing, chemistry, and biology [21, 23, 25, 32].

Fixed point theorems constitute a cornerstone of nonlinear analysis, providing a powerful and versatile framework for investigating the qualitative properties of solutions to various classes of differential equations, including ordinary, partial, and, more recently, fractional differential equations [29]. The fundamental principle involves reformulating a differential equation as an operator equation within an appropriate function space. Demonstrating that this operator satisfies the conditions of a fixed point theorem—such as those established by Banach, Schaefer, or Krasnoselskii—allows one to conclude the existence, uniqueness, and stability of solutions.

The application of this methodology has led to significant advancements in the theory of abstract fractional differential equations over the past two decades [18, 19, 20, 24, 31, 35]. Research has progressively addressed increasingly complex models, particularly those incorporating nonlocal conditions, which account for the history of a system rather than just an initial point, and impulsive dynamics, which describe sudden, discontinuous changes in a system's state. The development of theories involving resolvent operators has been instrumental in handling these sophisticated abstract problems.

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A substantial body of literature is dedicated to establishing not only the existence but also the stability of solutions in the Ulam-Hyers (UH) and Ulam-Hyers-Rassias (UHR) senses. For instance, the work in [10] leverages fixed point techniques to prove the existence and UH-/UHR-stability of solutions for functional abstract fractional differential inclusions that feature noninstantaneous impulses. Similarly, the authors of [26] extend this analysis by examining stability and the stronger property of logarithmic decay for solutions to both nonlinear and linear fractional differential problems. In a related vein, Balachandran and Kiruthika [15] investigated the solvability of nonlocal fractional integro-differential equations of Sobolev type, a class of problems that arises in numerous physical applications. These efforts are part of a broader research trend, extensively documented in [11, 12, 13, 14, 16, 17, 22, 30, 33, 34, 36, 37], that successfully applies fixed point methods to establish UH- and UHR-stability for a wide spectrum of linear and nonlinear fractional differential equations.

Beyond purely theoretical explorations, the framework of fixed point theorems is increasingly being applied to complex models in mathematical biology. A prominent area of interest is the analysis of spatial heterogeneous viral infection models. Understanding the existence, boundedness, and stability of solutions to these systems is critical for predicting disease dynamics, and many mathematicians are now actively engaged in this pursuit, as evidenced by the growing body of work in [27, 28, 37, 38, 1]. Thus, the utility of fixed point theory continues to expand, proving essential for both deriving fundamental theoretical results and ensuring the robustness of applied mathematical models.

In [27], the authors examined and investigated the following viral infection model:

(1.1)
$$\begin{cases} \frac{\partial u_1(t,x)}{\partial t} = \omega_1(x) - \nu_1(x)\, u_1(t,x) - \alpha(x)\, u_1(t,x)\, u_3(t,x), \\ \frac{\partial u_2(t,x)}{\partial t} = -\nu_2(x)\, u_2(t,x) + \beta(x)\, u_1(t-\tau,x)\, u_3(t-\tau,x)\, e^{-\delta t}, \\ \frac{\partial u_3(t,x)}{\partial t} = -du_3(t,x) - \nu_3(x)\, u_3(t,x) + \omega_2(x)\, u_2(t,x) + d\int_\Sigma f(x-y) u_3(t,y)\, dy, \end{cases}$$

with $(t,x) \in \mathbb{R}^+ \times \Sigma$ and $u_1(t,x)$, $u_2(t,x)$, and $u_3(t,x)$ denoting the concentrations of target cells, infected cells, and free virions at time t and location $x \in \overline{\Sigma}$; $d > 0$, $\omega_j(x) > 0$ ($j = 1, 2$), $\nu_i(x) > 0$ ($i = 1, 2, 3$), while $\beta(x)$ in (1.1) is uniformly continuous in $\overline{\Sigma}$. For (1.1), define a linear operator $\mathcal{A}$ and nonlinear operator $\mathcal{H}$ such that

$$\mathcal{A}[u] = \begin{pmatrix} \mathcal{A}_1 u_1 \\ \mathcal{A}_2 u_2 \\ \mathcal{A}_3 u_3 \end{pmatrix} = \begin{pmatrix} -\nu_1(x)\, u_1(t,\cdot) \\ -\nu_2(x)\, u_2(t,\cdot) \\ -(d+\nu_3(x)) u_3(t,\cdot) \end{pmatrix}, \tag{1.2}$$

and

$$\mathcal{H}[u] = \mathcal{H}(x, \mathcal{I}_\Sigma(u(x)), u(x)) = \begin{pmatrix} \omega_1(x) - \beta(x)\, u_1(t,\cdot)\, u_3(t,\cdot) \\ \beta(x)\, u_1(t-\tau,\cdot)\, u_3(t-\tau,\cdot)\, e^{-\delta t} \\ \omega_2(x)\, u_2(t,\cdot) + d\int_\Sigma f(x-y) u_3(t,y)\, dy \end{pmatrix}, \tag{1.3}$$

where $\mathcal{I}_\Sigma$ is an integral over the subset Σ.

In [2], the authors examined and investigated the uniqueness and stability results of the solutions to the following abstract fractional spatial heterogeneous viral infection model:

$$\mathcal{D}^\zeta_{0,x} \widetilde{\mathcal{Z}}(x) = \mathcal{B}\left[\widetilde{\mathcal{Z}}(x)\right] + \mathcal{W}_\eta\left(x, \mathcal{T}^\sigma_{0,x} \widetilde{\mathcal{Z}}(x), \widetilde{\mathcal{Z}}(x)\right), \quad x \in [0,\theta], \tag{1.4}$$

$$\widetilde{\mathcal{Z}}(0) = \widetilde{\mathcal{Z}}_0, \tag{1.5}$$

where $(t,x) \in \mathbb{R}^+ \times \Pi$, $\mathcal{D}^{\zeta}_{0,t}$ denotes the Caputo–Fabrizio fractional derivative of order ζ $(0 < \zeta < 1)$, $\mathcal{T}^{\sigma}_{0,x}$ is the corresponding fractional integral of order σ $(0 < \sigma < 1)$, $\mathcal{B}$ is a closed linear unbounded operator defining the system's linear dynamics, and $\mathcal{W}_\eta$ is a nonlinear operator describing the viral infection mechanics with a sufficiently large parameter η.

