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Generalized Density Convergence

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ABSTRACT. This paper introduces a unified framework for sequence convergence based on Chun's (1986) generalized density $d_{M,G,F}$. We define $d_{M,G,F}$-convergence, prove linearity and regularity, and show that classical, statistical, uniform statistical (Buck), filter, ideal, and matrix convergence arise as special cases for suitable choices of M, G, F. We also present explicit parameter choices that yield *new* convergence types, provide distinguishing examples establishing proper inclusions among the induced spaces, and develop the associated summability theory (linearity, regularity, and continuity in a natural topology).

1. INTRODUCTION

The study of sequence convergence has been a central theme in mathematical analysis and summability theory for over a century. While classical convergence provides a fundamental tool, its limitations in handling sequences that exhibit irregular behavior on "large" sets motivated the development of various generalized convergence concepts. The parallel development of statistical convergence, ideal convergence, filter convergence, and matrix methods has created a rich but fragmented landscape of generalized convergence theories, each with its own framework and applications.

Statistical convergence, independently introduced by Fast [12] and Steinhaus [19], allows a sequence to converge "almost everywhere" with respect to asymptotic density. This paradigm proved particularly valuable in analysis and probability theory, where it provides a natural framework for studying sequences that converge except on sparse sets. The subsequent extensions—uniform statistical convergence [4], filter convergence [3], ideal convergence [17], and matrix convergence [2]—each addressed specific limitations while finding applications in diverse areas including ergodic theory, number theory, and functional analysis.

In ergodic theory, for instance, statistical convergence appears naturally in the context of pointwise ergodic theorems where classical convergence fails. In approximation theory, statistical convergence provides the appropriate framework for studying the effectiveness of approximation processes that may fail on small sets. The increasing applications of these generalized convergence concepts across mathematics highlight the need for a unified framework that can systematically relate and compare these different approaches.

A significant breakthrough came with Chun's 1986 doctoral thesis [6], which introduced a remarkably general density concept $d_{M,G,F}$ defined through a sequence of non-negative regular matrices M and two filters G and F. This density subsumes most known densities, including asymptotic density, uniform density, various matrix densities, and 0-1 densities associated with ideals. Despite its unifying potential, Chun's framework has received limited attention in the decades since its introduction, with most research continuing along the separate development paths of statistical convergence, ideal convergence, and related concepts.

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Recent work has seen increased interest in the interactions between different convergence types. Balázs and Zamborsky [1] investigated I-convergence in metric spaces, while Das et al. [11] extended ideal convergence to double sequences. Connor [9] explored matrix summability with respect to a modulus, and Burgin and Duman [5] examined the relationship between statistical convergence and convergence in statistics. However, these works largely operate within their specialized frameworks without exploiting the full unifying potential of Chun's approach.

In this paper, we leverage Chun's generalized density to develop a comprehensive unified theory of sequence convergence that encompasses and extends the aforementioned concepts. Our work demonstrates that $d_{M,G,F}$-convergence provides not merely a theoretical unification but a practical framework for generating new convergence types tailored to specific applications. The flexibility in choosing parameters allows researchers to design convergence notions that precisely capture the "size" concepts relevant to their particular problems—whether in harmonic analysis, where different notions of density correspond to different summation methods for Fourier series, or in number theory, where various density concepts naturally arise.

Our main contributions are:

(1) We define $d_{M,G,F}$-convergence as a comprehensive convergence framework and establish its fundamental properties, including linearity and regularity.
(2) We demonstrate that most known convergence types emerge as special cases of our general definition through appropriate choices of parameters M, G, and F.
(3) We introduce several new convergence types not previously studied in the literature and provide concrete examples illustrating their distinct nature.
(4) We establish a comprehensive hierarchy of inclusion relationships among various convergence spaces, characterizing when one convergence type implies another.
(5) We develop the associated summability theory for $d_{M,G,F}$-convergence, proving linearity, regularity, and other desirable properties of the corresponding summability methods.

The unified perspective offered by our framework has significant implications for both theory and applications. Theoretically, it reveals the fundamental connections between seemingly disparate convergence concepts and provides a systematic approach to extending results from one convergence type to others. Practically, it offers researchers a flexible tool for designing convergence notions adapted to specific problems, particularly in areas like signal processing where different notions of "average behavior" correspond to different applications.

This diversity of convergence types, while enriching, also presents a challenge: how to select the most appropriate one for a given problem. For instance, in signal processing, one might want to ignore sparse, high-intensity noise, a goal naturally aligned with statistical convergence. In number theory, studying the local distribution of primes requires a density sensitive to arithmetic structures, which uniform density (Buck convergence) can provide. The true utility of a unified framework like Chun's $d_{M,G,F}$ lies not only in its power to subsume existing theories but also in its capacity to generate new, context-sensitive convergence notions. It provides a "toolbox" allowing researchers to tailor the concepts of 'largeness' and 'convergence' by strategically choosing M, G, and F to match the specific mathematical structure of their problem.

The paper is organized as follows: Section 2 contains preliminary definitions and results. Section 3 introduces our main concept of $d_{M,G,F}$-convergence and establishes its basic properties. Section 4 demonstrates how known convergence types are recovered as special cases. Section 5 investigates the hierarchy of convergence spaces. Section 6 develops the associated summability theory. Finally, Section 7 contains concluding remarks and directions for future research.

2. PRELIMINARIES

Throughout this paper, we employ the following notation: $\mathbb{N}$ denotes the set of positive integers, ω the space of all real sequences, and $2^{\mathbb{N}}$ the power set of $\mathbb{N}$. We assume familiarity with basic concepts in real analysis and summability theory.

We begin by recalling fundamental definitions that form the building blocks of our generalized framework.

Definition 2.1. *(Regular Matrix). ([16]) An infinite matrix $A = (a_{nk})$, where $n, k \in \mathbb{N}$, is called **regular** if it satisfies the Silverman-Toeplitz conditions:*

(1) There exists $M > 0$ such that $\sup_n \sum_{k=1}^{\infty} |a_{nk}| \leq M < \infty$,
(2) For each fixed $k \in \mathbb{N}$, $\lim_{n\to\infty} a_{nk} = 0$,
(3) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$.

A regular matrix preserves classical limits: if $x_n \to L$, then $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k \to L$.

Definition 2.2. *(Non-negative Regular Matrix). ([16]) A regular matrix $A = (a_{nk})$ is called **non-negative** if $a_{nk} \geq 0$ for all $n, k \in \mathbb{N}$. Such matrices preserve non-negativity and order relations between sequences.*

Definition 2.3. *(Filter). ([3]) A family $\mathcal{F} \subset 2^{\mathbb{N}}$ is called a **filter** on $\mathbb{N}$ if*

(1) $\emptyset \notin \mathcal{F}$,
(2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$ (closed under finite intersections),
(3) If $A \in \mathcal{F}$ and $A \subset B \subseteq \mathbb{N}$, then $B \in \mathcal{F}$ (closed under supersets).

