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Some New Results on GK-algebras. A Different Approach to the Internal Architecture of GK-algebras

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ABSTRACT. In this paper, by revising the claims about GK-algebras, we prove some new assertions about this class of logical algebras, we show the relationship between sub-algebras and ideals in them as well as about the connection of ideals and right congruences in this class of logical algebras. Also, the concept of self-distributability of GK-algebras is treated in this paper.

1. INTRODUCTION

Logical algebraic structures are crucial in various fields of mathematics and computer science. For example, algebraic structures like Boolean algebra form the foundation of classical logic or binary logic. Boolean algebra is widely used in the design and analysis of logical circuits and digital systems. A BCK/BCI-algebre are important classes of logical algebras introduced by Y. Imai and K. Iséki in 1966 ([5]) as a generalization of the concept of set-theoretic difference and propositional calculus, and was extensively investigated by several researchers. Since then, determinations of new logical algebras have appeared, such as, for example, d-algebras (1999, [13]), BE-algebras (2007, [10]), CI-algebras (2010, [12]) known in the literature as RME-algebra, BI-algebras (2017, [1]) and many others. In the spirit of the previous one, in 2018 the concept of GK-algebra was introduced in [2] by R. Gowri and J. Kavith as a subclass of the class GB-algebras ([11]). This type of logical algebra was the focus of articles [3, 4, 7, 8, 9]. While in [3, 7, 9] fuzzy techniques were applied to this class of logical algebras, in [4] and [8] the authors discuss the properties of GK-algebras.

In this paper, by reviewing the claims presented in the aforementioned articles, we prove some new assertions on this class of logical algebras. First of all, we show that the given axiomatic system, by which the concept of GK-algebras is determined, can be reduced (Theorem 3.1). Then, some new properties of this class of logical algebras were proved (Proposition 3.5 and Theorem 3.2). In the mentioned theorem it is proved that the direct product of any family of GK-algebras is again a GK-algebra. Also, apart from the fact that it was shown that every sub-algebra is an ideal (Theorem 3.3), it was proved that every ideal in the GK-algebra generates a right congruence in it. Conversely, the kernel of every right congruence on a GK-algebra is both a sub-algebra and an ideal in it (Theorem 3.7). On the other hand, it is proved that every closed ideal is a sub-algebra (Theorem 3.4). Further on, we investigated the concepts of strong, weak ideals and p-ideals in GK-algebras. Also, in this article, we prove that the GK-algebra can be neither left nor right distributive.

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2. Preliminaries

It should be emphasized here that the formulas in this text are written in a standard way, as is usual in mathematical logic, with the standard use of labels for logical functions. Thus, the labels $\wedge$, $\vee$, $\Longrightarrow$, $\Longleftrightarrow$, $\neg$, and so on, are labels for the logical functions of conjunction, disjunction, implication, equivalence, negation, and so on. Brackets in formulas are used in the standard way, too. All formulas appearing in this paper are closed by some quantifier. If one of the formulas is open, then the variables that appear in it should be seen as free variables. In addition to the previous one, the sign =:, in the use of $A =: B$, should be understood in the sense that the mark A is the abbreviation for the formula B.

In this text, to mark recognizable formulas, we will use, as far as possible, their standard abbreviations that appear in a very well-known paper [6].

Definition 2.1 ([2], Definition 3.1)**.** *A non-empty set A with fixed constant 1 and a binary operation $*$ is called GK-algebra if it satisfying the following axioms:*

(Re) $(\forall x \in A)(x * x = 1)$.

(M) $(\forall x \in A)(x * 1 = x)$.

(An) $(\forall x, y \in A)((x * y = 1 \wedge y * x = 1) \Longrightarrow x = y)$.

(GK) $(\forall x, y, z \in A)((x * z) * (y * z) = x * y)$.

(v) $(\forall x, y \in A)((x * y) * (1 * y) = x)$.

*We denote this axiomatic system by **GK** and the algebraic structure $\mathfrak{A} =: (A, *, 1)$ determined by it by GK-algebra.*

In order to gain insight into the connection between the GK-algebra and the BG-algebra, the determination of the BG-algebra is listed:

Definition 2.2 ([11], pp. 498)**.** *A BG-algebra is an algebra $\mathfrak{A} = (A, *, 1)$ of type $(2, 0)$ satisfying the axioms (Re), (M) and (v).*

Thus, the GK-algebra is a subclass of the BG-algebra. The reverse, of course, does not have to be the case: There are BG-algebras that are not GK-algebras as the following example shows.

Example 2.1. *Let $A = \{1, a, b, c\}$ and let the operation in A be determined as follows*

$*$	1	a	b	c
1	1	a	b	c
a	a	1	a	a
b	b	b	1	b
c	c	c	a	1

It is easy to verify that $\mathfrak{A} =: (A, *, 1)$ is a BG-algebra ([11], *Example 2.11*) *but it is not a GK-algebra because, for example, we have $(c * b) * (1 * b) = a * b = a \neq c = c * 1$.* □

The following proposition gives some properties of GK-algebras:

Proposition 2.1 ([2], Theorem 3.4)**.** *Let $\mathfrak{A} =: (A, *, 1)$ be a GK-algebra. Then the following holds:*

(1) $(\forall x \in A)(1 * (1 * x) = x)$.

(2) $(\forall x \in A)(x * (1 * (1 * x)) = 1)$.

(3) $(\forall x, y \in a)(1 * x = 1 * y \Longrightarrow x = y)$.

(4) $(\forall x \in A)((x * (1 * x)) * x = x)$.

(5) $(\forall x, y \in A)(1 * (x * y) = y * x)$.

The following two propositions describe two properties of GK-algebra, the second of which is of interest for this article.