Motivated by these works, this paper addresses the existence and uniqueness, as well as UH and UHR-stability, for the solutions of the abstract fractional spatial heterogeneous viral infection model:

$$\begin{cases} {}^{C}D^{p,\psi}_{0,t}u(t) = \mathcal{A}[u(t))] + \mathcal{H}_\eta(t, I^{\alpha,\psi}_{0,t}u(t), u(t)), \qquad t \in J = [0,T]. \\ u(0) = u_0. \end{cases} \tag{1.6}$$

Here ${}^{C}D^{p,\psi}_{0,t}$, and $I^{\alpha,\psi}_{0,t}$ represent the ψ-Caputo fractional derivative and ψ-Riemann-Liouville fractional integral

where $0 < p, \alpha < 1$, $\mathcal{A}$ is a closed linear unbounded operator and $\mathcal{H}_\eta \in C(J,\chi)$ is a continuous function with the parameter η large enough and $u_0 \in \chi$. The domains and ranges of the operators are subsets of a Banach space χ. The operator $\mathcal{A}$ is defined on $D(\mathcal{A})$, a domain endowed with the norm

$$\|u\|_{D(\mathcal{A})} = \|u\| + \|\mathcal{A}u\| \quad \text{and} \quad \|u\|_{C(J,E)} = \sup_{x\in J}\|u(x)\|$$

The tools employed in this article are consistent with those in [3, 4]. One of the primary differences of this study from the references above is that we use the Banach's fixed point theorem and we utilize an equivalent integral equation for (1.6) based on the Laplace transform. This facilitates the use of Mittage–Lefler function properties in order to reduce computations while deriving stability results. This method has been employed by [5] in order to analyze the stability of a fractional differential equation system. To the authors' knowledge, this method has not previously been employed to investigate stability via the fixed point method.

The paper is organized as follows. In Sect. 2, we introduce the basic concepts of fractional calculus and the theorems and lemmas needed to describe and examine equation (1.6). Also, we introduce the functions and spaces required for solutions of equation (1.6). In Sect. 3, by implementing the fixed point technique we investigate the stability of equation (1.6) and establish the existence of a integral equation. In Sect. 4, we consider two examples of equation (1.6), including some graphs. A conclusion section is given at the end.

2. Preliminaries

For ease, this section is devoted to providing an outline of few ideas, definitions and some fundamental outcomes from fractional calculus which are utilized throughout this article.

Let X be the Banach set of the all the continuous bound functions on $J = [0,T]$, and $C(J,X)$ is the Banach set of all continuous bound functions on J endowed with supremum norm:

$$\|x\| = \sup\{|x(t)|, t \in J\}.$$

Definition 2.1. [6] *Let $p > 0, u \in L^1([J,\mathbb{R})$ and $\psi \in C^n(J,\mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in J$. The ψ-Riemann-Liouville fractional integral at order p of the function u is given by*

$$I^{p,\psi}_{0^+}u(t) = \frac{1}{\Gamma(p)}\int_0^t (\psi(t) - \psi(s))^{p-1}\psi'(s)u(s)ds. \tag{2.7}$$

Remark 2.1. *Note that if $\psi(t) = t$ and $\psi(t) = \log(t)$, then the equation* (2.7) *is reduced to the Riemann-Liouville and Hadamard fractional integrals respectively.*

Definition 2.2. [6] *Allow $(n \in \mathbb{N}^*)$, $n-1 < p < n$. $u \in C^{n-1}(J,\mathbb{R})$ and $\psi \in C^n(J,\mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in J$. The ψ-Caputo fractional derivative at order p of the function u is given by*

$$(2.8) \qquad {}^C D_{0^+}^{p,\psi} u(t) = \frac{1}{\Gamma(n-p)} \int_0^t (\psi(t)-\psi(s))^{n-p-1} \psi'(s) u_\psi^{[n]}(s) ds.$$

where

$$u_\psi^{[n]}(s) = \left(\frac{1}{\psi'(s)} \frac{d}{ds} \right)^n u(s).$$

Remark 2.2. *In particular, note that if $\psi(t) = t$ and $\psi(t) = \log(t)$, then the equation* (2.8) *is reduced to the the Caputo fractional derivative and Caputo-Hadamard fractional derivative respectively.*

Remark 2.3. *If $p \in (0,1)$, then the equation* (2.8) *can be written as follows*

$${}^C D_{0^+}^{p,\psi} u(t) = \frac{1}{\Gamma(1-p)} \int_0^t (\psi(t)-\psi(s))^{-p} u'(s) ds.$$

In another way, we have

$${}^C D_{0^+}^{p,\psi} u(t) = I_{0^+}^{1-p,\psi} \left(\frac{u'(t)}{\psi'(t)} \right).$$

Proposition 2.1. [6] *Let $p > 0$, if $u \in C^{n-1}(J,\mathbb{R})$, then we have*

1) ${}^C D_{0^+}^{p,\psi} I_{0^+}^{p,\psi} u(t) = u(t)$.
2) $I_{0^+}^{p,\psi} {}^C D_{0^+}^{p,\psi} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]}(0)}{k!} (\psi(t)-\psi(0))^k$.
3) $I_{0^+}^{p,\psi}$ *is linear and bounded from $C(J,\mathbb{R})$ to $C(J,\mathbb{R})$.*

Definition 2.3. *For every $\epsilon > 0$ and with any solution $u \in C(J,\chi)$ of*

$$(2.9) \qquad \left| {}^C D_{0,t}^{p,\psi} u(t) - \mathcal{A}[u(t))] - \mathcal{H}_\eta(t, I_{0,t}^{\alpha,\psi} u(t), u(t)) \right| \leq \epsilon, \quad t \in J$$

the solution of the problems (1.6) *is deemed to be Ulam-Hyers stable. If we would discover a positive real number $C_{\mathcal{H}_\eta} > 0$ and a solution $v \in \mathcal{C}(J,\chi)$ of* (1.6) *meeting the inequality:*

$$(2.10) \qquad |u(t) - v(t)| \leq C_{\mathcal{H}_\eta} \epsilon, \quad t \in J$$