Definition 2.4. *(Fréchet Filter). The Fréchet filter $\mathcal{F}_0$ is defined as: $\mathcal{F}_0 = \{A \subseteq \mathbb{N} : A^c \text{ is finite}\}$ This filter consists of all cofinite subsets of $\mathbb{N}$ and plays a fundamental role in classical convergence.*

Definition 2.5. *(Cesàro Matrix). The Cesàro matrix $C = (c_{nk})$ is defined by*

$$c_{nk} = \begin{cases} \frac{1}{n} & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}$$

For any sequence $x = (x_n)$, the Cesàro transform is given by

$$(Cx)_n = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

The Cesàro matrix is regular and non-negative.

Definition 2.6. *(Forward Shift Operator). The forward shift operator N is defined by $(Nx)_n = x_{n+1}$ for any sequence $x = (x_n)$. This operator will be used in constructing matrices for uniform density.*

Definition 2.7. *(Ideal). ([17]) A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is called an **ideal** on $\mathbb{N}$ if*

(1) $\emptyset \in \mathcal{I}$;
(2) If $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$ (closed under finite unions);
(3) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$ (hereditary);
(4) $\mathbb{N} \notin \mathcal{I}$ (proper).

*An ideal is **non-trivial** if it contains all finite subsets of $\mathbb{N}$.*

Definition 2.8. *(Zeroclass / Ideal)([6]). A **zeroclass** (or non-trivial ideal) is a family $\mathcal{X} \subset 2^{\mathbb{N}}$ satisfying*

(1) Every finite set belongs to $\mathcal{X}$*;*
(2) If $A, B \in \mathcal{X}$ *then* $A \cup B \in \mathcal{X}$*;*
(3) If $A \subset B$ *and* $B \in \mathcal{X}$*, then* $A \in \mathcal{X}$*;*
(4) $\mathbb{N} \notin \mathcal{X}$*.*

Zeroclasses represent families of "small" subsets of $\mathbb{N}$ *in various generalized convergence theories.*

Proposition 2.1. *(Dual Filter). If* $\mathcal{X}$ *is a zeroclass, then the family*

$$\mathcal{F}_{\mathcal{X}} = \{A \subseteq \mathbb{N} : A^c \in \mathcal{X}\}$$

is a filter on $\mathbb{N}$ *finer than the Fréchet filter* $\mathcal{F}_0$*.*

Proof. We verify the filter axioms
(1) $\emptyset \notin \mathcal{F}_{\mathcal{X}}$ since $\emptyset^c = \mathbb{N} \notin \mathcal{X}$
(2) If $A, B \in \mathcal{F}_{\mathcal{X}}$, then $A^c, B^c \in \mathcal{X}$, so $(A \cap B)^c = A^c \cup B^c \in \mathcal{X}$, hence $A \cap B \in \mathcal{F}_{\mathcal{X}}$
(3) If $A \in \mathcal{F}_{\mathcal{X}}$ and $A \subset B$, then $B^c \subset A^c \in \mathcal{X}$, so $B^c \in \mathcal{X}$, hence $B \in \mathcal{F}_{\mathcal{X}}$

Since $\mathcal{X}$ contains all finite sets, if A is cofinite then A^c is finite, so $A \in \mathcal{F}_{\mathcal{X}}$. Thus, $\mathcal{F}_{\mathcal{X}}$ is finer than $\mathcal{F}_0$. □

The cornerstone of our unified framework is Chun's generalized density concept. We begin by recalling the axiomatic characterization of lower asymptotic densities.

Definition 2.9 (Upper asymptotic density)**.** *For* $A \subseteq \mathbb{N}$ *define*

$$\overline{\delta}(A) := \limsup_{n\to\infty} \frac{|A \cap \{1, \dots, n\}|}{n}.$$

Throughout this paper, we will use the notation $\overline{\delta}(A)$ *interchangeably with* $\overline{\delta}(A)$*.*

Definition 2.10 (Density–1 filter)**.** *Let*

$$\mathcal{D} := \{B \subseteq \mathbb{N} : \ \overline{\delta}(B) = 1\}.$$

Then $\mathcal{D}$ *is a filter on* $\mathbb{N}$ *(often called the density–1 filter).*

Definition 2.11 (Lower Density Axioms)**.** *A set function* $\delta : 2^{\mathbb{N}} \to [0,1]$ *is called a **lower density** if it satisfies the following axioms for all* $A, B \subseteq \mathbb{N}$*:*
(D1) $\delta(\emptyset) = 0$ *and* $\delta(\mathbb{N}) = 1$*;*
*(D2) **Monotonicity:** If* $A \subseteq B$*, then* $\delta(A) \leq \delta(B)$*;*
*(D3) **Finite perturbation invariance:** If* $A \triangle B$ *is finite (i.e., A and B differ by finitely many elements), then* $\delta(A) = \delta(B)$*;*
*(D4) **Finite sets have zero density:** If A is finite, then* $\delta(A) = 0$*.*

*The corresponding **upper density** is defined as* $\overline{\delta}(A) = 1 - \delta(A^c)$*.*

We now present the main density concept that underpins our entire framework.

Definition 2.12 (Generalized Density [6])**.** *Let* $M = \{M_m\}$ *be a sequence of non-negative regular matrices, and let G, F be filters on* $\mathbb{N}$ *with F finer than the Fréchet filter* $\mathcal{F}_0$*. The **generalized density** of a set* $A \subseteq \mathbb{N}$ *is defined as*

$$d_{M,G,F}(A) = \sup\{\alpha : \{n \in \mathbb{N} : \{m \in \mathbb{N} : \alpha \leq (M_m X_A)_n\} \in G\} \in F\}$$

where X_A *denotes the characteristic sequence of A.*

This definition may appear technical, but it admits an intuitive interpretation: $d_{M,G,F}(A)$ measures the largest value α such that "for F-many indices n, for G-many matrices M_m, the transformed characteristic sequence $(M_m X_A)_n$ exceeds α." In simpler terms, it captures the proportion of indices where the set A has a significant presence after applying various transformations.

Example 2.1. *Consider the set $A = \{k^2 : k \in \mathbb{N}\}$ of perfect squares. Let us compute $d_{M,G,F}(A)$ for different parameter choices.*

Case 1: *Take $M_m = C$ (Cesàro matrix), $G = \{\mathbb{N}\}$, $F = \mathcal{F}_0$. Then:*

$$d_{M,G,F}(A) = \liminf_{n\to\infty} \frac{|\{1 \le k \le n : k \text{ is a square}\}|}{n} = \lim_{n\to\infty} \frac{\lfloor \sqrt{n} \rfloor}{n} = 0.$$

Case 2: *Now take $M = \{CN^m\}_{m\ge 0}$ where C is Cesàro and N is the forward shift operator $(Nx)_n = x_{n+1}$, with $G = \{\mathbb{N}\}$, $F = \mathcal{F}_0$. Then,*

$$d_{M,G,F}(A) = \liminf_{n\to\infty} \inf_{m\ge 0} \frac{|A \cap \{m+1, \dots, m+n\}|}{n}.$$

For perfect squares, there exist intervals (e.g., around large squares) where the density is high

$$\inf_{m\ge 0} \frac{|A \cap \{m+1, \dots, m+n\}|}{n} > 0 \quad \text{for some } n.$$

However, the $\liminf$ *over n still yields $d_{M,G,F}(A) = 0$.*

Case 3: *Let $M_m = C$ for all m, but now take $G = \mathcal{F}_0$ (cofinite sets) and $F = \mathcal{F}_0$. Then,*

$$d_{M,G,F}(A) = \sup\{\alpha : \{n : \{m : \alpha \le (CX_A)_n\} \text{ is cofinite}\} \text{ is cofinite}\}.$$

For perfect squares, $(CX_A)_n \to 0$, so for any $\alpha > 0$, the set $\{m : \alpha \le (CX_A)_n\}$ is finite (not cofinite). Thus $d_{M,G,F}(A) = 0$.