Proposition 2.2 ([2], Theorem 3.5)**.** *Left and right cancellation law holds in any GK-algebra.*

Let $\mathfrak{A} =: (A, *, 1_A)$ and $\mathfrak{B} =: (B, \star, 1_B)$ be two GK algebras. Direct product $\mathfrak{A} \times \mathfrak{B}$ is defined ([8], Definition 2.1) as a structure $\mathfrak{A} \times \mathfrak{B} =: (A \times B, \odot, (1_A, 1_B))$, where:

$$(\forall x, y \in A)(\forall u, v \in B)((x, u) \odot (y, v) =: (x * y, u \star v)).$$

Then, we have:

Proposition 2.3 ([8], Theorem 2,2)**.** *Direct product of any two GK-algebras is again a GK algebra.*

3. THE MAIN RESULT

This section is the central part of this paper. It contains five sub-sections. In the first of them, besides proving that the system of axioms, which determines the class of GK-algebras, can be reduced, some new properties of this class of logical algebras were also shown. In addition to the above, it is proved that the direct product of any family of GK-algebras is again a GK-algebra. The next two subsections are devoted to sub-algebras, ideals and (right) congruences in GK-algebras. In subsection 4 we discuss about strong and weak ideals, and p-ideals in GK-algebras. The last subsection discusses about (left, right) distributivity in this class of logical algebras. In connection with this, it was shown that a non-trivial GK-algebra cannot be left (right) distributive.

3.1. Some new properties of GK-algebras.

We first prove an important theorem:

Theorem 3.1. *The formula (v) can be deduced from (GK) and (M).*

Proof. If we put $z = y$ and $y = 1$ in (GK), with respect to (M), we get $(x * y) * (1 * y) = x * 1 = x$. □

The following proposition recognizes another new property of GK-algebras.

Proposition 3.4. *Let $\mathfrak{A} =: (A, *, 1)$ be a GK-algebra. Then:*

(6) $(\forall x, y \in A)(x * y = 1 \iff y * x = 1)$.

Proof. Let $x, y \in A$ be arbitrary elements such that $x * y = 1$. Then, referring to (Re) and (5), we have $y * x = 1 * (x * y) = 1 * 1 = 1$. With this, the implication $x * y = 1 \implies y * x = 1$ is proven. Since the variables x and y are free in it, it is clear that the reverse implication is also valid. □

Remark 3.1. *The previously obtained result contradicts the claim made in* [2]*, Proposition 3.11.*

As a consequence of the previous proposition, we have:

Corollary 3.1. *Let $\mathfrak{A} =: (A, *, 1)$ be a GK-algebra. Then:*

(7) $(1) \iff (2)$.

If we introduce the relation $\leqslant$ in $\mathfrak{A}$ in the standard way

$$(\forall x, y \in A)(x \leqslant y \iff x * y = 1),$$

we get:

Corollary 3.2. *Let $\mathfrak{A} =: (A, *, 1)$ be a GK-algebra. Then:*

(8) $(\forall x, y \in A)(x \leqslant y \iff x = y)$.

Proof. Let $x, y \in A$ be such that $x \leqslant y$. Then $x * y = 1$. Thus $y * x = 1$ by (6). Now, $x * y = 1$ and $y * x = 1$ give $x = y$ according to (An). □

Remark 3.2. *Note that, for better understanding of the following text, the proven formula (8) can be viewed as the next valid formula*

$$(\forall u, v \in A)(u * v \neq 1 \iff u \neq v),$$

because it is the contraposition of formula (8) which can be written in the form

$$(\forall x, y \in A)(x * y = 1 \iff x = y).$$

In addition to the previously mentioned properties, GK-algebra has the following properties:

Proposition 3.5. *Let $\mathfrak{A} =: (A, *, 0)$ be a GK-algebra. Then the following holds:*
(9) $(\forall x, y \in A)(x \neq y \implies x * (x * y) \neq y * (x * y))$.
(10) $(\forall x, y \in A)((x * y) * y = ((x * y) * x) * (y * x))$.
(11) $(\forall x, y \in A)((y * x) * y = ((y * x) * x) * (y * x))$.
(12) $(\forall x, y \in A)((x * y) * ((x * y) * y) = x * (x * y))$.
(13) $(\forall x, y \in A)((x * y) * ((y * x) * y) = x * (y * x))$.
(14) $(\forall x, y \in A)((x * (x * y)) * (y * x) = x)$.
(15) $(\forall x, y \in A)((x * (y * x)) * (x * y) = x)$.

Proof. (9): Let $x, y, z \in A$ be arbitrary elements such that $x \neq y$. If we start with the axiom (GK) and of we put $z = x * y$ in it, we get

$$(x * (x * y)) * (y * (x * y)) = x * y.$$

Since $x \neq y$, it is certain that $x * y \neq 1$, because if $x * y = 1$, we would have $x = y$, according to Corollary 3.2, which is impossible. So, for arbitrary x, y such that $x \neq y$, we have $(x * (x * y)) * (y * (x * y)) \neq 1$. From here, it follows $x * (x * y) \neq y * (x * y)$ which had to be proven.

(10): If we put $x = x * y$ and $z = x$ in (GK), we get (9).

(11): If we put $x = y * x$ and $z = x$ in (GK), we get (10).

(12): If we put $y = x * y$ and $z = y$ in (GK), we get (11).

(13): If we put $y = y * x$ and $z = y$ in (GK), we get (12).

(14): If we put $y = x * y$ in (v), we get $(x * (x * y)) * (1 * (x * y)) = x$ from where, taking into account (5), we get (14).