Definition 2.4. *Assume that $\vartheta_{\mathcal{H}_\eta} \in \mathcal{C}(\mathbb{R}^+,\mathbb{R}^+)$ such that for any solution u of* (2.9)*, there exists a solution $v \in C(J,\chi)$ of* (1.6) *satisfying*

$$(2.11) \qquad \left| u(t) - v(t) \right| \leq \vartheta_{\mathcal{H}_\eta}(\epsilon), \quad t \in J.$$

Then, the solution of the problems (1.6) *is identified as generalized Ulam-Hyers stable.*

Definition 2.5. *For any $\epsilon > 0$ and for any solution u of* (1.6)*, the solution of the problems* (1.6) *is known as Ulam-Hyers-Rassias stable with respect to $\xi \in \mathcal{C}(J,\mathbb{R}^+)$ if*

$$(2.12) \qquad \left| {}^C D_{0,t}^{p,\psi} u(t) - \mathcal{A}[u(t))] - \mathcal{H}_\eta(t, I_{0,t}^{\alpha,\psi} u(t), u(t)) \right| \leq \epsilon \xi(t), \quad t \in J,$$

and there's a real number $C_{\mathcal{H}_\eta} > 0$ *and a solution* $v \in \mathcal{C}(J, \mathbb{R})$ *of* (1.6) *as long*

$$|u(t) - v(t)| \leq C_{\mathcal{H}_\eta} \epsilon \xi(t), t \in [0,1], \quad t \in J, \tag{2.13}$$

Lemma 2.1. *For* $f \in C(J, E)$*, we considere the problem*

$$\begin{cases} {}^C D_{0,t}^{p,\psi} u(t) = \mathcal{A}[u(t))] + f(t), & 0 < p < 1, t \in J = [0,T]. \\ u(0) = u_0. \end{cases} \tag{2.14}$$

Then, the solution of the problem (2.14) *is given by*

$$u(t) = u_0 + \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) \mathcal{A}[u(s))] ds + \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) f(s) ds.$$

Proof. Using the $\psi-$ Rieman -Liouville fractional integrale of ordre $0 < p < 1$ for the equation

$${}^C D_{0,t}^{p,\psi} u(t) = \mathcal{A}[u(t))] + f(t)$$

one finds

$$I_{0^+}^{p,\psi} {}^C D_{0,t}^{p,\psi} u(t) = I_{0^+}^{p,\psi} \mathcal{A}[u(t))] + I_{0^+}^{p,\psi} f(t)$$

Then

$$u(t) - u(0) = \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) \mathcal{A}[u(s))] ds + \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) f(s) ds$$

Then

$$u(t) = u_0 + \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) \mathcal{A}[u(s))] ds + \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) f(s) ds.$$

□

Remark 2.4. *The given problem* (2.14) *is equivalent to the integral equation of the following form*

$$u(t) = f(t) + \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) \mathcal{A}[u(s))] ds, \quad t \geq 0, \tag{2.15}$$

where $f(t) = u_0 + \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) f(s) ds$*. Let us assume that the integral equation* (2.15) *has an associated resolvent operator* $R(x)$*,* $t \geq 0$ *on* χ*.*

The resolvent operator for the integral equation (2.15) is defined as follows.

Definition 2.6. [7] *Let* $\{R(t)\}_{t \geq 0}$ *denote a set of bounded linear mappings on* χ*. Then it is said to be the resolvent map for* (2.15) *if it satisfies the following properties:*

i) $R(0)x = x$*, and* $R(\cdot)x \in C(\mathbb{R}^+, E)$*, for each* $x \in \chi$*.*
ii) $\mathcal{A}R(t)x = R(t)\mathcal{A}x$ *and* $R(t)D(\mathcal{A}) \subset D(\mathcal{A})$*, for every* $x \in D(\mathcal{A})$ *and* $t \geq 0$*.*
iii) *For all* $t \geq 0$ *and* $x \in D(\mathcal{A})$*, we have:*

$$R(t)x = x + \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) \mathcal{A}R(s) x ds.$$

The concept of a solution is defined as follows.

Definition 2.7. *A map* $u \in C([0,T], E)$ *is said to be a mild solution of the integral equation* (2.15) *on* $[0,T)$ *if* $\int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) u(s) ds \in D(\mathcal{A})$ *for all* $t \in J = [0,T]$*,* $f(t) \in C([0,T], \chi)$*, and*

$$u(t) = \frac{\mathcal{A}}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) u(s) ds + f(s), \quad t \in [0,T]. \tag{2.16}$$

The next result follows.

Lemma 2.2. [7]*: The following properties are valid under the constraints given in Definition 2.6:*

(1) *If u satisfies* (2.15) *on $[0,T]$, then the mapping $t \to \int_0^t R(t-s)f(s)ds$ is continuously differentiable on $[0,T]$ and we have:*

$$u(t) = \frac{d}{dt}\int_0^t R(t-s)f(s)ds, \quad t \in [0,T].$$

(2) *If $(R(t))_{t\geq 0}$ is differentiable and $f \in C([0,T], D(\mathcal{A}))$, then the mapping $u : [0,T] \to \chi$ defined as:*

$$u(t) = \int_0^t R'(t-s)f(s)ds + f(t), \quad t \in [0,T].$$

is a mild solution of (2.15).

Definition 2.8. *Let A be a nonempty subset of a Banach space χ. A mapping $T : A \to A$ is called a contraction if there exists $\lambda \in [0,1)$ such that for all $x, y \in A$,*

$$\| T(x) - T(y) \| \leq \lambda \| x - y \| .$$

Theorem 2.1. *(Banach's fixed point theorem, see* [8], [9]*). Let A be a nonempty closed subset of a Banach space χ . Then, any contraction mapping T from A to itself has one unique fixed point.*

3. Main results

We set and introduce the following hypotheses before beginning and demonstrating the actual objectives:

(A1) $\mathcal{A} : D(\mathcal{A}) :\longrightarrow \chi$ is a closed linear mapping.