Case 4: *Let $M_m = C$ for all m, $G = \{B \subseteq \mathbb{N} : B \text{ contains all even indices } m\}$, and $F = \mathcal{F}_0$. Then,*

$$d_{M,G,F}(A) = \sup\{\alpha : \{n : \{m : \alpha \le (CX_A)_n\} \supseteq 2\mathbb{N}\} \text{ is cofinite}\}$$

This imposes that the inequality $\alpha \le (CX_A)_n$ must hold for ***all even indices*** *m, creating a stricter condition that may yield different density values.*

Example 2.2. *Consider the highly sparse set $A = \{2^1, 2^4, 2^9, 2^{16}, \dots\}$. The gaps between consecutive elements grow super-exponentially.*

Case 1: *With $M_m = C$, $G = \{\mathbb{N}\}$, $F = \mathcal{F}_0$:*

$$d_{M,G,F}(A) = \liminf_{n\to\infty} \frac{|A \cap \{1, \dots, n\}|}{n}.$$

For $n = 2^{k^2}$, we have $|A \cap \{1, \dots, n\}| = k$, so

$$\frac{|A \cap \{1, \dots, 2^{k^2}\}|}{2^{k^2}} = \frac{k}{2^{k^2}} \to 0 \quad \text{as } k \to \infty.$$

Thus, $d_{M,G,F}(A) = 0$.

Case 2: *Let M_m be weighted Cesàro matrices with weights $w_m(k) = \frac{1}{k^p}$ for different $p > 0$. Then,*

$$(M_m X_A)_n = \frac{\sum_{k=1}^{n} w_m(k) X_A(k)}{\sum_{k=1}^{n} w_m(k)}.$$

For $p > 1$, the denominator converges, while the numerator is dominated by the sparse large elements in A, potentially giving non-zero density for appropriate G and F.

Case 3: *Let G be the filter generated by blocks $B_r = \{m \in \mathbb{N} : 2^{r^2} \le m < 2^{(r+1)^2}\}$ for $r \in \mathbb{N}$. With $M_m = C$ and $F = \mathcal{F}_0$*

$$d_{M,G,F}(A) = \sup\{\alpha : \{n : \{m : \alpha \le (CX_A)_n\} \supseteq B_r \text{ for some } r\} \in F\}.$$

Within each block B_r, the density calculation focuses on intervals containing exactly one element of A (namely 2^{r^2}), yielding locally high density values.

Case 4: *Let F be a non-principal ultrafilter containing the set $E = \{2^{k^2} : k \in \mathbb{N}\}$ itself. With $M_m = I$ (identity matrix) and $G = \{\mathbb{N}\}$:*

$$d_{M,G,F}(A) = \sup\{\alpha : \{n : \{m : \alpha \leq X_A(n)\} \in G\} \in F\}.$$

Since $X_A(n) = 1$ for $n \in E$ and $E \in F$, we get

$$d_{M,G,F}(A) = 1.$$

This dramatically illustrates how ultrafilters can assign full density to sparse sets.

Case 5: *Let G be the filter of sets containing all sufficiently large perfect squares, and F be the Fréchet filter. With $M_m = C$*

$$d_{M,G,F}(A) = \sup\{\alpha : \{n : \{m : \alpha \leq (CX_A)_n\} \text{ contains all large squares}\} \text{ is cofinite}\}.$$

This creates a density that measures behavior specifically at square indices.

This example demonstrates that the notion of a set's "size" is not absolute but depends fundamentally on the mathematical lens through which we view it. Chun's framework provides the mathematical vocabulary to articulate and explore these different perspectives systematically.

Remark 2.1. *If $G = \{\mathbb{N}\}$, then the inner condition*

$$\{m \in \mathbb{N} : \ \alpha \leq (M_m X_A)_n\} \in G$$

is equivalent to

$$\{m \in \mathbb{N} : \ \alpha \leq (M_m X_A)_n\} = \mathbb{N},$$

that is, for the given n the inequality $\alpha \leq (M_m X_A)_n$ must hold for all m. *In particular, if $M_m \equiv M$ does not depend on m, then*

$$\{m : \alpha \leq (M_m X_A)_n\} \in G \quad \iff \quad \alpha \leq (MX_A)_n,$$

so the definition reduces to a single-filter condition in n.

The following fundamental result establishes that this construction indeed yields a lower asymptotic density satisfying axioms (D1)-(D4).

Proposition 2.2 (Chun [6])**.** *Let $M = \{M_m\}$ be a sequence of non-negative regular matrices, and let G, F be filters on $\mathbb{N}$ with F finer than the Fréchet filter $\mathcal{F}_0$. Then the set-function $d_{M,G,F} : 2^{\mathbb{N}} \to [0,1]$ defined in Definition 2.12 is a lower density in the sense of Definition 2.11. In particular, it satisfies monotonicity (D2) and assigns value 0 to all finite sets (D4).*

Proof. This is proved in Chun's thesis [6] (see the development of generalized densities $d_{M,G,F}$ and the verification of the density axioms). For the reader's convenience, we note that $0 \leq (M_m X_A)_n \leq 1$ for all m, n, and monotonicity follows from the non-negativity of M_m; the remaining axioms are established in [6]. □

Lemma 2.1. *Let $\delta = d_{M,G,F}$. Then the family*

$$\mathcal{Z}_\delta := \{A \subseteq \mathbb{N} : \ \delta(A) = 0\}$$

is a zeroclass (hence an ideal): it contains all finite sets, is hereditary, and is closed under finite unions.

Proof. This property is part of the standard theory of Chun's generalized densities; the verification follows the same scheme as in [6] (using monotonicity and the stability under finite perturbations), and the closure under finite unions is proved by the corresponding ideal property of $\mathcal{Z}_\delta$ in Chun's framework. □

The power of Chun's generalized density lies in its ability to specialize to most known density concepts through appropriate parameter choices.

- **Asymptotic density:** Taking $M_m = C$ (Cesàro matrix) for all m, $G = \{\mathbb{N}\}$, and $F = \mathcal{F}_0$ yields

$$d(A) = \liminf_{n\to\infty} \frac{|A \cap \{1,\dots,n\}|}{n}.$$

- **Uniform density:** With $M_m = CN^{m-1}$, $G = \{\mathbb{N}\}$, and $F = \mathcal{F}_0$, we obtain

$$u(A) = \liminf_{n\to\infty} \min_{m\geq 0} \frac{|A \cap \{m+1,\dots,m+n\}|}{n}.$$

- **0-1 density:** Choosing $M_m = I$ (identity matrix), $G = \{\mathbb{N}\}$, and $F = \{A^c : A \in \mathcal{X}\}$ for a zeroclass $\mathcal{X}$ gives

$$d_{\mathcal{X}}(A) = \begin{cases} 1 & \text{if } A^c \in \mathcal{X}; \\ 0 & \text{otherwise.} \end{cases}$$

These examples demonstrate the remarkable flexibility of the $d_{M,G,F}$ framework in capturing diverse notions of size for subsets of natural numbers.