(15) can be obtained similarly to (14). □

Remark 3.3. *The formula (9) is logically equivalent to the formula*

$$(\forall x, y \in A)(x * (x * y) = y * (x * y) \implies x = y),$$

since it is the contraposition of (8). On the other hand, this last formula is a valid formula according to Proposition 2.2. Argument based on the previous one, one can prove the validity of the formula

$$(\forall x, y \in A)(y \neq x \implies (x * y) * y \neq (x * y) * x).$$

Let $\mathfrak{X} =: \{(A_i, *_i, 1_i) : i \in I\}$ be a family of GK-algebras. If on the set

$$\prod_{i \in I} A_i = \{f : I \longrightarrow \cup_{i \in I} A_i \mid (\forall i \in I)(f(i) \in A_i)\},$$

we define the operation $\odot$ as follows

$$(\forall f, g \in \prod_{i \in I} A_i)(\forall \in I)((f \odot g)(i) =: f(i) *_i g(i)),$$

we created the structure $\sqcap \mathfrak{X} = (\prod_{i \in I} A_i, \odot, f_0)$, where f_0 was chosen as follows

$$(\forall i \in I)(f_0(i) =: 1_i).$$

Before we start working with direct products of GK-algebras, we say that the operation determined in this way is well-defined.

Remark 3.4. *Although there is a small inconsistency in the writing of the direct product of GK-algebras compared to the our usual way of writing logical algebras, it should not lead to misunderstanding.*

If a priori we accept conditions that ensure the existence of non-empty direct product, we can prove the following theorem.

Theorem 3.2. *The direct product of any family of GK-algebras, determined as above, is a GK-algebra.*

Proof. By direct verification, it can be proved that this structure satisfies the axioms of GK-algebra:

Let $f, g, h \in \prod_{i \in I} A_i$ be arbitrary elements and $i \in I$. Then, we have:

(Re) $(f \odot f)(i) = f(i) *_i f(i) = 1_i = f_0(i)$.

(M) $(f \odot f_0)(i) = f(i) *_i f_0(i) = f(i) *_i 1_i = f(i)$.

(An) Assume that $f \odot g = f_0$ and $g \odot f = f_0$ hold for f and g. Let us prove that $f = g$ holds. Since for $i \in I$ we have $f_0(i) = (f \odot g)(i) = f(i) *_i g(i)$, we conclude that $f(i) = g(i)$ in accordance with (8). Therefore, $f = g$ holds.

(GK) Considering that

$$\begin{aligned} ((f \odot h) \odot (g \odot h))(i) &= (f(i) *_i h(i)) *_i (g(i) *_i h(i)) \\ &= f(i) *_i g(i) = (f \odot g)(i), \end{aligned}$$

we have that (GK) is a valid formula for the observed structure.

Therefore, the structure $(\prod_{i \in I} A_i, \odot, f_0)$ is a GK-algebra. □

The preceding theorem is a generalization of Proposition 2.3 ([8], Theorem 2,2).

3.2. **Sub-algebras and ideals.** The concept of sub-algebras in GK-algebras is determined by the standard way:

Definition 3.3 ([2], Definition 3.4). *Let $\mathfrak{A} =: (A, *, 1)$ be a GK-algebra. A non-empty subset S of A is called a sub-algebra of $\mathfrak{A}$ if*

(S1) $(\forall x, y \in A)((x \in S \wedge y \in S) \implies x * y \in S)$.

It can be immediately concluded that it is valid

(S0) $1 \in S$.

Indeed, if S is a non-empty subset of A, then there exists at least some $x \in A$ such that $x \in S$. Then $1 = x * x \in S$ according to (S1) and (Re).

We denote the family of all sub-algebras of the GK-algebra $\mathfrak{A}$ by $\mathfrak{S}(A)$. This family is not empty because $\{1\} \in \mathfrak{S}(A)$ and $A \in \mathfrak{S}(A)$.

The following definition introduces the concept of ideals in a GK-algebra.

Definition 3.4. *Let $\mathfrak{A} = (A, *, 1)$ be a GK-algebra. A non-empty subset J of A is called an ideal in $\mathfrak{A}$ if the following holds:*

(J0) $1 \in J$.

(J1) $(\forall x, y \in A)((x * y \in J \land y \in J) \implies x \in J)$.

The family of all ideals in the GK-algebra $\mathfrak{A} =: (A, *, 1)$ is denoted by $\mathfrak{J}(A)$. This family is not empty because $\{1\} \in \mathfrak{J}(A)$ and $A \in \mathfrak{J}(A)$.

Proposition 3.6. *Let J be an ideal in a GK-algebra $\mathfrak{A} = (A, *, 1)$. Then*

$$(\forall x \in A)(x \in J \implies x * (1 * x) \in J).$$

Proof. Let J be an ideal in a GK-algebra $\mathfrak{A}$ and let $x \in A$ be arbitrary elements such that $x \in J$. Then $(x * (1 * x)) * x = x \in J$ by (4). Thus $x * (1 * x) \in J$ according to (J1). □

Proposition 3.7. *Let J be an ideal in a GK-algebra $\mathfrak{A} = (A, *, 1)$. Then*

$$(\forall x, y \in A)((x * y \in J \land y \in J) \implies x * (y * x) \in J).$$

Proof. Let J be an ideal in $\mathfrak{A}$ and let $x, y \in A$ be such that $x * y \in J$ and $y \in J$. Then, taking into account (15) and, we have $(x * (y * x)) * (x * y) = x \in J$ by (J1). Thus $x * (y * x) \in J$ by (J1) again. □

Example 3.2. *Let $A = \{0, a, b\}$ and let the operation in A be determined as follows*

$*$	1	a	b
1	1	b	a
a	a	1	b
b	b	a	1

It is easy to verify that $\mathfrak{A} =: (A, *, 1)$ is a GK-algebra ([2], *Example 3.2).*