(A2) The resolvent mapping $\{R(t)\}_{t\geq 0}$ is analytic and $\exists \mu_{\mathcal{A}} \in L^1_{loc}([0,T], \mathbb{R}^+)$ such that:

$$\|R'(t)x\| \leq \mu_{\mathcal{A}}(t)\|x\|_{D(\mathcal{A})} \quad \forall t \geq 0.$$

(A3) $\exists \eta_0 > 0$, $\forall \eta \geq \eta_0$, the function $\mathcal{H}_\eta : [0,T] \times \chi^2 \longrightarrow \chi$ is complement continuous and there exist $L_\eta > 0$ such that:

$$\|\mathcal{H}_\eta(t, x_1, y_1) - \mathcal{H}_\eta(t, x_2, y_2)\|_{D(\mathcal{A})} \leq L_\eta \left(\|x_1 - x_2\| + \|y_1 - y_2\|\right).$$

$\forall (t, x_j, y_j) \in [0,T] \times \chi \times \chi; \quad j = 1, 2.$

(A4) $\exists \eta_0 > 0$, $\forall \eta \geq \eta_0$,

$$\rho_\eta = \frac{(\psi(T) - \psi(0))^p}{\Gamma(p+1)} L_\eta \left(\frac{(\psi(T) - \psi(0))^p}{\Gamma(\alpha+1)} + 1 \right) (1 + \|\mu_{\mathcal{A}}\|_{L^1}) < 1. \tag{3.17}$$

3.1. **Existence and uniqueness results.** In this subsection, we explore into the class of non-linear fractional initial value problem given in (1.6). Before commencing our main results and their proofs, we present the following lemma.

Lemma 3.3. *Let $0 < p < 1$, and $\psi \in \mathcal{C}^1(J, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in J$. then $u \in \mathcal{C}^1(J, \mathbb{R})$ is a solution of* (1.6) *if and only it is a solution of the integral equation*

$$\begin{aligned} u(t) =& u_0 + \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) \mathcal{A}[u(s))] ds \\ &+ \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) \mathcal{H}_\eta(t, I^{\alpha,\psi}_{0,s} u(s), u(s)) ds. \end{aligned} \tag{3.18}$$

Proof. Using the $\psi-$ Rieman -Liouville fractional integrale of ordre $0 < p < 1$ for the equation

$$^{C}D_{0,t}^{p,\psi}u(t) = \mathcal{A}[u(t))] + \mathcal{H}_\eta(t, I_{0,s}^{\alpha,\psi}u(t), u(t)).$$

one finds

$$I_{0^+}^{p,\psi C}D_{0,t}^{p,\psi}u(t) = I_{0^+}^{p,\psi}\mathcal{A}[u(t))] + I_{0^+}^{p,\psi}\mathcal{H}_\eta(t, I_{0,s}^{\alpha,\psi}u(t), u(t))$$

Then

$$\begin{aligned} u(t) - u(0) = &\frac{1}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{A}[u(s))]ds \\ &+ \frac{1}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi}u(s), u(s))ds. \end{aligned}$$

Then

$$\begin{aligned} u(t) = u(0) + &\frac{1}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{A}[u(s))]ds \\ &+ \frac{1}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi}u(s), u(s))ds. \end{aligned}$$

□

Definition 3.9. *A mapping $u \in C([0,T],\chi)$ is said to be a mild solution of* (1.1) *on* $[0,T]$ *if* $\int_0^t(\psi(t)-\psi(s))^{p-1}\psi'(s)u(s)ds \in D(\mathcal{A})$*, for each* $t \in [0,T]$*, and* u *satisfies the integral equation* (3.18).

Remark 3.5. *Assume that there is a resolvent mapping $R(t), t \geq 0$ which is differentiable, and $\mathcal{H}_\eta$ is continuous on χ. Then, due to point 2 of Lemma 2.2, one has*

(3.19)

$$\begin{aligned} u(t) = u_0 + &\frac{1}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi}u(s), u(s))ds \\ &+ \int_0^t R'(t-s)\left(u_0 + \frac{1}{\Gamma(p)}\int_0^s (\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi}u(\tau), u(\tau))d\tau\right)ds, \quad t \in [0,T]. \end{aligned}$$

Next, a theorem concerning the existence and uniqueness of a mild solution for the given problem (1.6) is established.

Theorem 3.2. *Suppose* $(A1)-(A4)$ *hold and let* $u_0 \in D(\mathcal{A})$*. Then, there is exactly one mild solution of* (1.6) *in* $[0,T]$.

Proof. We define the operator $T : C([0,T], E) \longrightarrow C([0,T],\chi)$ by

$$\begin{aligned} T[u(t)] = u_0 + &\frac{1}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi}u(s), u(s))ds \\ &+ \int_0^t R'(t-s)\left(u_0 + \frac{1}{\Gamma(p)}\int_0^s (\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi}u(\tau), u(\tau))d\tau\right)ds, \quad t \in [0,T]. \end{aligned}$$

First we show that T is a contraction mapping from the assumption on $\mathcal{H}_\eta$ and letting $u \in C([0,T],\chi)$, we see that

$$
\begin{aligned}
&\left\| R'(t-s)\left(u_0+\frac{1}{\Gamma(p)}\int_0^s(\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\mathcal{H}_\eta(\tau,I_{0,\tau}^{\alpha,\psi}u(\tau),u(\tau))d\tau\right)\right\|.\\
&\leq \|R'(t-s)\|u_0+\|R'(t-s)\|\frac{1}{\Gamma(p)}\int_0^s(\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\|\mathcal{H}_\eta(\tau,I_{0,\tau}^{\alpha,\psi}u(\tau),u(\tau))\|d\tau.\\
&\leq \|\mu_{\mathcal{A}}\|_{L^1}u_0+\frac{\|\mu_{\mathcal{A}}\|_{L^1}}{\Gamma(p)}\sup_{\tau\in[0,s]}\|\mathcal{H}_\eta(\tau,I_{0,\tau}^{\alpha,\psi}u(\tau),u(\tau))\|\int_0^s(\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)d\tau.\\
&\leq \left(u_0+\frac{(\psi(T)-\psi(0))^p}{p\Gamma(p)}\sup_{\tau\in[0,T]}\|\mathcal{H}_\eta(\tau,I_{0,\tau}^{\alpha,\psi}u(\tau),u(\tau))\|\right)\|\mu_{\mathcal{A}}\|_{L^1}.
\end{aligned}
$$

