

# Positive solutions to higher order fractional boundary value problems with non-homogeneous conditions

B. R. KUMAR<sup>1</sup>, N. SREEDHAR<sup>2</sup>, AND K. R. PRASAD<sup>3</sup>

**ABSTRACT.** In this study, we consider the higher order nonlinear three-point fractional order boundary value problem containing a non-homogeneous term

$$\begin{aligned} D_{0+}^q u(t) + a(t)f(u(t)) &= 0, \quad t \in (0, b), \\ u^{(j)}(0) &= 0, \quad j = 0, 1, \dots, n-2, \\ u^{(p)}(b) - \alpha u^{(p)}(\xi) &= \lambda, \quad p \in \{1, 2, \dots, n-2\} \text{ is a fixed integer,} \end{aligned}$$

where  $D_{0+}^q$  is the Riemann-Liouville fractional derivative of order  $q \in (n-1, n]$  with  $n > 2$  being a natural number,  $\alpha$  is a positive real constant such that  $0 \leq \alpha < (\frac{b}{\xi})^{q-p-1}$ ,  $\lambda \in [0, \infty)$  is a parameter and  $0 < \xi < b < \infty$ . We establish the existence as well as nonexistence of positive solutions by utilizing the Guo-Krasnosel'skii fixed point theorem.

## 1. INTRODUCTION

Fractional differential equations have emerged as powerful tools for modeling and analyzing complex systems across various scientific and engineering disciplines. These equations involve derivatives of both integer and non-integer orders. This unique feature that enables them to effectively capture memory and hereditary properties inherent in many physical and biological processes. They are particularly valuable in fields such as physics, chemistry, aerodynamics, complex media electrodynamics, polymer rheology, and related areas, where traditional models often fail to accurately describe anomalous behaviors.

Due to their ability to represent such intricate dynamics, fractional differential equations have generated substantial interest in both theoretical studies and applied research, with the aim of solving real-world problems more accurately. This dual focus has driven the development of advanced methods specifically designed for fractional order models, making them more useful and effective in modern science, we refer the books [10, 12, 4] for fractional calculus.

In 2011, Saadi and Benbachir [14] established the positive solutions for three-point nonlinear fractional boundary value problem (BVP)

$$\begin{aligned} D_{0+}^q u(t) + a(t)f(u(t)) &= 0, \quad 0 < t < 1, \quad 2 < q \leq 3, \\ u(0) &= 0, \quad u'(0) = 0, \quad u'(1) - \mu u'(\eta) = \lambda, \end{aligned}$$

where  $0 < \eta < 1$ , using Guo-Krasnosel'skii and Schauder's fixed point theorems. Tidke and Patil [19] in 2023, studied the existence and uniqueness of solutions to the following fractional BVP

$$\begin{aligned} (D_{*a}^q)u(t) &= f(t, u(t)), \quad t \in [a, b], \\ u^{(j)}(a) &= c_j, \quad j = 0, 1, \dots, n-2, \quad u^{(n-1)}(b) = c_b, \end{aligned}$$

---

Received: 11.06.2025. In revised form: 03.07.2025. Accepted: 08.01.2026

2020 *Mathematics Subject Classification.* 26A33, 34A12.

Key words and phrases. *Fractional order, kernel, fixed point theorem, positive solutions.*

Corresponding author: Sreedhar Namburi; [sreedharnamburi@rediffmail.com](mailto:sreedharnamburi@rediffmail.com)

where  $q \in (n - 1, n]$ , by using  $S$ -iteration method. In 2024, Shivanian [15] proved the presence and uniqueness of solutions to the higher order three-point fractional BVP

$${}_cD_t^q u(t) + f(t, u(t)) = 0, \quad t \in (0, 1),$$

$$u^{(j)}(0) = 0, \quad j = 0, 1, \dots, m - 2, \quad u(1) = \gamma u(\eta),$$

where  $q \in (m - 1, m]$ ,  $0 < \eta < 1$  and  ${}_cD_t^q$  is Caputo fractional derivative, by Banach fixed theorem. For more information, here are a few relevant papers, see [2, 20, 1, 6, 11].

Based on the above studies, we consider the higher order fractional differential equation

$$(1.1) \quad D_{0+}^q u(t) + a(t)f(u(t)) = 0, \quad t \in (0, b),$$

satisfying the conditions with non-homogeneous type

$$(1.2) \quad \begin{aligned} &u^{(j)}(0) = 0, \quad j = 0, 1, \dots, n - 2, \\ &u^{(p)}(b) - \alpha u^{(p)}(\xi) = \lambda, \quad p \in \{1, 2, \dots, n - 2\} \text{ is a fixed integer,} \end{aligned}$$

where  $D_{0+}^q$  denotes Riemann-Liouville fractional derivative of order  $q \in (n - 1, n]$  with  $n > 2$  being a natural number,  $\lambda \in [0, \infty)$  is a parameter,  $0 < \xi < b < \infty$ ,  $\alpha$  is a positive real constant such that  $0 \leq \alpha < (\frac{b}{\xi})^{q-p-1}$ , and establish the existence as well as nonexistence of positive solutions. The problem described in (1.1)-(1.2) is a generalization of several existing problems of both integer and non-integer orders found in the literature; see [17, 14, 18, 8, 13, 16, 7, 9]. We now suppose that the following conditions are valid:

- (C1)  $f(u) \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$ ,  $a(t) \in \mathcal{C}([0, b], \mathbb{R}^+)$  and  $a(t)$  is not identically zero on any closed subinterval of  $[0, b]$ ,
- (C2) the constant  $\alpha$  is a positive real number such that  $0 \leq \alpha < (\frac{b}{\xi})^{q-p-1}$ ,
- (C3) the nonnegative extended real numbers  $f_0$  and  $f_\infty$  exist and are defined as

$$f_0 = \lim_{u \rightarrow 0^+} \frac{f(u)}{u} \text{ and } f_\infty = \lim_{u \rightarrow \infty} \frac{f(u)}{u}.$$

We say that  $f$  is superlinear if it satisfies  $f_0 = 0$  and  $f_\infty = \infty$  and, that  $f$  is sublinear if it satisfies  $f_0 = \infty$  and  $f_\infty = 0$ .