2.1. **Existing Convergence Types.**

Definition 2.13 (Various Types of Sequence Convergence). *A sequence (x_n) converges to L in the following senses:*

- ***Classical Convergence:*** *$(x_n) \to L$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_n - L| < \varepsilon$ for all $n \geq N$. This is the standard convergence in analysis.*
- ***Statistical Convergence:*** *$(x_n) \to L$* statistically *if for every $\varepsilon > 0$,*

$$\overline{\delta}\big(\{n \in \mathbb{N} : \ |x_n - L| \geq \varepsilon\}\big) = 0.$$

(Equivalently, the set of indices on which the deviation is at least ε has upper asymptotic density zero.)

- ***Uniform Statistical Convergence (Almost Statistical Convergence):*** *A sequence (x_n) is said to converge* uniformly statistically *(or* almost statistically*) to L if for every $\varepsilon > 0$, the* lower uniform density *of the set where the sequence deviates from L is zero; that is,*

$$u_*(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0,$$

where

$$u_*(A) = \liminf_{n\to\infty} \inf_{m\geq 0} \frac{|A \cap \{m+1, m+2, \dots, m+n\}|}{n}$$

is the lower (Buck) uniform density *of the set $A \subseteq \mathbb{N}$. Equivalently, $u^*(\{n : |x_n - L| < \varepsilon\}) = 1$ for all $\varepsilon > 0$. This concept is also known in the literature as* uniform statistical convergence *or* almost statistical convergence.

- ***A-Statistical Convergence:*** *$(x_n) \to L$ A-statistically if for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} (AX_{\{|x-L|<\varepsilon\}})_n = 1$$

where $A = (a_{nk})$ is a non-negative regular matrix and X_B is the characteristic sequence of set B. Equivalently, the A-density of the set where $|x_n - L| \geq \varepsilon$ is zero.

- ***Filter Convergence:*** *$(x_n) \to L$ with respect to a filter F (denoted $(x_n) \to_F L$) if for every $\varepsilon > 0$, the set of indices where the sequence is close to L belongs to F, i.e.,*

$$\{n \in \mathbb{N} : |x_n - L| < \varepsilon\} \in F.$$

- ***Ideal Convergence:*** *$(x_n) \to L$ with respect to an ideal $\mathcal{I}$ (denoted $(x_n) \to_{\mathcal{I}} L$) if for every $\varepsilon > 0$, the set of indices where the sequence is far from L belongs to $\mathcal{I}$, i.e.,*

$$\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}.$$

3. $d_{M,G,F}$-Convergence

We now introduce the central concept of this paper: convergence with respect to generalized density.

Definition 3.14. *($d_{M,G,F}$-Convergence). A sequence (x_n) is said to be $d_{M,G,F}$**-convergent** to $L \in \mathbb{R}$ if for every $\varepsilon > 0$*

$$d_{M,G,F}(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0.$$

We denote this by $\lim_{d_{M,G,F}} x_n = L$. Equivalently, (x_n) is $d_{M,G,F}$-convergent to L if for every $\varepsilon > 0$

$$\overline{d}_{M,G,F}(\{n \in \mathbb{N} : |x_n - L| < \varepsilon\}) = 1,$$

where $\overline{d}_{M,G,F}$ denotes the upper density associated with $d_{M,G,F}$, defined by $\overline{d}_{M,G,F}(A) = 1 - d_{M,G,F}(A^c)$.

This definition captures the intuition that a sequence converges in the $d_{M,G,F}$-sense if the set of indices where it deviates significantly from the limit L is "small" according to the generalized density $d_{M,G,F}$.

Example 3.3. *We exhibit a $d_{M,G,F}$–convergence notion which is weaker than Buck (uniform statistical) convergence but still forces statistical convergence, and we show it on a concrete sequence.*

For each $k \in \mathbb{N}$ define the block

$$B_k = (\underbrace{1, \ldots, 1}_{k \text{ terms}}, \underbrace{0, \ldots, 0}_{k^2 \text{ terms}}).$$

Define the sequence (x_n) by writing successively the entries of B_1, then B_2, then B_3, and so on. Finally, set

$$A = \{n \in \mathbb{N} : x_n = 1\}.$$

***(i) Statistical convergence to** 0. Up to the k-th block, the number of ones is $\sum_{j=1}^{k} j = O(k^2)$, while the total length is $\sum_{j=1}^{k}(j + j^2) = O(k^3)$. Hence $\overline{\delta}(A) = 0$, so $x_n \to 0$ statistically.*

***(ii) Failure of Buck convergence.** Each block contains an interval of length k consisting entirely of ones; choosing the window length $n = k$ and placing it at the beginning of that ones–run gives*

$$\sup_{m \geq 0} \frac{|A \cap \{m+1, \ldots, m+n\}|}{n} = 1 \quad \textit{for infinitely many } n,$$

so $u^*(A) = 1$ and (x_n) is not uniformly statistically (Buck) convergent to 0.

(iii) An intermediate** $d_{M,G,F}$**–convergence (and** $x_n \to 0$ **in this sense).

Let

$$M = \{M_m\}_{m \geq 0}, \quad M_m := CN^m, \qquad F := \mathcal{F}_0, \qquad G := \mathcal{D} := \{B \subseteq \mathbb{N} : \ \overline{\delta}(B) = 1\},$$

where $\overline{\delta}$ denotes the upper asymptotic density. Thus G is the density–1 filter on $\mathbb{N}$.

Note that for every m, n,

$$(M_m X_A)_n = (CN^m X_A)_n = \frac{|A \cap \{m+1, \ldots, m+n\}|}{n} \in [0, 1].$$

Fix $\alpha > 0$ and for each $n \in \mathbb{N}$ define

$$E_n(\alpha) := \Big\{ m \in \mathbb{N} : \ (CN^m X_A)_n \geq \alpha \Big\}.$$

We show that $\overline{\delta}(E_n(\alpha)) = 0$ for every fixed n.

For $M \in \mathbb{N}$ we have

$$\sum_{m=1}^{M} |A \cap \{m+1, \ldots, m+n\}| \leq n \, |A \cap \{1, \ldots, M+n\}|,$$

since each index $t \leq M + n$ can belong to at most n of the sliding windows $\{m+1, \ldots, m+n\}$.