Then the mapping

$$
s\longrightarrow R'(t-s)\left(u_0+\frac{1}{\Gamma(p)}\int_0^s(\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\mathcal{H}_\eta(\tau,I_{0,\tau}^{\alpha,\psi}u(\tau),u(\tau))d\tau\right).
$$

is integrable on $[0,t]$, for eah $t\in[0,T]$. This implies that $T(u)\in C([0,T],\chi)$ and T is well defined. Furthermore for each $u,v\in C([0,T],\chi)$ and $t\in[0,T]$, we have:

$$
\begin{aligned}
&\left\|T[u](t)-T[v](t)\right\| = \left\|\frac{1}{\Gamma(p)}\int_0^t(\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{H}_\eta(s,I_{0,s}^{\alpha,\psi}u(s),u(s))ds\right.\\
&+\int_0^t R'(t-s)\left(u_0+\frac{1}{\Gamma(p)}\int_0^s(\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\mathcal{H}_\eta(\tau,I_{0,\tau}^{\alpha,\psi}u(\tau),u(\tau))d\tau\right)ds\\
&\quad-\frac{1}{\Gamma(p)}\int_0^t(\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{H}_\eta(s,I_{0,s}^{\alpha,\psi}v(s),v(s))ds\\
&\left.-\int_0^t R'(t-s)\left(u_0+\frac{1}{\Gamma(p)}\int_0^s(\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\mathcal{H}_\eta(\tau,I_{0,\tau}^{\alpha,\psi}v(\tau),v(\tau))d\tau\right)ds\right\|\\
&\quad\leq\frac{1}{\Gamma(p)}\int_0^t(\psi(t)-\psi(s))^{p-1}\psi'(s)\|\mathcal{H}_\eta(s,I_{0,s}^{\alpha,\psi}u(s),u(s))-\mathcal{H}_\eta(s,I_{0,s}^{\alpha,\psi}v(s),v(s))\|_{D(\mathcal{A})}ds\\
&+\int_0^t R'(t-s)\frac{1}{\Gamma(p)}\int_0^s(\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\|\mathcal{H}_\eta(\tau,I_{0,\tau}^{\alpha,\psi}u(\tau),u(\tau))-\mathcal{H}_\eta(\tau,I_{0,\tau}^{\alpha,\psi}v(\tau),v(\tau))\|_{D(\mathcal{A})}d\tau ds.\\
&\quad\leq\frac{1}{\Gamma(p)}\int_0^t(\psi(t)-\psi(s))^{p-1}\psi'(s)dsK_\eta\left(\|I_{0,s}^{\alpha,\psi}u(s)-I_{0,s}^{\alpha,\psi}v(s)\|+\|u-v\|\right)ds\\
&+\int_0^t\mu_{\mathcal{A}}(t-s)\frac{1}{\Gamma(p)}\int_0^s(\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)L_\eta\left(\|I_{0,\tau}^{\alpha,\psi}u(\tau)-I_{0,\tau}^{\alpha,\psi}v(\tau)\|+\|u-v\|\right)d\tau ds.\\
&\leq\frac{(\psi(T)-\psi(0))^p}{p\Gamma(p)}L_\eta\left(\|I_{0,s}^{\alpha,\psi}u(s)-I_{0,s}^{\alpha,\psi}v(s)\|+\|u-v\|\right)\\
&+\frac{(\psi(T)-\psi(0))^p}{p\Gamma(p)}L_\eta\left(\|I_{0,\tau}^{\alpha,\psi}u(\tau)-I_{0,\tau}^{\alpha,\psi}v(\tau)\|+\|u-v\|\right)\|\mu_{\mathcal{A}}\|_{L^1}.\\
&\quad\leq\frac{(\psi(T)-\psi(0))^p}{\Gamma(p+1)}L_\eta\left(\|I_{0,t}^{\alpha,\psi}u(t)-I_{0,t}^{\alpha,\psi}v(t)\|+\|u-v\|\right)(1+\|\mu_{\mathcal{A}}\|_{L^1}).
\end{aligned}
$$

Since

(3.20)

$$\begin{aligned}
&\|I_{0,t}^{\alpha,\psi}u(t) - I_{0,t}^{\alpha,\psi}v(t)\| \\
&= \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)u(s)ds - \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)v(s)ds \right\|. \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\|u(s)-v(s)\|ds. \\
&\leq \frac{(\psi(t)-\psi(0))^{\alpha}}{\alpha\Gamma(\alpha)}\|u(s)-v(s)\|. \\
&\leq \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)}\|u-v\|.
\end{aligned}$$

On has

$$\|T[u](t)-T[v](t)\| \leq \frac{(\psi(T)-\psi(0))^{p}}{\Gamma(p+1)} L_{\eta} \left(\frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} + 1 \right) (1+\|\mu_{\mathcal{A}}\|_{L^1})\|u-v\|.$$

Thus, by hypothesis (A4) T is a contraction mapping, consequently , there is one mild solution of the problem (1.6). □

3.2. **Stability results.** In this part, we will study the UH- and UHR- stability of the problem (1.6).