Subset $S_0 = \{1\}$ is a sub-algebra in $\mathfrak{A}$, while the subsets $\{1, a\}$, $\{1, b\}$ are not sub-algebras in $\mathfrak{A}$. So, $\mathfrak{S}(A) = \{S_0, A\}$. Also $\mathfrak{J}(A) = \{S_0, A\}$. □

Example 3.3. *Let $A = \{0, a, b, c\}$ and let the operation in A be determined as follows*

$*$	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

It is easy to verify that $\mathfrak{A} =: (A, *, 1)$ is a GK-algebra ([7], *Example 2.1).*

Subsets $S_0 = \{1\}$, $S_1 =: \{1, a\}$, $S_2 = \{1, b\}$, $S_3 = \{1, c\}$ are sub-algebras in $\mathfrak{A}$, while the subsets $S_4 =: \{1, a, b\}$, $S_5 =: \{1, a, c\}$ and $A_6 =: \{1, b, c\}$ are not sub-algebras in $\mathfrak{A}$. So, $\mathfrak{S}(A) = \{S_0, S_1, S_2, S_3, A\}$. All subalgebras in $\mathfrak{A}$ are ideals in $\mathfrak{A}$ and there are no other ideals in $\mathfrak{A}$. □

In the following statements we show the relations of sub-algebras and ideals for this class of logical algebras as answers to the generally accepted commitment of logical algebras researchers to analyze the relations between sub-algebras and ideals.

Theorem 3.3. *Every sub-algebra of a GK-algebra is an ideal of it.*

Proof. Let S be a sub-algebra in a GK-algebra $\mathfrak{A} = (A, *, 1)$ and let $x, y \in A$ be such that $x * y \in S$ and $y \in S$. Then $1 * y \in S$ because $1 \in S$ by (S0). Thus $x = (x * y) * (1 * y) \in S$ in accordance with (S1) and (v). Therefore, S is an ideal in $\mathfrak{A}$. This means that $\mathfrak{S}(A) \subseteq \mathfrak{J}(A)$. □

It is quite reasonable to ask the question: Does $\mathfrak{S}(A) = \mathfrak{J}(A)$ or $\mathfrak{S}(A) \subset \mathfrak{J}(A)$ hold?

If we accept that the ideal J in the GK-algebra $\mathfrak{A} = (A, *, 1)$ is a sub-algebra in $\mathfrak{A}$, then we can prove:

Proposition 3.8. *For the ideal J in the GK-algebra $\mathfrak{A} = (A, *, 1)$ which is also a sub-algebra in $\mathfrak{A}$, the following holds:*

(16) $(\forall x, y \in A)(x * y \in J \implies y * x \in J)$.

(17) $1 * J =: \{1 * x : x \in J\} \subseteq J$.

Proof. Let $x, y \in A$ be arbitrary elements such that $x * y \in J$. Then $y * x = 1 * (x * y) \in J$ with respect to (8), (J0) and (S1).

The validity of statement (17) immediately follows from the validity of statement (16) for $y = 1$. □

The previous proposition is a motive for the following definition:

Definition 3.5. *For an ideal J in a GK-algebra $\mathfrak{A}$, we say that it is a closed ideal in $\mathfrak{A}$ if $1 * J \subseteq J$ holds.*

Now we can prove the converse of Theorem 3.3.

Theorem 3.4. *Every closed ideal in a GK-algebra is a sub-algebra in it.*

Proof. Let J be a closed ideal in a GK-algebra $\mathfrak{A} = (A, *, 1)$ and let $x \in J$ and $y \in J$ be arbitrary elements. Then $1 * y \in J$ by the accepted assumption. On the other hand, we have $(x * y) * (1 * y) = x \in J$ according to (v). From here we get $x * y \in J$ in accordance with (J1). This proves that J is a sub-algebra in $\mathfrak{A}$. □

Also, we have:

Proposition 3.9. *Let $\{(A_i, *_i, 1_i) : i \in I\}$ be a family of GK-algebras, K be a subset of I and let S_i be a sub-algebra (an ideal) in $(A_i, *_i, 1_i)$ for each $i \in K$. Then $\prod_{i\in I} T_i$, where $T_i = S_i$ for $i \in K$ and $T_i = A_i$ for $i \in I \setminus K$, is a sub-algebra (an ideal, respectively) in the GK-algebra $\prod_{i\in I} A_i$.*

Proof. If $K = \emptyset$, then $\prod_{i\in I} T_i = \prod_{i\in I} A_i$, so $\prod_{i\in I} T_i$ is certainly a sub-algebra (an ideal) in $\prod_{i\in I} A_i$. Assume, therefore, that $K \neq \emptyset$.

Assume that S_i is an ideal in $(A_i, *_i, 1_i)$ for every $i \in K$. Let $x, y \in \prod_{i\in I} A_i$ be such that $x \odot y \in \prod_{i\in I} T_i$ and $y \in \prod_{i\in I} T_i$. Then $(x \odot y)(i) = x(i) *_i y(i) \in S_i$ and $y(i) \in S_i$ for each $i \in K$. Thus $x(i) \in S_i$ since S_i is an ideal in $(A_i, *_i, 1_i)$ for each $i \in K$. Hence, $x \in \prod_{i\in I} T_i$. So, $\prod_{i\in I} T_i$ is an ideal in $\prod_{i\in I} A_i$.