An overview of the remaining portion of the study is mentioned here. Section 2 contains the solution of (1.1)-(1.2), which can be expressed as the solution of the corresponding integral equation involving kernel functions. Furthermore, explicit bounds for a specific kernel function are derived. Section 3 focuses on establishing the existence as well as nonexistence of positive solutions to the problem (1.1)-(1.2). The examples are constructed to demonstrate the applicability of the main results in the last section.

## 2. ESTABLISHING KERNELS AND BOUNDS

The solution to the problem (1.1)-(1.2) is represented into the solution of the corresponding integral equation, which contains kernel functions and the inequalities for kernel functions are established. These are helpful in illustrating our discoveries.

Let  $\Psi(t) \in \mathcal{C}([0, b], \mathbb{R}^+)$ . Then the solution to the problem

$$(2.3) \quad D_{0+}^q u(t) + \Psi(t) = 0, \quad t \in (0, b),$$

with the conditions specified in (1.2), is established.

**Lemma 2.1.** Let  $\chi = b^{q-p-1} - \alpha\xi^{q-p-1}$ . If the condition (C2) holds, then the solution to the equations (2.3) and (1.2) is given by

$$(2.4) \quad u(t) = \frac{\lambda t^{q-1}}{[\prod_{j=1}^p (q-j)] \chi} + \int_0^b \mathcal{M}(t, s) \Psi(s) ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) \Psi(s) ds, \quad t \in [0, b],$$

where

$$(2.5) \quad \mathcal{M}(t, s) = \begin{cases} \frac{t^{q-1}(b-s)^{q-p-1}}{b^{q-p-1}\Gamma(q)} - \frac{(t-s)^{q-1}}{\Gamma(q)}, & 0 \leq s \leq t \leq b, \\ \frac{t^{q-1}(b-s)^{q-p-1}}{b^{q-p-1}\Gamma(q)}, & 0 \leq t \leq s \leq b, \end{cases}$$

and

$$(2.6) \quad \mathcal{N}(\xi, s) = \begin{cases} \frac{\xi^{q-p-1}(b-s)^{q-p-1}}{b^{q-p-1}\Gamma(q)} - \frac{(\xi-s)^{q-1}}{\Gamma(q)}, & 0 \leq s \leq \xi \leq b, \\ \frac{\xi^{q-p-1}(b-s)^{q-p-1}}{b^{q-p-1}\Gamma(q)}, & 0 \leq \xi \leq s \leq b. \end{cases}$$

*Proof.* Let  $u(t) \in \mathcal{C}^{[q]+1}[0, b]$  be a solution of fractional BVP (2.3), (1.2). Then the solution of the equation (2.3) is

$$(2.7) \quad u(t) = e_1 t^{q-1} + e_2 t^{q-2} + \dots + e_n t^{q-n} - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Psi(s) ds,$$

where  $e_1, e_2, \dots, e_n$  are arbitrary constants. Using the conditions (1.2), one can obtain

$$e_n = 0, \quad e_{n-1} = 0, \dots, e_2 = 0, \quad \text{and}$$

$$e_1 = \frac{\lambda}{[\prod_{j=1}^p (q-j)] (b^{q-p-1} - \alpha\xi^{q-p-1})} + \frac{1}{(b^{q-p-1} - \alpha\xi^{q-p-1}) \Gamma(q)} \int_0^b (b-s)^{q-p-1} \Psi(s) ds - \frac{\alpha}{(b^{q-p-1} - \alpha\xi^{q-p-1}) \Gamma(q)} \int_0^\xi (\xi-s)^{q-p-1} \Psi(s) ds.$$

Substituting these values in (2.7), we have

$$\begin{aligned} u(t) &= \frac{\lambda t^{q-1}}{[\prod_{j=1}^p (q-j)] (b^{q-p-1} - \alpha\xi^{q-p-1})} + \frac{t^{q-1}}{(b^{q-p-1} - \alpha\xi^{q-p-1}) \Gamma(q)} \\ &\quad \int_0^b (b-s)^{q-p-1} \Psi(s) ds - \frac{\alpha t^{q-1}}{(b^{q-p-1} - \alpha\xi^{q-p-1}) \Gamma(q)} \int_0^\xi (\xi-s)^{q-p-1} \Psi(s) ds \\ &\quad - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Psi(s) ds \\ &= \frac{\lambda t^{q-1}}{[\prod_{j=1}^p (q-j)] (b^{q-p-1} - \alpha\xi^{q-p-1})} + \frac{[b^{q-p-1} - \alpha\xi^{q-p-1} + \alpha\xi^{q-p-1}] t^{q-1}}{b^{q-p-1} (b^{q-p-1} - \alpha\xi^{q-p-1}) \Gamma(q)} \\ &\quad \int_0^b (b-s)^{q-p-1} \Psi(s) ds - \frac{\alpha t^{q-1}}{(b^{q-p-1} - \alpha\xi^{q-p-1}) \Gamma(q)} \int_0^\xi (\xi-s)^{q-p-1} \Psi(s) ds \\ &\quad - \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Psi(s) ds \end{aligned}$$