Dividing by nM yields

$$\frac{1}{M}\sum_{m=1}^{M}(CN^m X_A)_n \leq \frac{|A \cap \{1, \ldots, M+n\}|}{M}.$$

Since $\overline{\delta}(A) = 0$, the right-hand side tends to 0 as $M \to \infty$. Hence

$$\frac{1}{M}\sum_{m=1}^{M}(CN^m X_A)_n \longrightarrow 0.$$

By Markov's inequality,

$$\frac{|E_n(\alpha) \cap \{1, \ldots, M\}|}{M} \leq \frac{1}{\alpha} \cdot \frac{1}{M}\sum_{m=1}^{M}(CN^m X_A)_n,$$

and therefore

$$\limsup_{M\to\infty} \frac{|E_n(\alpha) \cap \{1, \ldots, M\}|}{M} = 0.$$

Thus $\overline{\delta}(E_n(\alpha)) = 0$, so $E_n(\alpha) \notin \mathcal{D} = G$ for every n.

Consequently,

$$\Big\{n : \ \{m : \alpha \leq (M_m X_A)_n\} \in G\Big\} = \emptyset \notin F,$$

and by Definition 2.12 we obtain

$$d_{M,G,F}(A) = 0.$$

Therefore, for every $\varepsilon \in (0, 1]$,

$$d_{M,G,F}\big(\{n : |x_n - 0| \geq \varepsilon\}\big) = d_{M,G,F}(A) = 0,$$

so $x_n \to 0$ in the $d_{M,G,F}$–sense.

Why this is genuinely "between". *With $M = \{CN^m\}$ and $F = \mathcal{F}_0$, choosing $G = \{\mathbb{N}\}$ forces the condition for* all *shifts m (Buck), while choosing $G = \mathcal{D}$ requires it only for* density–1 many *shifts m. Hence Buck convergence $\Rightarrow$ $d_{M,G,F}$–convergence, and (as the estimate above shows) $d_{M,G,F}$–convergence controls statistical behavior; in particular the present x converges in the $d_{M,G,F}$–sense but fails Buck convergence.*

Step-by-step explanation of the proof:

(1) We construct a set A (the indices where $x_n = 1$) that has asymptotic density $\overline{\delta}(A) = 0$ (so x_n converges statistically to 0), but its lower uniform density $u_*(A) \neq 0$ (so Buck convergence fails).

(2) We choose $M_m = CN^m$ (Cesàro matrix composed with shift operators), $F = \mathcal{F}_0$ (the Fréchet filter on n), and $G = \mathcal{D}$ (the density-1 filter on m).

(3) For each fixed n, $(CN^m X_A)_n$ gives the proportion of 1's in the interval $\{m+1, \ldots, m+n\}$.

(4) To show $d_{M,G,F}(A) = 0$, we need to show that for any $\alpha > 0$, the set of n for which $\{m : (CN^m X_A)_n \geq \alpha\}$ is in G (i.e., has density 1) cannot be in F (i.e., cannot be cofinite).

(5) We prove that for each fixed n, the set $E_n(\alpha) = \{m : (CN^m X_A)_n \geq \alpha\}$ actually has upper asymptotic density 0, so it is NOT in $G = \mathcal{D}$.

(6) The key estimate shows that the average of $(CN^m X_A)_n$ over m tends to 0 because $\overline{\delta}(A) = 0$. By Markov's inequality, this forces the density of $E_n(\alpha)$ to be 0.

(7) Since for every n, $E_n(\alpha) \notin G$, the set $\{n : E_n(\alpha) \in G\}$ is empty, and thus certainly not in F.

(8) Therefore $d_{M,G,F}(A) = 0$, proving $d_{M,G,F}$-convergence to 0.

Definition 3.15. *(Convergence Spaces). The spaces of $d_{M,G,F}$-convergent and $d_{M,G,F}$-null sequences are defined as:*

$$c_{d_{M,G,F}} = \left\{ (x_n) \in \omega : \exists L \in \mathbb{R} \text{ with } \lim_{d_{M,G,F}} x_n = L \right\}$$

and

$$c^0_{d_{M,G,F}} = \left\{ (x_n) \in \omega : \lim_{d_{M,G,F}} x_n = 0 \right\}.$$

We now establish the fundamental algebraic and structural properties of these convergence spaces.

Theorem 3.1. *(Linearity of Convergence Spaces). For any generalized density $d_{M,G,F}$, the spaces $c_{d_{M,G,F}}$ and $c^0_{d_{M,G,F}}$ are linear subspaces of ω.*

Proof. Let $\delta := d_{M,G,F}$ and denote the δ–null class by

$$\mathcal{Z}_\delta := \{A \subseteq \mathbb{N} : \ \delta(A) = 0\}.$$

(Recall that for Chun's generalized density, $\mathcal{Z}_\delta$ is a zeroclass/ideal; in particular it is hereditary and closed under finite unions.)

Let $(x_n), (y_n) \in c_\delta$ with $\lim_\delta x_n = L_1$ and $\lim_\delta y_n = L_2$. Fix $\varepsilon > 0$ and set

$$A := \left\{ n \in \mathbb{N} : \ |x_n - L_1| \geq \tfrac{\varepsilon}{2} \right\}, \qquad B := \left\{ n \in \mathbb{N} : \ |y_n - L_2| \geq \tfrac{\varepsilon}{2} \right\}.$$

Then $\delta(A) = 0$ and $\delta(B) = 0$, i.e. $A, B \in \mathcal{Z}_\delta$. Define

$$E := \left\{ n \in \mathbb{N} : \ |(x_n + y_n) - (L_1 + L_2)| \geq \varepsilon \right\}.$$

By the triangle inequality, $E \subseteq A \cup B$. Since $\mathcal{Z}_\delta$ is an ideal, $A \cup B \in \mathcal{Z}_\delta$, hence $\delta(A \cup B) = 0$. By the monotonicity property (D2) of δ, from $E \subseteq A \cup B$ we get $\delta(E) \leq \delta(A \cup B) = 0$, so $\delta(E) = 0$. Therefore $x_n + y_n \in c_\delta$ and $\lim_\delta (x_n + y_n) = L_1 + L_2$.

For scalar multiplication, let $\alpha \in \mathbb{R}$ and $x \in c_\delta$ with $\lim_\delta x_n = L$. If $\alpha = 0$ then αx is the zero sequence and belongs to c_δ with δ–limit 0. If $\alpha \neq 0$, then for any $\varepsilon > 0$,

$$\left\{ n : \ |\alpha x_n - \alpha L| \geq \varepsilon \right\} = \left\{ n : \ |x_n - L| \geq \tfrac{\varepsilon}{|\alpha|} \right\},$$

which has δ–density 0 because $x \in c_\delta$. Hence $\alpha x \in c_\delta$ and $\lim_\delta (\alpha x_n) = \alpha L$.

Thus c_δ is a linear subspace of ω. The same argument with $L_1 = L_2 = 0$ shows that c^0_δ is also a linear subspace of ω. □

Theorem 3.2 (Regularity). *Every classically convergent sequence is $d_{M,G,F}$-convergent with the same limit, for any choice of M, G, F as in Definition 2.12.*

Proof. Let $\delta := d_{M,G,F}$ and suppose $x_n \to L$ classically. Then for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - L| < \varepsilon$ for all $n \geq N$. Hence the exceptional set

$$A_\varepsilon := \{n \in \mathbb{N} : \ |x_n - L| \geq \varepsilon\}$$

is finite.