Theorem 3.3. *Assume that (A1)-(A4) hold. Then, the solution of the problem* (1.6) *is UH and generalized UH-stable.*

Proof. Let $v \in C([0,T],E)$ a solution of inequality (2.9)(approximate solution of problem).i.e

$$\left| {}^{C}D_{0,t}^{p,\psi}v(t) - \mathcal{A}[v(t))] - \mathcal{H}_{\eta}(t, I_{0,t}^{\alpha,\psi}v(t), v(t)) \right| \leq \epsilon, \quad t\in[0,T], \epsilon>0. \tag{3.21}$$

and let $u \in C([0,T],\chi)$ a solution of problem :

$$\begin{cases} {}^{C}D_{0,t}^{p,\psi}u(t) = \mathcal{A}[u(t))] + \mathcal{H}_{\eta}(t, I_{0,t}^{\alpha,\psi}u(t), u(t)), & t\in J=[0,T]. \\ u(0)=v(0). \end{cases} \tag{3.22}$$

The solution of equation (3.22) is given by:

$$\begin{aligned}
u(t) &= u(0) + \frac{1}{\Gamma(p)} \int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{H}_{\eta}(s, I_{0,s}^{\alpha,\psi}u(s), u(s))ds \\
&+ \int_0^t R'(t-s)\left(u(0) + \frac{1}{\Gamma(p)} \int_0^s (\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\mathcal{H}_{\eta}(\tau, I_{0,\tau}^{\alpha,\psi}u(\tau), u(\tau))d\tau \right) ds, \quad t\in[0,T].
\end{aligned}$$

By integrating inequality (3.21), we get

$$\begin{aligned}
&\Bigg| v(t) - v(0) - \frac{1}{\Gamma(p)} \int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{H}_{\eta}(s, I_{0,s}^{\alpha,\psi}v(s), v(s))ds \\
&- \int_0^t R'(t-s)\left(v(0) + \frac{1}{\Gamma(p)} \int_0^s (\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\mathcal{H}_{\eta}(\tau, I_{0,\tau}^{\alpha,\psi}v(\tau), v(\tau))d\tau \right) ds \Bigg| \\
&\leq \frac{\epsilon}{p\Gamma(p)}(\psi(t)-\psi(0))^{p} \leq \frac{\epsilon}{\Gamma(p+1)}(\psi(T)-\psi(0))^{p}
\end{aligned}$$

On the other hand , we have $u(0) = v(0)$ then

$$v(t) - u(t) = v(t) - u(0) - \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) \mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi} u(s), u(s)) ds$$
$$- \int_0^t R'(t-s) \left(u(0) + \frac{1}{\Gamma(p)} \int_0^s (\psi(s) - \psi(\tau))^{p-1} \psi'(\tau) \mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi} u(\tau), u(\tau)) d\tau \right) ds,$$
$$+ \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) \left(\mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi} v(s), v(s) - \mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi} v(s), v(s)) \right) ds$$
$$+ \int_0^t R'(t-s) \left[\frac{1}{\Gamma(p)} \int_0^s (\psi(s) - \psi(\tau))^{p-1} \psi'(\tau) \left(\mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi} v(\tau), v(\tau)) - \mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi} v(\tau), v(\tau)) \right) d\tau \right] ds.$$

On the other hand

$$\left| v(t) - u(t) \right| \leq \left| v(t) - v(0) - \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) \mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi} v(s), v(s)) ds \right.$$
$$\left. - \int_0^t R'(t-s) \left(v(0) + \frac{1}{\Gamma(p)} \int_0^s (\psi(s) - \psi(\tau))^{p-1} \psi'(\tau) \mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi} v(\tau), v(\tau)) d\tau \right) ds \right|$$
$$+ \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) \| \mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi} v(s), v(s) - \mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi} u(s), u(s)) \|_{D(\mathcal{A})} ds$$
$$+ \int_0^t R'(t-s) \left[\frac{1}{\Gamma(p)} \int_0^s (\psi(s) - \psi(\tau))^{p-1} \psi'(\tau) \| \mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi} v(\tau), v(\tau)) \right.$$
$$\left. - \mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi} u(\tau), u(\tau)) \|_{D(\mathcal{A})} d\tau \right] ds$$
$$\leq \frac{\epsilon}{p\Gamma(p)} (\psi(T) - \psi(0))^p + \frac{1}{\Gamma(p)} \int_0^t (\psi(t) - \psi(s))^{p-1} \psi'(s) K_\eta \left(\| I_{0,s}^{\alpha,\psi} v(s) - I_{0,s}^{\alpha,\psi} u(s) \| + \| v(s) - u(s) \| \right) ds$$
$$+ \int_0^t \| R'(t-s) \| \left[\frac{1}{\Gamma(p)} \int_0^s (\psi(s) - \psi(\tau))^{p-1} \psi'(\tau) K_\eta \left(\| I_{0,s}^{\alpha,\psi} v(s) - I_{0,s}^{\alpha,\psi} u(s) \| + \| v(s) - u(s) \| \right) d\tau \right] ds.$$

Then, due to (3.20),

$$\left| v(t) - u(t) \right| \leq \frac{\epsilon \left(\psi(T) - \psi(0) \right)^p}{p\Gamma(p)} + \frac{(\psi(T) - \psi(0))^p}{p\Gamma(p)} K_\eta \left(\| I_{0,\tau}^{\alpha,\psi} v(\tau) - I_{0,\tau}^{\alpha,\psi} u(\tau) \| + \| v - u \| \right) \left(1 + \| \mu_{\mathcal{A}} \|_{L^1} \right).$$
$$\leq \frac{\epsilon \left(\psi(T) - \psi(0) \right)^p}{\Gamma(p+1)} + \frac{(\psi(T) - \psi(0))^p}{\Gamma(p+1)} L_\eta \left(\frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)} + 1 \right) \left(1 + \| \mu_{\mathcal{A}} \|_{L^1} \right) \| v - u \|.$$

Hence

$$\left(1 - \frac{(\psi(T) - \psi(0))^p}{\Gamma(p+1)} L_\eta \left(\frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)} + 1 \right) \left(1 + \| \mu_{\mathcal{A}} \|_{L^1} \right) \right) \| v - u \| \leq \frac{\epsilon \left(\psi(T) - \psi(0) \right)^p}{\Gamma(p+1)}$$

and so for each $t \in [0, T]$, we have

$$\left\| v(t) - u(t) \right\| \leq \frac{\epsilon \left(\psi(T) - \psi(0) \right)^p}{\Gamma(p+1) \left(1 - \frac{(\psi(T) - \psi(0))^p}{\Gamma(p+1)} L_\eta \left(\frac{(\psi(T) - \psi(0))^\alpha}{\Gamma(\alpha+1)} + 1 \right) \left(1 + \| \mu_{\mathcal{A}} \|_{L^1} \right) \right)} = C_{\mathcal{H}_\eta} \epsilon$$

Therefore, the solution of the problem (1.6) is UH-stable.