$$\begin{aligned}
 &= \frac{\lambda t^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] (b^{q-p-1} - \alpha \xi^{q-p-1})} + \int_0^t \left[ \frac{t^{q-1}(b-s)^{q-p-1}}{b^{q-p-1}\Gamma(q)} - \frac{(t-s)^{q-1}}{\Gamma(q)} \right] \Psi(s) ds \\
 &+ \int_t^b \frac{t^{q-1}(b-s)^{q-p-1}}{b^{q-p-1}\Gamma(q)} \Psi(s) ds + \frac{\alpha t^{q-1}}{(b^{q-p-1} - \alpha \xi^{q-p-1})} \\
 &\left\{ \int_0^\xi \left[ \frac{\xi^{q-p-1}(b-s)^{q-p-1}}{b^{q-p-1}\Gamma(q)} - \frac{(\xi-s)^{q-p-1}}{\Gamma(q)} \right] \Psi(s) ds + \int_\xi^b \frac{\xi^{q-p-1}(b-s)^{q-p-1}}{b^{q-p-1}\Gamma(q)} \Psi(s) ds \right\} \\
 &= \frac{\lambda t^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] \chi} + \int_0^b \mathcal{M}(t, s) \Psi(s) ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) \Psi(s) ds.
 \end{aligned}$$

□

**Lemma 2.2.** *The kernels  $\mathcal{M}(t, s)$  and  $\mathcal{N}(t, s)$  given in (2.5) and (2.6) respectively, fulfills the following stated inequalities:*

- (i)  $\mathcal{M}(t, s) \geq 0$  and  $\mathcal{N}(t, s) \geq 0$  for all  $t, s \in [0, b]$ ,
- (ii)  $\mathcal{M}(t, s) \leq \mathcal{M}(b, s)$  for all  $t, s \in [0, b]$ ,
- (iii)  $\mathcal{M}(t, s) \geq \frac{1}{4^{q-1}} \mathcal{M}(b, s)$  for all  $t \in I$  and  $s \in [0, b]$ , where  $I = [\frac{b}{4}, \frac{3b}{4}]$ .

*Proof.* The inequalities are obtained by using algebraic computations. □

The Guo–Krasnosel’skii fixed point theorem is the analytical tool utilized to establish the main findings of this study.

**Theorem 2.1.** [3, 5] *Let  $\mathcal{P}$  be a cone in a Banach Space  $\mathcal{B}$ . Let  $\Theta_1$  and  $\Theta_2$  be two open subsets of  $\mathcal{B}$  with  $0 \in \Theta_1$  and  $\bar{\Theta}_1 \subset \Theta_2$ . If the function  $\mathcal{T} : \mathcal{P} \cap (\bar{\Theta}_2 \setminus \Theta_1) \rightarrow \mathcal{P}$  is completely continuous operator satisfying the following conditions, either*

- (i)  $\|\mathcal{T}u\| \leq \|u\|$ , for  $u \in \mathcal{P} \cap \partial\Theta_1$  and  $\|\mathcal{T}u\| \geq \|u\|$ , for  $u \in \mathcal{P} \cap \partial\Theta_2$ , or
- (ii)  $\|\mathcal{T}u\| \geq \|u\|$ , for  $u \in \mathcal{P} \cap \partial\Theta_1$  and  $\|\mathcal{T}u\| \leq \|u\|$ , for  $u \in \mathcal{P} \cap \partial\Theta_2$ ,

*then the operator  $\mathcal{T}$  has a fixed point in  $\mathcal{P} \cap (\bar{\Theta}_2 \setminus \Theta_1)$ .*

### 3. EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS

In this section, we establish both the existence and nonexistence of positive solutions for the BVP (1.1)-(1.2). To facilitate our analysis, we consider the Banach space  $\mathcal{B} = \{u : u \in \mathcal{C}[0, b]\}$  equipped with the norm

$$\|u\| = \max_{t \in [0, b]} |u(t)|.$$

Next, we define a cone  $\mathcal{P}$  within  $\mathcal{B}$  as follows:

$$\mathcal{P} = \left\{ u \in \mathcal{B} : u(t) \geq 0 \text{ on } t \in [0, b] \text{ and } \min_{t \in I} u(t) \geq \frac{1}{4^{q-1}} \|u\| \right\}.$$

Based on the integral equation (2.4), we introduce the operator  $\mathcal{T} : \mathcal{P} \rightarrow \mathcal{B}$ , defined as follows:

$$\begin{aligned}
 (3.8) \quad \mathcal{T}u(t) &= \frac{\lambda t^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] \chi} \\
 &+ \int_0^b \mathcal{M}(t, s) a(s) f(u(s)) ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) f(u(s)) ds,
 \end{aligned}$$

for  $u \in \mathcal{P}$  and  $t \in [0, b]$ .