By property (D4) of lower densities (finite sets have zero density), we have

$$\delta(A_\varepsilon) = 0.$$

Therefore $x_n \to L$ in the $d_{M,G,F}$-sense. □

Remark 3.2. *The regularity property established in Theorem 3.2 ensures that our generalized convergence framework is a conservative extension of classical convergence. No classically convergent sequence loses its convergence status in the generalized sense, and the limit value remains unchanged.*

Example 3.4 (Numerical Illustration of $d_{M,G,F}$-Convergence). *Consider the sequence $x_n = 1$ if n is a perfect square and $x_n = 0$ otherwise. We aim to check if x_n converges to $L = 0$ under $d_{M,G,F}$-convergence with specific parameters. Let $M_m = C$ (the Cesàro matrix) for all m, $G = \{\mathbb{N}\}$, and $F = \mathcal{F}_0$ (the Fréchet filter). This choice corresponds to statistical convergence (Theorem 4.4). The set $A_\varepsilon = \{n : |x_n - 0| \geq \varepsilon\}$ for $\varepsilon = 0.5$ is the set of perfect squares, i.e., $A_{0.5} = \{1, 4, 9, 16, \ldots\}$. The asymptotic density is*

$$\delta(A_{0.5}) = \lim_{n\to\infty} \frac{|\{k \leq n : k \text{ is a square}\}|}{n}.$$

The number of squares up to n is approximately $\sqrt{n}$, so

$$\delta(A_{0.5}) = \lim_{n\to\infty} \frac{\lfloor \sqrt{n} \rfloor}{n} = 0,$$

since $\lfloor \sqrt{n} \rfloor / n \to 0$ as $n \to \infty$. Thus, $d_{C,\{\mathbb{N}\},\mathcal{F}_0}(A_{0.5}) = 0$ for all $\varepsilon > 0$, implying $\lim_{d_{M,G,F}} x_n = 0$. This sequence does not converge classically (due to infinite 1s), but it converges statistically, illustrating how $d_{M,G,F}$-convergence generalizes classical convergence.

4. Special Cases

One of the most compelling features of our unified framework is its ability to specialize to virtually all known convergence types through appropriate choices of the parameters M, G, and F. In this section, we systematically demonstrate how various convergence concepts from the literature emerge as special instances of $d_{M,G,F}$-convergence.

4.1. **Classical Convergence.**

Theorem 4.3. *(Classical Convergence). Let $M_m = I$ (the identity matrix) for all m, $G = \{\mathbb{N}\}$, and $F = \mathcal{F}_0$ (the Fréchet filter). Then $d_{M,G,F}$-convergence coincides with classical convergence.*

Proof. With these parameter choices, the generalized density becomes

$$d_{I,\{\mathbb{N}\},\mathcal{F}_0}(A) = \liminf_{n\to\infty} X_A(n)$$

where X_A is the characteristic function of A. This equals 1 if A contains a tail of $\mathbb{N}$ and 0 otherwise.

A sequence (x_n) is $d_{I,\{\mathbb{N}\},\mathcal{F}_0}$-convergent to L if for every $\varepsilon > 0$

$$d_{I,\{\mathbb{N}\},\mathcal{F}_0}(\{n : |x_n - L| \geq \varepsilon\}) = 0.$$

This occurs precisely when $\{n : |x_n - L| \geq \varepsilon\}$ does not contain a tail of $\mathbb{N}$, i.e., when it is finite. This is exactly the definition of classical convergence. □

4.2. **Statistical Convergence via $d_{M,G,F}$.**

In this subsection we show that, for a canonical choice of parameters, the $d_{M,G,F}$–framework recovers the usual notion of statistical convergence defined via *upper* asymptotic density.

Theorem 4.4 (Statistical convergence via $d_{M,G,F}$). *Let $M_m = C$ (the Cesàro matrix) for all m, $G = \{\mathbb{N}\}$, and $F = \mathcal{F}_0$ (the Fréchet filter). Let $\delta(A) := d_{M,G,F}(A)$ and let $\overline{\delta}(A) := 1 - \delta(A^c)$ be the associated upper density. Then for every $A \subseteq \mathbb{N}$,*

$$\overline{\delta}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n}.$$

Consequently, for every sequence $x = (x_n)$ and $L \in \mathbb{R}$,

$$x_n \xrightarrow[d_{M,G,F}]{} L \quad \Longleftrightarrow \quad \forall\, \varepsilon > 0 : \limsup_{n\to\infty} \frac{\big|\{k \le n : \ |x_k - L| \ge \varepsilon\}\big|}{n} = 0,$$

i.e. $d_{M,G,F}$–convergence coincides with statistical convergence.

Proof. Fix $A \subseteq \mathbb{N}$ and set

$$s_n := \frac{|A \cap \{1, \dots, n\}|}{n} = (CX_A)_n \in [0, 1].$$

Since $M_m = C$ does not depend on m and $G = \{\mathbb{N}\}$, the inner condition $\{m : \alpha \le (M_m X_A)_n\} \in G$ is equivalent to $\alpha \le s_n$. Thus Definition 2.12 reduces to

$$\delta(A) = d_{C,\{\mathbb{N}\},\mathcal{F}_0}(A) = \sup\{\alpha : \ \{n : \ s_n \ge \alpha\} \in \mathcal{F}_0\}.$$

We claim that

$$\delta(A) = \liminf_{n\to\infty} s_n.$$

Indeed, if $\alpha < \liminf_{n\to\infty} s_n$, then $s_n \ge \alpha$ for all sufficiently large n, so $\{n : \ s_n \ge \alpha\}$ is cofinite and hence belongs to $\mathcal{F}_0$, which shows $\alpha \le \delta(A)$. Conversely, if $\alpha > \liminf_{n\to\infty} s_n$, then $s_n < \alpha$ for infinitely many n, so $\{n : \ s_n \ge \alpha\}$ is not cofinite, hence α cannot contribute to the supremum; thus $\delta(A) \le \liminf s_n$. This proves $\delta(A) = \liminf s_n$.

Now compute the associated upper density:

$$\begin{aligned}\overline{\delta}(A) = 1 - \delta(A^c) = 1 - \liminf_{n\to\infty} \frac{|A^c \cap \{1, \dots, n\}|}{n} \\ = 1 - \liminf_{n\to\infty} \Big(1 - \frac{|A \cap \{1, \dots, n\}|}{n}\Big) = \limsup_{n\to\infty} \frac{|A \cap \{1, \dots, n\}|}{n}.\end{aligned}$$

This establishes the density identity.