Assume, now, that S_i is a sub-algebra in $(A_i, *_i, 1_i)$ for every $i \in K$. Let $x, y \in \prod_{i\in I} A_i$ be such that $x \in \prod_{i\in I} T_i$ and $y \in \prod_{i\in I} T_i$. This means $x(i) \in S_i$ and $y(i) \in S_i$ for each $i \in K$. Then $(x \odot y)(i) = x(i) *_i y_i \in S_i$ since S_i is a sub-algebra in $(A_i, *_i, 1_i)$ for each $i \in K$. Hence $x \odot y \in \prod_{i\in I} T_i$. □

Example 3.4. *Let $\mathfrak{A} = \{1, a, b\}$ as in Example 3.2. Then $(A, *, 1)$ is a GK-algebra. Also, the products $\mathfrak{A} \times \mathfrak{A}$ and $\mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}$ are GK-algebras according to Theorem 3.2. Since $\{1\}$ is a sub-algebra (an ideal) in $\mathfrak{A}$, then:*

(i) *Subsets $\{1\} \times A$, $A \times \{1\}$, $\{(1, 1)\}$ and $A \times A$ are sub-algebras (ideals, respectively) in $\mathfrak{A} \times \mathfrak{A}$.*

(ii) *Subsets $\{1\} \times A \times A$, $\{1\} \times \{1\} \times A$, $\{(1, 1, 1)\}$, $A \times \{1\} \times A$, $A \times \{1\} \times \{1\}$, $A \times A \times \{1\}$, $\{1\} \times A \times \{1\}$ and $A \times A \times A$ are sub-algebras (ideals) in the GK-algebra $\mathfrak{A} \times \mathfrak{A} \times \mathfrak{A}$.* □

Therefore, in both previously observed cases, $\mathfrak{J}(A)$ is not empty and besides trivial ideals it has some non-trivial ideals.

It can be demonstrated that:

Theorem 3.5. *The families $\mathfrak{S}(A)$ and $\mathfrak{J}(A)$ are complete lattices.*

Proof. Let $\{T_i\}_{i\in I}$ be a family of sub-algebras (ideals) in the GK-algebra $\mathfrak{A} =: (A, *, 1)$. We will demonstrate the proof for the family $\mathfrak{S}(A)$. The proof for the family $\mathfrak{J}(A)$ can be demonstrated analogously.

Suppose T_i are sub-algebras for each $i \in I$. If $x, y \in A$ are such that $x \in \cap_{i\in I}T_i$ and $y \in \cap_{i\in I}T_i$, then $x \in T_i$ and $y \in T_i$ for every $i \in I$. From here, we get $x * y \in T_i$, since T_i is a sub-algebra in $\mathfrak{A}$. Thus, $x * y \in \cap_{i\in I}T_i$.

Let $\mathcal{Z}$ be the family of all sub-algebras in $\mathfrak{A}$ that contain $\cup_{i\in I}T_i$. Then, according to the first part of this proof, $\cap\mathcal{Z}$ is a sub-algebra in $\mathfrak{A}$.

If we put $\sqcap_{\imath\in I}T_i = \cap_{i\in I}T_i$ and $\sqcup_{u\in I} = \cap\mathcal{Z}$, then $(\mathfrak{S}(A), \sqcap, \sqcup)$ is a complete lattice. □

Corollary 3.3. *Let $\mathfrak{A} =: (A, *, 0)$ be a GK-algebra. For each $x \in A$ there exists a smallest sub-algebra/ideal S_x in $\mathfrak{A}$ such that $x \in S_x$.*

Proof. If $\mathcal{Z}$ is the family of all sub-algebras/ideals in $\mathfrak{A}$ that contain x, then $S_x = \cap\mathcal{Z}$ is a sub-algebra (an ideal) in $\mathfrak{A}$ that contains x according to the previous theorem. Let S be a sub-algebra (an ideal) in $\mathfrak{A}$ containing x. Then $S \in \mathcal{Z}$. Thus $S_x \subseteq S$. Therefore, S_x is a minimal sub-algebra/ideal in $\mathfrak{A}$ containing x. □

3.3. **Ideals and congruences.** For an equivalence ρ on the support A of the GK-algebra $\mathfrak{A} =: (A, *, 1)$ we say that it is a right congruence on $\mathfrak{A}$ if the following holds

$$(\forall x, y, z \in A)((x, y) \in \rho \implies (x * z, y * z) \in \rho).$$

The concept of left congruence is defined analogously. For an equivalence on the GK-algebra $\mathfrak{A}$, we say that it is a congruence on $\mathfrak{A}$ if it is a left and right congruence on $\mathfrak{A}$.

The following theorem relates any ideal in the GK-algebra $\mathfrak{A} =: (A, *, 1)$ to a right congruence on $\mathfrak{A}$.

Theorem 3.6. *For a given ideal J of the GK-algebra $\mathfrak{A} =: (a, *, 1)$, the relation ρ_J, defined as follows*

$$(\forall x, y \in A)((x, y) \in \rho_J \iff (x * y \in J \wedge y * x \in J)),$$

is a right congruence on $\mathfrak{A}$ such that for the kernel $[1]_{\rho_J} =: \{x \in A : (x, 1) \in \rho_J\}$ of the right congruence ρ_J, the following $[1]_{\rho_J} \subseteq J$ is valid.

Proof. Since it is obvious that the relation ρ_J is reflexive and symmetric, it remains to prove transitivity. Let $x, y, z \in A$ be such that $(x, y) \in \rho_J$ and $(y, z) \in \rho_J$. This means $x * y \in J$, $y * x \in J$, $y * z \in J$ and $z * y \in J$. Now, from $(x * z) * (y * z) = x * y \in J$ and $y * z \in J$ we get $x * z \in J$ according to (J1). Also, from $(z * x) * (y * x) = z * y \in J$ and $y * x \in J$, in accordance with (J1), we get $z * x \in J$. Therefore, $(x, z) \in \rho_J$ whereby the transitivity of the relation ρ_J is proved. On the other hand, for arbitrary elements $x, y, z \in A$ such that $(x, y) \in \rho_J$, according to (GK), we have $(x * z) * (y * z) = x * y \in J$ and $(y * z) * (x * z) = y * x \in J$. So, $(x * z, y * z) \in \rho_J$. Therefore, ρ_J is a right congruence on $\mathfrak{A}$.