**Lemma 3.3.** *The operator  $\mathcal{T} : \mathcal{P} \rightarrow \mathcal{B}$  defined in equation (3.8) is a self map on the cone  $\mathcal{P}$ .*

*Proof.* By positivity of kernels  $\mathcal{M}(t, s)$  and  $\mathcal{N}(\xi, s)$ , we have  $\mathcal{T}u(t) \geq 0$  on  $t \in [0, b]$  and for  $u \in \mathcal{P}$ . Applying Lemma 2.2 and for  $u \in \mathcal{P}$ , to obtain

$$\begin{aligned} \mathcal{T}u(t) &= \frac{\lambda t^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] \chi} + \int_0^b \mathcal{M}(t, s) a(s) f(u(s)) ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) f(u(s)) ds \\ &\leq \frac{\lambda b^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] \chi} + \int_0^b \mathcal{M}(b, s) a(s) f(u(s)) ds + \frac{\alpha b^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) f(u(s)) ds. \end{aligned}$$

So,

$$(3.9) \quad \begin{aligned} \|\mathcal{T}u\| &\leq \frac{\lambda b^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] \chi} + \int_0^b \mathcal{M}(b, s) a(s) f(u(s)) ds \\ &\quad + \frac{\alpha b^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) f(u(s)) ds, \end{aligned}$$

for any  $u \in \mathcal{P}$ . Further, from the inequality (3.9) and for  $u \in \mathcal{P}$ , we obtain

$$\begin{aligned} \min_{t \in I} \mathcal{T}u(t) &= \min_{t \in I} \left\{ \frac{\lambda t^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] \chi} + \int_0^b \mathcal{M}(t, s) a(s) f(u(s)) ds \right. \\ &\quad \left. + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) f(u(s)) ds \right\} \\ &\geq \frac{1}{4^{q-1}} \left\{ \frac{\lambda b^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] \chi} + \int_0^b \mathcal{M}(b, s) a(s) f(u(s)) ds \right. \\ &\quad \left. + \frac{\alpha b^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) f(u(s)) ds \right\} \\ &\geq \frac{1}{4^{q-1}} \|\mathcal{T}u\|. \end{aligned}$$

Therefore,  $\mathcal{T} : \mathcal{P} \rightarrow \mathcal{P}$  and hence, it is proved.

Moreover, according to Arzela-Ascoli theorem,  $\mathcal{T}$  is a completely continuous operator.

Let

$$(3.10) \quad \mathcal{L}_1 = \frac{1}{2} \left\{ \int_0^b \mathcal{M}(b, s) a(s) ds + \frac{\alpha b^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) ds \right\}^{-1}$$

and

$$(3.11) \quad \mathcal{L}_2 = 4^{2q-2} \left\{ \int_{s \in I} \mathcal{M}(b, s) a(s) ds + \frac{\alpha b^{q-1}}{\chi} \int_{s \in I} \mathcal{N}(\xi, s) a(s) ds \right\}^{-1}.$$

□

**Theorem 3.2.** Assume that conditions (C1) and (C2) are fulfilled. If the function  $f$  fulfills the properties  $f_0 = 0$  and  $f_\infty = \infty$ , then the BVP (1.1)-(1.2) admits at least one positive solution for sufficiently small values of  $\lambda$ .

*Proof.* We will now demonstrate that the BVP (1.1)-(1.2) possesses at least one positive solution when  $\lambda > 0$  is sufficiently small.

By the definition of  $f_0 = 0$ , there exist  $\mathcal{L}_1 > 0$  and  $\mathcal{H}_1 > 0$  such that

$$(3.12) \quad f(u) \leq \mathcal{L}_1 u, \quad 0 < u \leq \mathcal{H}_1.$$

Let  $\lambda$  satisfy

$$(3.13) \quad 0 < \lambda \leq \frac{\mathcal{H}_1 \left[ \prod_{j=1}^p (q-j) \right] \chi}{2b^{q-1}}.$$

Choose  $u \in \mathcal{P}$  and  $\|u\| = \mathcal{H}_1$ , then from Lemma 2.2 and by the inequalities (3.12) and (3.13), we have

$$\begin{aligned} \mathcal{T}u(t) &= \frac{\lambda t^{q-1}}{\left[ \prod_{j=1}^p (q-j) \right] \chi} + \int_0^b \mathcal{M}(t, s) a(s) f(u(s)) ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) f(u(s)) ds \\ &\leq \frac{\lambda b^{q-1}}{\left[ \prod_{j=1}^p (q-j) \right] \chi} + \int_0^b \mathcal{M}(b, s) a(s) f(u(s)) ds + \frac{\alpha b^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) f(u(s)) ds \\ &\leq \frac{\mathcal{H}_1}{2} + \mathcal{L}_1 \left[ \int_0^b \mathcal{M}(b, s) a(s) u(s) ds + \frac{\alpha b^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) u(s) ds \right] \\ &\leq \frac{\mathcal{H}_1}{2} + \mathcal{L}_1 \left[ \int_0^b \mathcal{M}(b, s) a(s) ds + \frac{\alpha b^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) ds \right] \|u\| \\ &\leq \frac{\mathcal{H}_1}{2} + \frac{\mathcal{H}_1}{2} = \mathcal{H}_1, \end{aligned}$$

so that

$$\mathcal{T}u(t) \leq \mathcal{H}_1, \text{ for } t \in [0, b].$$

Therefore,  $\|\mathcal{T}u\| \leq \mathcal{H}_1 = \|u\|$ . If we take

$$\Omega_1 = \{u \in \mathcal{B} : \|u\| < \mathcal{H}_1\},$$

then

$$(3.14) \quad \|\mathcal{T}u\| \leq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_1.$$