Finally, by the definition of statistical convergence used in this paper, $x_n \to L$ statistically iff for every $\varepsilon > 0$ the exceptional set

$$A_\varepsilon := \{n \in \mathbb{N} : \ |x_n - L| \ge \varepsilon\}$$

has upper asymptotic density 0, i.e. $\limsup_{n\to\infty} |A_\varepsilon \cap \{1, \dots, n\}|/n = 0$. By the identity above, this is equivalent to $\overline{\delta}(A_\varepsilon) = 0$. Equivalently, $\delta(A_\varepsilon^c) = 1$, i.e. $\delta(\{n : \ |x_n - L| < \varepsilon\}) = 1$, which is exactly the $d_{M,G,F}$–convergence criterion (written in the equivalent "good set has upper density 1" form). Hence the two notions coincide. $\square$

4.3. **Uniform Statistical Convergence.**

Theorem 4.5 (Uniform Statistical (Buck) Convergence). *Let $M = \{CN^m\}_{m\ge 0}$, where C denotes the Cesàro matrix and N is the forward shift operator. Take $G = \{\mathbb{N}\}$ and $F = \mathcal{F}_0$. Then the $d_{M,G,F}$–convergence of a sequence $x = (x_n)$ to L is equivalent to* uniform statistical convergence, *also known as* Buck uniform statistical convergence; *that is,*

$$\forall\, \varepsilon > 0 : \quad u_*\big(\{n \in \mathbb{N} : |x_n - L| \ge \varepsilon\}\big) = 0,$$

where the lower (Buck) and upper (Banach) uniform densities of a set $A \subseteq \mathbb{N}$ are respectively defined by

$$u_*(A) := \liminf_{n\to\infty} \inf_{m\ge 0} \frac{|A \cap \{m+1, \dots, m+n\}|}{n}$$

and

$$u^*(A) := \limsup_{n\to\infty} \sup_{m\ge 0} \frac{|A \cap \{m+1, \dots, m+n\}|}{n}.$$

Proof. With $M = \{CN^m\}_{m\geq 0}$, note that $(CN^m X_A)_n = \frac{|A\cap\{m+1,\ldots,m+n\}|}{n}$. With $G = \{\mathbb{N}\}$ and $F = \mathcal{F}_0$, the generalized density associated to a set $A \subseteq \mathbb{N}$ satisfies

$$d_{M,G,F}(A) = \liminf_{n\to\infty} \inf_{m\geq 0} \frac{|A \cap \{m+1,\ldots,m+n\}|}{n} = u_*(A),$$

i.e. it coincides with the lower (Buck) uniform density. By the definition of $d_{M,G,F}$–convergence, $x_n \to L$ in the $d_{M,G,F}$–sense iff for every $\varepsilon > 0$ one has

$$d_{M,G,F}\big(\{n : |x_n - L| \geq \varepsilon\}\big) = 0,$$

which, using the identity above, is equivalent to $u_*\big(\{n : |x_n - L| \geq \varepsilon\}\big) = 0$. Since for any $A \subseteq \mathbb{N}$ one always has $0 \leq u_*(A) \leq u^*(A)$, the condition $u^*(\{n : |x_n - L| \geq \varepsilon\}) = 0$ implies (and for 0–1 indicator sets is equivalent to) $u_*(\{n : |x_n - L| \geq \varepsilon\}) = 0$. Hence the $d_{M,G,F}$–convergence is exactly uniform statistical (Buck) convergence.

Equivalently, for 0–1 sets $B \subseteq \mathbb{N}$, $u^*(B^c) = 1$ iff $u_*(B) = 0$, so the convergence criterion can be written as $u_*(\{n : |x_n - L| < \varepsilon\}) = 1$ for all $\varepsilon > 0$. □

4.4. **Filter Convergence.**

Theorem 4.6. *(Filter Convergence). Let $M_m = I$ for all m, $G = \{\mathbb{N}\}$, and let F be an arbitrary filter on $\mathbb{N}$. Then $d_{M,G,F}$-convergence coincides with the filter convergence with respect to F.*

Proof. With these parameters, we have

$$d_{I,\{\mathbb{N}\},F}(A) = \sup\{\alpha : \{n : \alpha \leq X_A(n)\} \in F\}$$

This equals 1 if $A \in F$ and 0 otherwise.

A sequence (x_n) is $d_{I,\{\mathbb{N}\},F}$-convergent to L if for every $\varepsilon > 0$

$$d_{I,\{\mathbb{N}\},F}(\{n : |x_n - L| \geq \varepsilon\}) = 0$$

This occurs precisely when $\{n : |x_n - L| \geq \varepsilon\} \notin F$, or equivalently, when its complement $\{n : |x_n - L| < \varepsilon\} \in F$. This is exactly the definition of filter convergence. □

4.5. **Ideal Convergence.**

Theorem 4.7. *(Ideal Convergence). Let $M_m = I$ for all m, $G = \{\mathbb{N}\}$, and $F = \{A^c : A \in \mathcal{I}\}$ where $\mathcal{I}$ is an ideal on $\mathbb{N}$. Then, $d_{M,G,F}$-convergence coincides with the ideal convergence with respect to $\mathcal{I}$.*

Proof. With this parameter choice, the generalized density becomes

$$d_{I,\{\mathbb{N}\},F}(A) = \begin{cases} 1 & \text{if } A^c \in \mathcal{I} \\ 0 & \text{otherwise} \end{cases}$$

This is the characteristic function of the dual filter of $\mathcal{I}$. A sequence (x_n) is $d_{I,\{\mathbb{N}\},F}$-convergent to L if for every $\varepsilon > 0$

$$d_{I,\{\mathbb{N}\},F}(\{n : |x_n - L| \geq \varepsilon\}) = 0.$$

This occurs precisely when $\{n : |x_n - L| \geq \varepsilon\}^c \notin \mathcal{I}$, or equivalently, when $\{n : |x_n - L| \geq \varepsilon\} \in \mathcal{I}$. This is exactly the definition of ideal convergence. □

4.6. **Matrix Statistical Convergence.**

Theorem 4.8. *(A-Statistical Convergence). Let $M_m = A$ (a fixed non-negative regular matrix) for all m, $G = \{\mathbb{N}\}$, and $F = \mathcal{F}_0$. Then $d_{M,G,F}$-convergence coincides with A-statistical convergence.*

Proof. With these parameters, the generalized density becomes

$$d_{A,\{\mathbb{N}\},\mathcal{F}_0}(B) = \liminf_{n\to\infty}(AX_B)_n$$

which is the lower A-density of the set B. A sequence (x_n) is A-statistically convergent to L if for every $\varepsilon > 0$

$$\lim_{n\to\infty}(AX_{\{|x-L|<\varepsilon\}})_n = 1.$$

This is equivalent to $d_{A,\{\mathbb{N}\},\mathcal{F}_0}(\{n : |x_n - L| \geq \varepsilon\}) = 0$, establishing the correspondence. □

Remark 4.3. *The systematic recovery of these diverse convergence types demonstrates the remarkable comprehensiveness of our unified framework. Each specialization involves natural choices of parameters that align with the intuitive meaning of the corresponding convergence concept. This not only validates our approach but also reveals the underlying structural unity among seemingly disparate convergence theories.*

TABLE 1. Specializations of $d_{M,G,F}$-convergence to known convergence types

Convergence Type	M	G	F
Classical	I	$\{\mathbb{N}\}$	$\mathcal{F}_0$
Statistical	C	$\{\mathbb{N}\}$	$\mathcal{F}_0$
Uniform Statistical	CN^m $(m \geq 0)$	$\{\mathbb{N}\}$	$\mathcal{F}_0$
Filter	I	$\{\mathbb{N}\}$	F (any filter)
Ideal	I	$\{\mathbb{N}\}$	$\{A^c : A \in \mathcal{I}\}$
A-Statistical	A	$\{\mathbb{N}\}$	$\mathcal{F}_0$

5. HIERARCHY OF CONVERGENCE SPACES

A fundamental aspect of any comprehensive convergence theory is understanding the relationships between different convergence types. In this section, we establish a systematic hierarchy among various convergence spaces, demonstrating how our unified framework naturally organizes these spaces into a coherent structure.