For arbitrary $u \in [1]_{\rho_J}$, we have $(u, 1) \in \rho_J$ which means $u * 1 = u \in J$ and $1 * u \in J$. Thus, $[1]_{\rho_J} \subseteq J$. □

If we consider the ideal J in a GK-algebra $\mathfrak{A} = (A, *, 1)$ which is also a sub-algebra in $\mathfrak{A}$, then the relation ρ_J can be defined by

$$(\forall x, y \in A)((x, y) \in \rho_J \iff x * y \in J)$$

according to (14).

Let us now consider what we can learn about the kernel $[1]_\rho$ of a right congruence ρ on a GK-algebra $\mathfrak{A} =: (A, *, 1)$.

Theorem 3.7. *If ρ is a right congruence on a GK-algebra $\mathfrak{A} =: (A, *, 1)$, then the kernel $[1]_\rho$ of this congruence is both a sub-algebra and an ideal in $\mathfrak{A}$.*

Proof. Surely $1 \in [1]_\rho$ because $(1, 1) \in \rho$ since ρ is reflexive. Let $x, y \in A$ be such that $x \in [1]_\rho$ and $y \in [1]_\rho$. This means $(x, 1) \in \rho$ and $(y, 1) \in \rho$. Then $(x * y, y) \in \rho$ because ρ is a right congruence and $(y, 1) \in \rho$. Thus $(x * y, 1) \in \rho$ since ρ is a transitive relation. Hence, $x * y \in [1]_\rho$. This proves that $[1]_\rho$ is a sub-algebra in $\mathfrak{A}$.

Let $x, y \in A$ be such that $x * y \in [1]_\rho$ and $y \in [1]_\rho$. Then $1 * y \in [1]_\rho$ because $[1]_\rho$ is a sub-algebra in $\mathfrak{A}$. Also, we have $(x * y) * (1 * y) \in [1]_\rho$ since $[1]_\rho$ is a sub-algebra in $\mathfrak{A}$. Since $(x * y) * (1 * y) = x$ according to (v), we get $x \in [1]_\rho$, which proves that (J1) holds. Therefore, $[1]_\rho$ is an ideal in $\mathfrak{A}$. □

However, if ρ is a congruence on the GK-algebra $\mathfrak{A} =: (A, *, 1)$, then we can create the structure $\mathfrak{A}/\rho =: (A/\rho, \otimes, [1]_\rho)$. Since ρ is a congruence on $\mathfrak{A}$, the interior operation $\otimes$ on $\mathfrak{A}/\rho$, defined as follows

$$(\forall x, y \in A)([x]_\rho \otimes [y]_\rho =: [x * y]_\rho),$$

is well-defined. Indeed, if $[x]_\rho = [u]_\rho$ and $[y]_\rho = [v]_\rho$, i.e. if $(x, u) \in \rho$ and $(y, v) \in \rho$, then we have $(x * y, u * y) \in \rho$ and $(u * y, u * v) \in \rho$. From here, we get $(x * y, u * v) \in \rho$. This means $[x * y]_\rho = [u * v]_\rho$, i.e. $[x]_\rho \otimes [y]_\rho = [u]_\rho \otimes [v]_\rho$.

By direct checking, with a little more attention, it can be proven:

Theorem 3.8. *If ρ is a congruence on a GK-algebra $\mathfrak{A} =: (A, *, 1)$, then the structure $\mathfrak{A}/\rho =: (A/\rho, \otimes, [1]_\rho)$ is a GK-algebra.*

Proof. Since the validity of propositions (Re), (M) and (GK) can be proved by direct verification, let us show how the validity of proposition (An) can be demonstrated. According to Proposition 3.4 and the comment in Remark 3.2, it suffices to prove the implication $[x]_\rho \otimes [y]_\rho = [1]_\rho \implies [x]_\rho = [y]_\rho$.

Let $x, y \in A$ be such that $[x]_\rho \otimes [y]_\rho = [1]_\rho$. Then $[x * y]_\rho = [1]_\rho$. This means $(x * y, 1) \in \rho$, Thus $((x * y) * (1 * y), 1 * (1 * y)) \in \rho$ since ρ is a congruence on $\mathfrak{A}$. Hence, $(x, y) \in \rho$ according to (5) and (1). So, $[x]_\rho = [y]_\rho$. □

In what follows, we need the following definition:

Definition 3.6 ([8], Definition 2.6). *Let $\mathfrak{A} =: (A, *, 1_A)$ and $\mathfrak{B} = (B, \star, 1_B)$ be GK-algebras. A mapping $f : A \longrightarrow B$ is called homomorphism if the following holds:*

(f1) $(\forall x, y \in A)(f(x * y) = f(x) \star f(y))$.

It can immediately be concluded that it is also valid

(f0) $f(1_A) = 1_B$

Indeed, for an arbitrary $x \in A$, we have $f(1_A) = f(x * x) = f(x) \star f(x) = 1_B$.

We denote the homomorphism between GK-algebras $\mathfrak{A}$ and $\mathfrak{B}$ by $f : \mathfrak{A} \longrightarrow \mathfrak{B}$. The terms epimorphism and monomorphism are understood here in the usual way.