Since  $f_\infty = \infty$ , there exist  $\mathcal{L}_2 > 0$  and  $\bar{\mathcal{H}}_2 > 0$  such that

$$(3.15) \quad f(u) \geq \mathcal{L}_2 u, \quad u \geq \bar{\mathcal{H}}_2.$$

Let  $\mathcal{H}_2 = \max\{2\mathcal{H}_1, 4^{q-1}\bar{\mathcal{H}}_2\}$ . Choose  $u \in \mathcal{P}$  with  $\|u\| = \mathcal{H}_2$ , then

$$(3.16) \quad \min_{t \in I} u(t) \geq \frac{1}{4^{q-1}} \|u\| \geq \bar{\mathcal{H}}_2.$$

Then from Lemma 2.2, and by inequalities (3.15) and (3.16), one can obtain

$$\begin{aligned} \mathcal{T}u(t) &= \frac{\lambda t^{q-1}}{\left[ \prod_{j=1}^p (q-j) \right] \chi} + \int_0^b \mathcal{M}(t, s) a(s) f(u(s)) ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) f(u(s)) ds \\ &\geq \int_0^b \mathcal{M}(t, s) a(s) f(u(s)) ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) f(u(s)) ds \\ &\geq \min_{t \in I} \left\{ \int_0^b \mathcal{M}(t, s) a(s) f(u(s)) ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s) a(s) f(u(s)) ds \right\} \\ &\geq \frac{1}{4^{q-1}} \left\{ \int_{s \in I} \mathcal{M}(b, s) a(s) f(u(s)) ds + \frac{\alpha b^{q-1}}{\chi} \int_{s \in I} \mathcal{N}(\xi, s) a(s) f(u(s)) ds \right\} \end{aligned}$$

$$\begin{aligned} &\geq \frac{\mathcal{L}_2}{4^{q-1}} \left\{ \int_{s \in I} \mathcal{M}(b, s)a(s)u(s)ds + \frac{\alpha b^{q-1}}{\chi} \int_{s \in I} \mathcal{N}(\xi, s)a(s)u(s)ds \right\} \\ &\geq \frac{\mathcal{L}_2}{4^{2q-2}} \left\{ \int_{s \in I} \mathcal{M}(b, s)a(s)ds + \frac{\alpha b^{q-1}}{\chi} \int_{s \in I} \mathcal{N}(\xi, s)a(s)ds \right\} \|u\| \\ &= \|u\|. \end{aligned}$$

So that

$$\mathcal{T}u(t) \geq \|u\|, \text{ for } t \in [0, b].$$

Hence,  $\|\mathcal{T}u\| \geq \|u\|$ . Now, if we take

$$\Omega_2 = \{u \in \mathcal{B} : \|u\| < \mathcal{H}_2\},$$

then

$$(3.17) \quad \|\mathcal{T}u\| \geq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_2.$$

Consequently, applying Theorem 2.1 to the equations (3.14) and (3.17), the operator  $\mathcal{T}$  possesses at least one fixed point  $u \in \mathcal{P} \cap (\Omega_2 \setminus \bar{\Omega}_1)$ , which serves as the positive solution to the problem (1.1)-(1.2).  $\square$

**Theorem 3.3.** *Assume that the conditions (C1) and (C2) are fulfilled. If the function  $f$  fulfills the properties  $f_0 = 0$  and  $f_\infty = \infty$ , then the BVP (1.1)-(1.2) admits no positive solution when  $\lambda$  is sufficiently large.*

*Proof.* We verify that when  $\lambda$  is sufficiently large, the problem (1.1)-(1.2) admits no positive solution. For this, for any positive integer  $m$ , there exist  $0 < \lambda_1 < \lambda_2 < \dots < \lambda_m < \dots$  with  $\lim_{m \rightarrow \infty} \lambda_m = \infty$  such that the problem

$$\begin{aligned} D_{0+}^q u(t) + a(t)f(u(t)) &= 0, \quad t \in (0, b), \\ u^{(j)}(0) &= 0, \quad j = 0, 1, \dots, n-2, \end{aligned}$$

$$u^{(p)}(b) - \alpha u^{(p)}(\xi) = \lambda_m, \quad p \in \{1, 2, \dots, n-2\} \text{ is a fixed integer,}$$

possesses a positive solution  $u_m(t)$ . From equation (2.4), one can obtain

$$\begin{aligned} u_m(b) &= \frac{\lambda_m b^{q-1}}{\left[ \prod_{j=1}^p (q-j) \right] \chi} + \int_0^b \mathcal{M}(b, s)a(s)f(u_m(s))ds + \frac{\alpha b^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s)a(s)f(u_m(s))ds \\ &\geq \frac{\lambda_m b^{q-1}}{\left[ \prod_{j=1}^p (q-j) \right] \chi} \rightarrow \infty \text{ as } m \rightarrow \infty. \end{aligned}$$

Thus,  $\|u_m\| \rightarrow \infty$  as  $m \rightarrow \infty$ .