Taking $\vartheta_{\mathcal{H}_\eta}(\epsilon) = C_{\mathcal{H}_\eta} \epsilon$, $\vartheta_{\mathcal{H}_\eta}(0) = 0$ yields that the solution of (1.6) is generalized UH-stable. □

The UHR-stability of the solution of the abstract problem with the fractional spatial heterogeneous viral infection equation (1.6) is now studied. Additionally, suppose that the following assumption holds:

(A5) There $\xi \in C([0,T],\mathbb{R}^+)$ and $\exists\ \lambda_\xi > 0$ such that:

$$\frac{1}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\xi(s)ds \le \lambda_\xi \xi(t), \quad \forall t\in[0,T].$$

Theorem 3.4. *Assume (A1)–(A3), and (A5) holds, then, the solution of the problem* (1.6) *is UHR stable.*

Proof. Let $v \in C([0,T],E)$ a solution of inequality (2.9),

$$\left|{}^C D_{0,t}^{p,\psi} v(t) - \mathcal{A}[v(t))] - \mathcal{H}_\eta(t, I_{0,t}^{\alpha,\psi} v(t), v(t))\right| \le \epsilon\xi(t), \quad t\in[0,T], \epsilon>0. \tag{3.23}$$

and let $u \in C([0,T],\chi)$ a solution of problem :

$$\begin{cases} {}^C D_{0,t}^{p,\psi} u(t) = \mathcal{A}[u(t))] + \mathcal{H}_\eta(t, I_{0,t}^{\alpha,\psi} u(t), u(t)), & t\in J=[0,T]. \\ u(0)=v(0). \end{cases} \tag{3.24}$$

The solution of equation (3.24) is given by:

$$\begin{aligned} u(t) &= u(0) + \frac{1}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi} u(s), u(s))ds \\ &+ \int_0^t R'(t-s)\left(u(0) + \frac{1}{\Gamma(p)}\int_0^s (\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi} u(\tau), u(\tau))d\tau\right)ds, \quad t\in[0,T]. \end{aligned}$$

By integrating inequality (3.23), we get

$$\begin{aligned} &\left|v(t) - v(0) - \frac{1}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi} v(s), v(s))ds \right. \\ &\left. - \int_0^t R'(t-s)\left(v(0) + \frac{1}{\Gamma(p)}\int_0^s (\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi} v(\tau), v(\tau))d\tau\right)ds\right| \\ &\le \frac{\epsilon}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\xi(s)ds. \end{aligned}$$

On the other hand

$$\begin{aligned} \left|v(t)-u(t)\right| &\le \left|v(t) - v(0) - \frac{1}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi} u(s), u(s))ds \right. \\ &\left. - \int_0^t R'(t-s)\left(u(0) + \frac{1}{\Gamma(p)}\int_0^s (\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi} u(\tau), u(\tau))d\tau\right)ds\right| \\ &+ \frac{1}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\|\mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi} v(s), v(s) - \mathcal{H}_\eta(s, I_{0,s}^{\alpha,\psi} v(s), v(s))\|_{D(\mathcal{A})}ds \\ &+ \int_0^t R'(t-s)\left[\frac{1}{\Gamma(p)}\int_0^s (\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)\|\mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi} v(\tau), v(\tau)) \right. \\ &\left. - \mathcal{H}_\eta(\tau, I_{0,\tau}^{\alpha,\psi} v(\tau), v(\tau))\|_{D(\mathcal{A})}d\tau\right]ds \\ &\le \frac{\epsilon}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)\xi(s)ds \\ &+ \frac{1}{\Gamma(p)}\int_0^t (\psi(t)-\psi(s))^{p-1}\psi'(s)_L\eta\left(\|I_{0,s}^{\alpha,\psi} v(s) - I_{0,s}^{\alpha,\psi} u(s)\| + \|v(s)-u(s)\|\right)ds \\ &+ \int_0^t \|R'(t-s)\|\left[\frac{1}{\Gamma(p)}\int_0^s (\psi(s)-\psi(\tau))^{p-1}\psi'(\tau)L_\eta\left(\|I_{0,\tau}^{\alpha,\psi} v(\tau) - I_{0,\tau}^{\alpha,\psi} u(\tau)\| + \|v(\tau)-u(\tau)\|\right)d\tau\right]ds. \end{aligned}$$

Then, due to (3.20),

$$\left|v(t)-u(t)\right| \leq \epsilon\lambda_\xi\xi(t)+\frac{(\psi(T)-\psi(0))^p}{p\Gamma(p)}L_\eta\left(\|I_{0,\tau}^{\alpha,\psi}v(\tau)-I_{0,\tau}^{\alpha,\psi}u(\tau)\|+\|v-u\|\right)(1+\|\mu_{\mathcal{A}}\|_{L^1}).$$
$$\leq \epsilon\lambda_\xi\xi(t)+\frac{(\psi(T)-\psi(0))^p}{\Gamma(p+1)}L_\eta\left(\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}+1\right)(1+\|\mu_{\mathcal{A}}\|_{L^1})\|v-u\|$$

Hence

$$\left(1-\frac{(\psi(T)-\psi(0))^p}{\Gamma(p+1)}L_\eta\left(\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}+1\right)(1+\|\mu_{\mathcal{A}}\|_{L^1})\right)\|v-u\|\leq\epsilon\lambda_\xi\xi(t)$$

and so for each $t\in[0,T]$, we have

$$\left\|v(t)-u(t)\right\|\leq\frac{\epsilon\lambda_\xi\xi(t)}{\left(1-\frac{(\psi(T)-\psi(0))^p}{\Gamma(p+1)}L_\eta\left(\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}+1\right)(1+\|\mu_{\mathcal{A}}\|_{L^1})\right)}=C_{\mathcal{H}_\eta}\epsilon\xi(t).$$

where

$$C_{\mathcal{H}_\eta}=\frac{\lambda_\xi}{\left(1-\frac{(\psi(T)-\psi(0))^p}{\Gamma(p+1)}L_\eta\left(\frac{(\psi(T)-\psi(0))^\alpha}{\Gamma(\alpha+1)}+1\right)(1+\|\mu_{\mathcal{A}}\|_{L^1})\right)}>0.$$