The following theorem is related to the previous one:

Theorem 3.9. *Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism between GK-algebras.*

(a) *If S is a sub-algebra in $\mathfrak{A}$, then $f(S)$ is a sub-algebra in $\mathfrak{B}$.*

(b) *$f(\mathfrak{A})$ is a sub-algebra in $\mathfrak{B}$.*

(c) *If K is an ideal in $\mathfrak{B}$, then $f^{-1}(K) = \{x \in A : f(x) \in K\}$ is an ideal in $\mathfrak{A}$.*

(d) *Kernel $Ker(f) = f^{-1}(\{1_B\})$ of the homomorphism f is an ideal in $\mathfrak{A}$.*

(e) *The relation ρ_f, defined as follows*

$$(\forall x, y \in A)((x, y) \in \rho_f \iff f(x) = f(y)),$$

is a congruence on $\mathfrak{A}$.

(f) *The function $\pi : \mathfrak{A} \longrightarrow \mathfrak{A}/\rho$, defined by $\pi(x) =: [x]_\rho$ is an epimorphism.*

(g) *The function $g : \mathfrak{A}/\rho \longrightarrow f(\mathfrak{A})$, defined by $g([x]_\rho) =: f(x)$ is a monomorphism.*

(h) *The homomorphism f can be represented as a superposition $f = i \circ g \circ \pi$, where $i : f(\mathfrak{A}) \longrightarrow \mathfrak{B}$ is the inclusion.*

Proof. Let $u, v \in B$ be arbitrary elements such that $u \in f(S)$ and $v \in f(S)$. Then there are elements $x, y \in S$ such that $u = f(x)$ and $v = f(y)$. This $u \star v = f(x) \star f(y) = f(x * y) \in f(S)$ since $x * y \in S$ because S is a sub-algebra in $\mathfrak{A}$.

The claim (b) is a special case of the claim (a), so its validity follows from the validity of the claim (a).

Let $x, y \in A$ be arbitrary elements such that $x * y \in f^{-1}(K)$ and $y \in f^{-1}(K)$. This means $f(x) \star f(y) = f(x * y) \in K$ and $f(y) \in K$. This $f(x) \in K$ because K is an ideal in $\mathfrak{B}$. Hence $x \in f^{-1}(K)$.

The claim (d) is a special case of the claim (c), so its validity follows from the validity of the claim (c) since $\{1_B\}$ is an ideal in $\mathfrak{B}$.

The proofs of statements (e)-(h) can be demonstrated by direct verification. For example, the validity of statement (h) follows from the following equalities: For arbitrary $x \in A$, we have

$$(i \circ g \circ \pi)(x) = (i \circ g)([x]_{\rho_f}) = i(g([x]_{\rho_f})) = i(f(x)) = f(x). \qquad \square$$

In the following example we give one of the ways of creating right congruences on the GK-algebra $\prod_{i \in I} A_i$.

Example 3.5. *Let I be a set and $\mathfrak{J}$ be a subfamily of $\mathcal{P}(I)$ such that*

$$\emptyset \in \mathfrak{J},\ (A \subseteq B \wedge B \in \mathfrak{J}) \Longrightarrow A \in \mathfrak{J},\ (A \in \mathfrak{J} \wedge B \in \mathfrak{J}) \Longrightarrow A \cup B \in \mathfrak{J}.$$

If $\{(A_i, *_i, 1_i)\}_{i \in I}$ is a family of GK-algebras, then the relation q on $\prod_{i \in I} A_i$, defined by

$$(x, y) \in q \iff \{i \in I : x(i) \neq y(i)\} \in \mathfrak{J},$$

is an equality relation on the product $\prod_{i \in I} A_i$ right compatible with the operation in $\prod_{i \in I} A_i$.

Let $x \in \prod_{i \in I} A_i$ be arbitrary element. Then $\{i \in I : x(i) \neq x(i)\} = \emptyset \in \mathfrak{J}$. This means $(x, x) \in q$ for every $x \in \prod_{i \in I} A_i$, that is, the relation q is reflexive. It is obvious that q is a symmetric relation. Let us prove that q is transitive. Let $x, y, z \in \prod_{i \in I} A_i$ be such that $(x, y) \in q$ and $(y, z) \in q$. This means $\{i \in I : x(i) \neq y(i)\} \in \mathfrak{J}$ and $\{i \in I : y(i) \neq z(i)\} \in \mathfrak{J}$. Then

$$\{i \in I : x(i) \neq z(i)\} \subseteq \{i \in I : x(i) \neq y(i)\} \cup \{i \in I : y(i) \neq z(i)\} \in \mathfrak{J}.$$

Thus $\{i \in I : x(i) \neq z(i)\} \in \mathfrak{J}$. Hence $(x, z) \in q$. So, q is an equivalence on $\prod_{i \in I} A_i$.

Let $x, y, z \in \prod_{i \in I} A_i$ be arbitrary elements such that $(x, y) \in q$, i.e. $\{i \in I : x(i) \neq y(i)\} \in \mathfrak{J}$. Recall that holds

$$(\forall i \in I)(x(i) = y(i) \Longrightarrow x(i) *_i z(i) = y(i) *_i z(i))$$

since the operation $*_i$ is extensive with respect to the equality $=$, and that the contraposition also holds

$$(\forall i \in I)((x \odot z)(i) \neq (y \odot z)(i) \implies x(i) \neq y(i)).$$

Now, we have

$$\{i \in I : (x \odot z)(i) \neq (y \odot z)(i)\} \subseteq \{i \in I : x(i) \neq y(i)\} \in \mathfrak{J}.$$

Thus $\{i \in I : (x \odot z)(i) \neq (y \odot z)(i)\} \in \mathfrak{J}$. *Therefore* $(x \odot z, y \odot z) \in q$. □

3.4. **Some types of ideals.** Here we introduce and analyze three types of ideals in GK-algebras.