From the definition of  $f_\infty = \infty$ , there exist  $\mathcal{L}_2 > 0$  and  $\bar{\mathcal{H}} > 0$  such that

$$f(u) \geq 2\mathcal{L}_2 u, \quad u \geq \bar{\mathcal{H}}.$$

Let  $m$  be sufficiently large such that  $\|u_m\| \geq \bar{\mathcal{H}}$ . Then

$$\begin{aligned} \|u_m\| &\geq u_m(b) \\ &= \frac{\lambda_m b^{q-1}}{\left[ \prod_{j=1}^p (q-j) \right] \chi} + \int_0^b \mathcal{M}(b, s)a(s)f(u_m(s))ds + \frac{\alpha b^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s)a(s)f(u_m(s))ds \\ &\geq \int_0^b \mathcal{M}(b, s)a(s)f(u_m(s))ds + \frac{\alpha b^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s)a(s)f(u_m(s))ds \\ &\geq 2\mathcal{L}_2 \left\{ \int_{s \in I} \mathcal{M}(b, s)a(s)u_m(s)ds + \frac{\alpha b^{q-1}}{\chi} \int_{s \in I} \mathcal{N}(\xi, s)a(s)u_m(s)ds \right\} \end{aligned}$$

$$\begin{aligned} &\geq \frac{2\mathcal{L}_2}{4^{2q-2}} \left\{ \int_{s \in I} \mathcal{M}(b, s)a(s)ds + \frac{\alpha b^{q-1}}{\chi} \int_{s \in I} \mathcal{N}(\xi, s)a(s)ds \right\} \|u_m\| \\ &= 2\|u_m\|, \end{aligned}$$

which leads to contradiction and hence the theorem.  $\square$

**Theorem 3.4.** *Assume that the conditions (C1) and (C2) are fulfilled. If the function  $f$  fulfills the properties  $f_0 = \infty$  and  $f_\infty = 0$ , then the BVP (1.1)-(1.2) admits at least one positive solution for any  $\lambda \in (0, \infty)$ .*

*Proof.* Since  $f_0 = \infty$ , there exist  $\mathcal{L}_3 > 0$  and  $\mathcal{J}_1 > 0$  such that

$$f(u) \geq \mathcal{L}_3 u, \quad 0 < u \leq \mathcal{J}_1,$$

where  $\mathcal{L}_3 \geq \mathcal{L}_2$  and  $\mathcal{L}_2$  is mentioned in (3.11). Then for any  $u \in \mathcal{P}$ ,  $\|u\| = \mathcal{J}_1$ , we get

$$\begin{aligned} \mathcal{T}u(t) &= \frac{\lambda t^{q-1}}{\left[ \prod_{j=1}^p (q-j) \right] \chi} + \int_0^b \mathcal{M}(t, s)a(s)f(u(s))ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s)a(s)f(u(s))ds \\ &\geq \int_0^b \mathcal{M}(t, s)a(s)f(u(s))ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s)a(s)f(u(s))ds \\ &\geq \min_{t \in I} \left\{ \int_0^b \mathcal{M}(t, s)a(s)f(u(s))ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi, s)a(s)f(u(s))ds \right\} \\ &\geq \frac{1}{4^{q-1}} \left\{ \int_{s \in I} \mathcal{M}(b, s)a(s)f(u(s))ds + \frac{\alpha b^{q-1}}{\chi} \int_{s \in I} \mathcal{N}(\xi, s)a(s)f(u(s))ds \right\} \\ &\geq \frac{\mathcal{L}_3}{4^{q-1}} \left\{ \int_{s \in I} \mathcal{M}(b, s)a(s)u(s)ds + \frac{\alpha b^{q-1}}{\chi} \int_{s \in I} \mathcal{N}(\xi, s)a(s)u(s)ds \right\} \\ &\geq \frac{\mathcal{L}_2}{4^{2q-2}} \left\{ \int_{s \in I} \mathcal{M}(b, s)a(s)ds + \frac{\alpha b^{q-1}}{\chi} \int_{s \in I} \mathcal{N}(\xi, s)a(s)ds \right\} \|u\| \\ &= \|u\|. \end{aligned}$$

Therefore,  $\|\mathcal{T}u\| \geq \|u\|$ . Now, if we set  $\Omega_3 = \{u \in \mathcal{B} : \|u\| < \mathcal{J}_1\}$ , then

$$(3.18) \quad \|\mathcal{T}u\| \geq \|u\|, \quad \text{for } u \in \mathcal{P} \cap \partial\Omega_3.$$

Further, since  $f^\infty = 0$ , there exist  $\mathcal{L}_4 > 0$  and  $\bar{\mathcal{J}}_2 > 0$  such that

$$f(u) \leq \mathcal{L}_4 u, \quad u \geq \bar{\mathcal{J}}_2,$$

where  $\mathcal{L}_4 \leq \mathcal{L}_1$  and  $\mathcal{L}_1$  is mentioned in (3.10). The result is demonstrated by examining the below cases.

Case (i): Suppose the function  $f$  is bounded. Then there is a positive constant  $\mathcal{R}$  with

$$f(u) \leq \mathcal{R}, \quad 0 < u < \infty.$$

We now take

$$\mathcal{J}_2 \geq \max \left\{ 2\mathcal{J}_1, \frac{\mathcal{R}}{\mathcal{L}_1}, \frac{2\lambda b^{q-1}}{\left[ \prod_{j=1}^p (q-j) \right] \chi} \right\},$$

and then for any  $u \in \mathcal{P}$  with  $\|u\| = \mathcal{J}_2$ , we obtain

$$\begin{aligned} \mathcal{T}u(t) &= \frac{\lambda t^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] \chi} + \int_0^b \mathcal{M}(t,s)a(s)f(u(s))ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi,s)a(s)f(u(s))ds \\ &\leq \frac{\lambda b^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] \chi} + \mathcal{R} \int_0^b \mathcal{M}(b,s)a(s)ds + \frac{\alpha b^{q-1} \mathcal{R}}{\chi} \int_0^b \mathcal{N}(\xi,s)a(s)ds \\ &\leq \frac{\mathcal{J}_2}{2} + \frac{\mathcal{J}_2}{2} = \mathcal{J}_2. \end{aligned}$$

So,  $\|\mathcal{T}u\| \leq \|u\|$ .