Then by definition 2.5, the solution of the abstract problem (1.6) is UHR stable. □

4. Applications

Example 4.1. *Consider the following problem with ψ-Caputo derivative in $\chi=C([0,1],\mathbb{R})$:*

$$\begin{cases} {}^CD_{0,t}^{\frac14,e^t}u(t)=\left(1+\frac{1}{10^3}\right)u(t)+\frac{1}{10^3}I_{0,t}^{\frac12,e^t}u(t), & t\in J=[0,1].\\ u(0)=u_0\in\mathbb{R}.\end{cases}\tag{4.25}$$

with $\mathcal{A}u=u$, $\eta=10^3$ and

$$\mathcal{H}_\eta(t,I_{0,t}^{\frac12,e^t}u(t),u(t))=\frac{1}{10^3}u(t)+\frac{1}{10^3}I_{0,t}^{\frac12,e^t}u(t)$$

for every $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$, we have

$$\|\mathcal{H}_\eta(t,x_1,y_1)-\mathcal{H}_\eta(t,x_2,y_2)\|_{D(\mathcal{A})}\leq\frac{1}{10^3}\left(\|x_1-x_2\|+\|y_1-y_2\|\right).$$

thus $L_\eta=\frac{1}{10^3}$ and $\|\mu_{\mathcal{A}}\|_{L^1}=e$ we have

$$\rho_\eta=\frac{(\psi(T)-\psi(0))^p}{\Gamma(p+1)}L_\eta\left(\frac{(\psi(T)-\psi(0))^p}{\Gamma(\alpha+1)}+1\right)(1+\|\mu_{\mathcal{A}}\|_{L^1})=0.0117<<1.$$

Then, the problem (4.25) *possesses one solution on* $[0,1]$.

Hence by theorem 3.3, the solution of the problem (4.25) *is UH stable.*

Also, let $\xi(t)=kt,\quad k\in\mathbb{R}^+$. then we have

$$\frac{1}{\Gamma(p)}\int_0^t(\psi(t)-\psi(s))^{p-1}\psi'(s)\xi(s)ds=\frac{k}{\Gamma(p)}\int_0^t(\psi(t)-\psi(s))^{p-1}\psi'(s)sds$$
$$\leq\frac{(\psi(T)-\psi(0))^p}{\Gamma(p+1)}kt=\lambda_\xi\xi(t).$$

then the condition (A5) is satisfied with $\xi(t)=kt$ and $\lambda_\xi=\frac{(\psi(T)-\psi(0))^p}{\Gamma(p+1)}$, by theorem 3.4, the solution of the problem (4.25) *is UHR-stable.*

Example 4.2. *Consider a pharmacokinetic model with fractional metabolism:*

$$(4.26)\qquad \begin{cases} {}^{C}D_{0,t}^{p,\psi}u(t) = (\frac{1}{10} - K)u(t) + MI_{0,t}^{\alpha,\psi}u(t), & t \in J = [0,\pi]. \\ u(0) = u_0 \in \mathbb{R}. \end{cases}$$

where $u(t)$ is the drug dose, $K = 0.1$ is the elimination rate, and $M = \frac{1}{10^2}$ is a feedback term. with $\mathcal{A}u = -Ku$, $\eta = 10$, $p = \frac{1}{3}$, $\alpha = 0.9$, $T = \pi$, $\|\mu_{\mathcal{A}}\|_{L^1} = 0.5$ and the nonlinear operator:

$$\mathcal{H}_\eta(t, I_{0,t}^{\frac{1}{3},e^t}u(t), u(t)) = u(t) + MI_{0,t}^{\alpha,\psi}u(t).$$

which satisfies the Lipschitz condition: for every $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, we have

$$\|\mathcal{H}_\eta(t, x_1, y_1) - \mathcal{H}_\eta(t, x_2, y_2)\|_{D(\mathcal{A})} \leq \frac{1}{10}\left(\|x_1 - x_2\| + \|y_1 - y_2\|\right).$$

For $L_\eta = \frac{1}{10}$ and $\psi(t) = \sqrt{t}$, we have:

$$\rho_\eta = \frac{(\psi(T) - \psi(0))^p}{\Gamma(p+1)} L_\eta \left(\frac{(\psi(T) - \psi(0))^p}{\Gamma(\alpha+1)} + 1\right)(1 + \|\mu_{\mathcal{A}}\|_{L^1}) = 0.558 < 1.$$

Then, the problem (4.26) *possesses one solution on* $[0,\pi]$.
Hence by theorem 3.3, the solution of the problem (4.26) *is UH stable.*
Also, let $\xi(t) = kt^2$, $\quad k \in \mathbb{R}^+$. then we have

$$\begin{aligned} \frac{1}{\Gamma(p)}\int_0^t (\psi(t) - \psi(s))^{p-1}\psi'(s)\xi(s)ds &= \frac{k}{\Gamma(p)}\int_0^t (\psi(t) - \psi(s))^{p-1}\psi'(s)s^2 ds \\ &\leq \frac{(\psi(T) - \psi(0))^p}{\Gamma(p+1)} kt^2 = \lambda_\xi \xi(t). \end{aligned}$$

then the condition (A5) is satisfied with $\xi(t) = kt^2$ and $\lambda_\xi = \frac{(\psi(T)-\psi(0))^p}{\Gamma(p+1)}$, by theorem 3.4, the solution of the problem (4.26) *is UHR stable.*

5. Conclusion

In conclusion, this study successfully derived existence results and stability criteria for solutions to a class of fractional-order differential equations by utilizing fixed point theorems. The existence of solutions was established through Schauder's fixed point theorem and the Banach contraction principle, which provided a robust theoretical framework for analyzing such equations. A significant focus was placed on applying Krasnoselskii's fixed point theorem to develop stability criteria for solutions to a specific class of fractional-order differential equations, introducing a novel approach to addressing stability challenges. Furthermore, a practical example was presented to demonstrate the real-world applicability and effectiveness of the derived stability results, highlighting the relevance and utility of the theoretical findings in practical scenarios.

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