Definition 3.7. *A non-empty subset J of a GK-algebra $\mathfrak{A} =: (A, *, 1)$ is called a strong ideal in $\mathfrak{A}$ if it satisfies (J0) and the following condition:*

(StJ) $(\forall x, y, z \in A)(((x * y) * z \in J \wedge y \in J) \implies x * z \in J)$.

The family of all strong pre-ideals in a GK-algebra $\mathfrak{A}$ is denoted by $\mathfrak{J}_s(A)$.

Proposition 3.10. *Any strong ideal in a GK-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$. This means that $\mathfrak{J}_s(A) \subseteq \mathfrak{J}(A)$.*

Proof. Putting $z = 0$ in (StJ), we obtain (J1). □

It is obvious that the following holds:

Proposition 3.11. *In every GK-algebra $\mathfrak{A} =: (A, *, 1)$, the subset $S_0 =: \{1\}$ is a strong ideal in $\mathfrak{A}$.*

Definition 3.8. *Let $\mathfrak{A} =: (A, *, 1)$ be a GK-algebra. A nonempty subset J of the set A is a weak ideal in $\mathfrak{A}$ if it satisfies (J0) and the following condition*

(WJ) $(\forall x, y, z \in A)((x * (y * z) \in J \wedge y \in J) \implies x * z \in J)$.

We denote the family of all weak ideals in a GK-algebra $\mathfrak{A}$ by $\mathfrak{J}_w(A)$.

Proposition 3.12. *Any weak ideal in a GK-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$.*

Proof. If we put $z = 1$ in (WJ), we get (J1). □

Proposition 3.13. *Every weak ideal in a GK-algebra is closed.*

Proof. Let J be a weak ideal in a GK-algebra $\mathfrak{A} =: (A, *, 1)$ and let $x \in J$ be an arbitrary element. Then $1 * (1 * x) = x \in J$ and $1 \in J$ in accordance with (1) and (J0). Thus $1 * x \in J$ by (WJ). □

Corollary 3.4. *Every weak ideal in a GK-algebra is a sub-algebra.*

Proof. Let J be a weak ideal in a GK-algebra $\mathfrak{A}$. Then J is an closed ideal in $\mathfrak{A}$ according previous proposition and Proposition 3.12. Thus, J is a sub-algebra in $\mathfrak{A}$ by Theorem 3.4. □

Using standard techniques, the following theorem can be proven without much difficulty:

Theorem 3.10. *Let $\mathfrak{A} =: (A, *, 1)$ be a GK-algebra. The families $\mathfrak{J}_s(A)$ and $\mathfrak{J}_w(A)$ are complete lattices.*

Since determination of p-ideals in many logical algebras is well known, we introduce the notion of p-pre-ideals in GK-algebras by a standard way:

Definition 3.9. *A nonempty subset J in a GK-algebra $\mathfrak{A} =: (A, *, 0)$ is called a p-ideal in $\mathfrak{A}$ if it satisfies:*

(J0) $0 \in J$.

(PJ) $(\forall x, y, z \in A)((x * z) * (y * z) \in J \wedge y \in J) \implies x \in J)$.

We immediately conclude:

Proposition 3.14. *Every ideal in GK-algebra $\mathfrak{A}$ is a p-ideal in $\mathfrak{A}$ and vice versa.*

Proof. If we put $z = 1$ in (PJ), we get (J1).

Conversely, let J be an ideal in $\mathfrak{A}$ and let $x, y, z \in A$ be such that $(x * z) * (y * z) \in J$ and $y \in J$. Then $x * y = (x * z) * (y * z) \in J$ and $y \in J$ according (GK). Thus $x \in J$ by (J1). Hence, J is a p-ideal in $\mathfrak{A}$. □

3.5. **Self-distributive GK-algebras.** In the article [2], the concept of self-distributability in GK-algebras was introduced as follows:

Definition 3.10 ([2], Definition 3.9)**.** *Let $\mathfrak{A} =: (A, *, 1)$ be a GK-algebra.*

(a) For $\mathfrak{A}$ we say that it is a right-distributive GK-algebra if, additionally, it holds

(RD) $(\forall x, y, z \in A)((x * y) * z = (x * z) * (y * z))$.

(b) For $\mathfrak{A}$ we say that it is a left distributive GK-algebra if, in addition, it holds

(LD) $(\forall x, y, z \in A)(x * (y * z) = (x * y) * (x * z))$.

However, we have:

Proposition 3.15. *GK-algebra cannot be right distributive.*

Proof. If $\mathfrak{A} =: (A, *, 1)$ were a right distributive GK-algebra, then we would have $(\forall z \in A)(1 * z = 1)$ for arbitrary $z \in A$. Indeed, from (GK) and (RD), it follows that $(x * y) * z = x * y$. If we choose $x = y$, from here, with respect to (Re), we get $1 * z = 1$. The obtained result is a contradiction to (5). Therefore, any non-trivial GK-algebra cannot be right distributive. □

Also, we have:

Proposition 3.16. *A non-trivial GK-algebra $\mathfrak{A} =: (A, *, 1)$ cannot be left distributive.*

Proof. Let $\mathfrak{A} =: (A, *, 1)$ be a left distributive GK-algebra and let $x \in A$ be arbitrary element. Then $x * 1 = x * (x * x) = (x * x) * (x * x) = 1$ holds, which contradicts to (M). Therefore, any GK-algebra cannot be left distributive. □

4. Final comments and space of potential research

The concept of GK-algebras was introduced in 2018 by R. Gowri and J. Kavitha. Although this class of logical algebras has not met with much interest from researchers, it still seems to open up space for further research. So, for example, one should find an example of an ideal that is not a sub-algebra, if, at all, such a thing exists. Further, determinations of some new types of ideals in GK-algebras could be investigated. In addition to the above, it could be interesting to study some extensions of GK-algebras such as, for example, pseudo/hyper-extension.

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