Case (ii): Suppose the function  $f$  is unbounded. Choose

$$\mathcal{J}_2 \geq \max \left\{ 2\mathcal{J}_1, \bar{\mathcal{J}}_2, \frac{2\lambda b^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] \chi} \right\},$$

with

$$f(u) \leq f(\mathcal{J}_2), \quad 0 \leq u \leq \mathcal{J}_2.$$

Then for  $u \in \mathcal{P}$  and  $\|u\| = \mathcal{J}_2$ , we have

$$\begin{aligned} \mathcal{T}u(t) &= \frac{\lambda t^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] \chi} + \int_0^b \mathcal{M}(t,s)a(s)f(u(s))ds + \frac{\alpha t^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi,s)a(s)f(u(s))ds \\ &\leq \frac{\lambda b^{q-1}}{\left[\prod_{j=1}^p (q-j)\right] \chi} + \int_0^b \mathcal{M}(b,s)a(s)f(\mathcal{J}_2)ds + \frac{\alpha b^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi,s)a(s)f(\mathcal{J}_2)ds \\ &\leq \frac{\mathcal{J}_2}{2} + \mathcal{L}_4 \mathcal{J}_2 \left\{ \int_0^b \mathcal{M}(b,s)a(s)ds + \frac{\alpha b^{q-1}}{\chi} \int_0^b \mathcal{N}(\xi,s)a(s)ds \right\} \\ &= \frac{\mathcal{J}_2}{2} + \frac{\mathcal{J}_2}{2} = \mathcal{J}_2. \end{aligned}$$

So,  $\|\mathcal{T}u\| \leq \|u\|$ . Therefore, in both cases, we can define

$$\Omega_4 = \{u \in \mathcal{P} : \|u\| < \mathcal{J}_2\},$$

then

$$(3.19) \quad \|\mathcal{T}u\| \leq \|u\|, \text{ for } u \in \mathcal{P} \cap \partial\Omega_4.$$

Hence by Theorem 2.1, the inequalities (3.18) and (3.19), the operator  $\mathcal{T}$  has at least one fixed point  $u \in \mathcal{P} \cap (\Omega_4 \setminus \bar{\Omega}_3)$ , which serves as the positive solution to the problem (1.1)-(1.2).  $\square$

#### 4. EXAMPLES

Examples are offered to support our established results.

**Example 4.1.** Consider the fractional differential equation

$$(4.20) \quad D_{0+}^{\frac{5}{2}}u + 2(t+1)u^3 = 0, \quad t \in (0, 1),$$

with

$$(4.21) \quad u(0) = 0, \quad u'(0) = 0 \text{ and } u'(1) - \frac{1}{\sqrt{2}}u'\left(\frac{1}{2}\right) = \lambda.$$

Here

$$f(u(t)) = (u(t))^3 \text{ and } a(t) = 2(t+1), \text{ for } t \in [0, 1].$$

Clearly,  $f(u)$  is a continuous function on  $\mathbb{R}^+$  and  $a(t)$  is a continuous function on  $[0, 1]$  such that  $a(t) \neq 0$  on any closed subinterval of  $[0, 1]$ . Also,

$$\left(\frac{b}{\xi}\right)^{q-p-1} = \sqrt{2} > \alpha = \frac{1}{\sqrt{2}}.$$

Then the conditions (C1) and (C2) are satisfied. Further,

$$f_0 = \lim_{u \rightarrow 0^+} \frac{u^3}{u} = 0 \text{ and } f_\infty = \lim_{u \rightarrow \infty} \frac{u^3}{u} = \infty.$$

So, all the conditions of Theorem 3.2 are satisfied and hence, the problem (4.20) and (4.21) has at least one positive solution by choosing  $\lambda$  that  $0 < \lambda \leq \frac{3}{8}\mathcal{H}_1$ , where  $\mathcal{H}_1 > 0$ . According to Theorem 3.3, the problem (4.20) and (4.21) has no positive solution for sufficiently large values of  $\lambda$ .

**Example 4.2.** Consider the fractional differential equation

$$(4.22) \quad D_{0^+}^{\frac{7}{2}}u + \frac{1}{2}(t+2)e^{-2u} = 0, \quad t \in (0, 1),$$

with

$$(4.23) \quad u(0) = 0, \quad u'(0) = 0, \quad u''(0) = 0 \text{ and } u''(1) - \frac{1}{4\sqrt{2}}u''\left(\frac{1}{4}\right) = \lambda.$$

Here

$$f(u(t)) = e^{-2u(t)} \text{ and } a(t) = \frac{1}{2}(t+2), \text{ for } t \in [0, 1].$$

Clearly,  $f(u)$  is a continuous function on  $\mathbb{R}^+$  and  $a(t)$  is a continuous function on  $[0, 1]$  such that  $a(t) \neq 0$  on any closed subinterval of  $[0, 1]$ . Also,

$$\left(\frac{b}{\xi}\right)^{q-p-1} = 2 > \alpha = \frac{1}{4\sqrt{2}}.$$

Then the conditions (C1) and (C2) are satisfied. Further,

$$f_0 = \lim_{u \rightarrow 0^+} \frac{e^{-2u}}{u} = \infty \text{ and } f_\infty = \lim_{u \rightarrow \infty} \frac{e^{-2u}}{u} = 0.$$

Therefore, all the assumptions of Theorem 3.4 are satisfied and implying that the problem (4.22) and (4.23) admits at least one positive solution for any  $\lambda \in (0, \infty)$ .

## 5. CONCLUSIONS

This paper establishes the existence of positive solutions to the fractional order problems with three-point boundary conditions involving a non-homogeneous term, using the Guo–Krasnosel'skii fixed point theorem. These results unify the theory of integer and non-integer order differential equations and highlight the effectiveness of fractional calculus.

## ACKNOWLEDGMENTS

The authors sincerely thank the referees for their valuable suggestions and insightful comments.

## REFERENCES

- [1] Chai, G.; Hu, S. Existence of positive solutions for a fractional high-order three-point boundary value problem. *Adv. Differ. Equ.* **2014** (2014), 1–17.
- [2] Goodrich, C. S. Existence of a positive solution to a class of fractional differential equations. *Appl. Math. Lett.* **23** (2010), no. 9, 1050–1055.
- [3] Guo, D.; Lakshmikantham, V. *Nonlinear problems in abstract cone*. Academic Press, Orlando, 1988.
- [4] Kilbas, A. A.; Srivastava, H. M.; Trujillo, J. J. *Theory and applications of fractional differential equations*. North-Holland Mathematics Studies, **204**, Elsevier Science, B. V. Amsterdam, 2006.
- [5] Krasnosel'skii, M. A. *Positive solutions of operator equations*. P. Noordhoff Ltd, Groningen, The Netherlands, 1964.
- [6] Kumar, S.; Kumar, V. R.; Kumar, N. H. Existence and uniqueness results for three-point nonlinear fractional (arbitrary order) boundary value problem. *Mat. Vesn.* **70** (2018), 314–325.
- [7] Krushna, B. M. B. Extremal points for a  $(n, p)$ -type Riemann-Liouville fractional order boundary value problems. *TWMS J. App. Eng. Math.* **14** (2024), no. 1, 247–258.
- [8] Lakoud, A. G.; Zenkoufi, L. Existence of positive solutions for a fourth order three-point boundary value problem. *J. Appl. Math. Comput.* **50** (2016), no. 1, 139–155.
- [9] Madhubabu, B.; Sreedhar, N.; Prasad, K. R. The existence of solutions to higher order differential equations with non-homogeneous conditions. *Lith. Math. J.* **64** (2024), no. 1, 53–66.
- [10] Miller, K. S.; Ross, B. *An introduction to the fractional calculus and differential equations*. John Wiley, Newyork, 1993.
- [11] Ntouyas, S. K.; Pourhadi, E. Positive solutions of nonlinear fractional three-point boundary value problem, *Matematiche.* **73** (2018), no. 1, 139–154.
- [12] Podlubny, I. *Fractional differential equations*. Mathematics in Sciences and Engineering, **198**, Academic Press, San Diego, 1999.
- [13] Prasad, K. R.; Sreedhar, N.; Rashmita, M. Existence and nonexistence results for  $n^{th}$  order non-homogeneous three-point boundary value problems. *Montes Taurus J. Pure Appl. Math.* **5** (2023), no. 1, 34–42.
- [14] Saadi, A.; Benbachir, M. Positive solutions for three-point nonlinear fractional boundary value problems. *Electron. J. Qual. Theory Differ. Equ.* **2011** (2011), 1–19.
- [15] Shivanian, E. On the solutions of Caputo fractional high-order three-point boundary value problem with applications to optimal control. *J. Nonlinear Math. Phys.* **31** (2024), 1–14.
- [16] Sreedhar, N.; Madhubabu, B.; Prasad, K. R. Existence results for differential equations of fourth order with non-homogenous boundary conditions. *Contemp. Math.* **4** (2023), no. 1, 118–131.
- [17] Sun, Y. Positive solutions for third order three-point non-homogeneous boundary value problems. *Appl. Math. Lett.* **22** (2009), no. 1, 45–51.
- [18] Sun, Y.; Zhu, C. Existence of positive solutions for singular fourth order three-point boundary value problems. *Adv. Differ. Equ.* **2013** (2013), 1–13.
- [19] Tidke, H. L.; Patil, G. S. Existence and uniqueness of solutions of a boundary vale problem of fractional order via S-iteration. *Creat. Math. Inform.* **32** (2023), no. 1, 97–120.
- [20] Wang, G.; Ntouyas, S. K.; Zhang, L. Positive solutions of the three-point boundary value problem for fractional order differential equations with an advanced argument. *Adv. Differ. Equ.* **2011** (2011), 1–11.

<sup>1</sup> DEPARTMENT OF MATHEMATICS, GITAM (DEEMED TO BE UNIVERSITY), VISAKHAPATNAM, 530 045, INDIA

*Email address:* bangaru.rk@gmail.com

<sup>2</sup> DEPARTMENT OF MATHEMATICS, GITAM (DEEMED TO BE UNIVERSITY), VISAKHAPATNAM, 530 045, INDIA

*Email address:* sreedharnamburi@rediffmail.com

<sup>3</sup> DEPARTMENT OF APPLIED MATHEMATICS, ANDHRA UNIVERSITY, VISAKHAPATNAM, 530 003, INDIA

*Email address:* rajendra92@rediffmail.com