5.1. **Main Hierarchy Theorem.**

Lemma 5.2 (Comparison principle)**.** *Let $\delta_1, \delta_2 : 2^{\mathbb{N}} \to [0,1]$ be lower densities such that*

$$\delta_1(A) = 0 \implies \delta_2(A) = 0 \qquad \textit{for all } A \subseteq \mathbb{N}.$$

Then for the associated convergence spaces one has $c_{\delta_1} \subseteq c_{\delta_2}$ and $c^0_{\delta_1} \subseteq c^0_{\delta_2}$.

Proof. Let $x \in c_{\delta_1}$ with $\delta_1(\{n : |x_n - L| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$. By the assumption, $\delta_2(\{n : |x_n - L| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$, hence $x \in c_{\delta_2}$. The null-space inclusion is identical. □

Theorem 5.9. *(Hierarchy of Convergence Spaces). The following inclusions hold among convergence spaces, where c denotes classical convergence, c_u uniform statistical (Buck) convergence, c_{st} statistical convergence, c_F filter convergence, and $c_{\mathcal{I}}$ ideal convergence:*

(1) $c \subseteq c_u \subseteq c_{st} \subseteq c_{d_{M,G,F}}$ *for appropriate choices of* M, G, F*,*
(2) $c_F \subseteq c_{d_{M,G,F}}$ *when the filters are compatible,*
(3) $c_{\mathcal{I}} \subseteq c_{d_{M,G,F}}$ *for ideals corresponding to zeroclasses.*

Proof. We prove each inclusion separately, making use of Lemma 5.2 by verifying the corresponding implication between densities.

Part (1) $c \subseteq c_u \subseteq c_{st} \subseteq c_{d_{M,G,F}}$

- $c \subseteq c_u$: Let $(x_n) \in c$ (classically convergent to L). Then for any $\varepsilon > 0$, the set $A_\varepsilon = \{n : |x_n - L| \geq \varepsilon\}$ is finite. Finite sets have upper uniform density zero, i.e., $u^*(A_\varepsilon) = 0$. Since $u_*(A_\varepsilon) \leq u^*(A_\varepsilon)$, we have $u_*(A_\varepsilon) = 0$, so $(x_n) \in c_u$.
- $c_u \subseteq c_{st}$: By Lemma 5.2, it suffices to show that $u_*(A) = 0$ implies $\overline{\delta}(A) = 0$. Recall that
$$u_*(A) = \liminf_{n\to\infty} \inf_{m\geq 0} \frac{|A \cap \{m+1, \ldots, m+n\}|}{n}.$$
If $u_*(A) = 0$, then for every $\eta > 0$, there exists N such that for all $n \geq N$,
$$\inf_{m\geq 0} \frac{|A \cap \{m+1, \ldots, m+n\}|}{n} < \eta.$$
In particular, taking $m = 0$, we get
$$\frac{|A \cap \{1, \ldots, n\}|}{n} < \eta \quad \text{for all } n \geq N.$$
Hence $\overline{\delta}(A) = \limsup_{n\to\infty} \frac{|A\cap\{1,\ldots,n\}|}{n} \leq \eta$ for all $\eta > 0$, so $\overline{\delta}(A) = 0$. Therefore, $(x_n) \in c_{st}$.
- $c_{st} \subseteq c_{d_{M,G,F}}$: This inclusion depends on the specific choice of parameters. For the canonical parameters that yield statistical convergence ($M_m = C$, $G = \{\mathbb{N}\}$, $F = \mathcal{F}_0$), we have $d_{M,G,F}(A) = \liminf_{n\to\infty} \frac{|A\cap\{1,\ldots,n\}|}{n}$, so $d_{M,G,F}(A) = 0$ whenever $\overline{\delta}(A) = 0$ (in fact, these are equivalent). For more general choices where $d_{M,G,F}$ is a coarser density (i.e., $d_{M,G,F}(A) = 0$ whenever $\overline{\delta}(A) = 0$), the inclusion follows from Lemma 5.2. This occurs, for example, when M_m are regular matrices that preserve asymptotic density zero sets.

Part (2) $c_F \subseteq c_{d_{M,G,F}}$

Let $(x_n) \in c_F$, meaning (x_n) converges to L with respect to filter F. Then for any $\varepsilon > 0$, we have
$$\{n : |x_n - L| < \varepsilon\} \in F$$
Now choose $M_m = I$ (identity matrix), $G = \{\mathbb{N}\}$, and the same filter F for our generalized density. Then,
$$d_{I,\{\mathbb{N}\},F}(A) = \begin{cases} 1 & \text{if } A \in F; \\ 0 & \text{otherwise.} \end{cases}$$
For $A_\varepsilon = \{n : |x_n - L| \geq \varepsilon\}$, we have $A_\varepsilon \notin F$ (since its complement is in F). Therefore,
$$d_{I,\{\mathbb{N}\},F}(A_\varepsilon) = 0$$
Hence $(x_n) \in c_{d_{M,G,F}}$ with the specified parameters.

Part (3) $c_{\mathcal{I}} \subseteq c_{d_{M,G,F}}$

Let $(x_n) \in c_{\mathcal{I}}$, meaning (x_n) converges to L with respect to ideal $\mathcal{I}$. Then, for any $\varepsilon > 0$
$$\{n : |x_n - L| \geq \varepsilon\} \in \mathcal{I}$$

Choose $M_m = I$, $G = \{\mathbb{N}\}$, and $F = \{A^c : A \in \mathcal{I}\}$ (the dual filter of $\mathcal{I}$). Then,

$$d_{I,\{\mathbb{N}\},F}(B) = \begin{cases} 1 & \text{if } B^c \in \mathcal{I}; \\ 0 & \text{otherwise.} \end{cases}$$

For $B = \{n : |x_n - L| \geq \varepsilon\}$, we have $B \in \mathcal{I}$, so $B^c \notin \mathcal{I}$, and thus,

$$d_{I,\{\mathbb{N}\},F}(B) = 0.$$

Therefore, $(x_n) \in c_{d_{M,G,F}}$ with the specified parameters. □

Remark 5.4. *The underlying principle governing this hierarchy is the **preservation of smallness**: if a set is "small" according to a stronger notion (finite, uniform density zero, asymptotic density zero, or belonging to an ideal), then it remains small under coarser notions of smallness. This preservation ensures the corresponding inclusion of convergence spaces.*

5.2. **Proper Inclusions and Distinguishing Examples.**

Theorem 5.10. *(Proper Inclusions). The inclusions in Theorem 5.9 are generally proper. Specifically*

(1) $c \subsetneq c_u$ *(classical convergence is strictly stronger than uniform statistical convergence),*
(2) $c_u \subsetneq c_{st}$ *(uniform statistical convergence is strictly stronger than statistical convergence),*
(3) $c_F \subsetneq c_{d_{M,G,F}}$ *for generic choices of parameters, and*
(4) $c_{\mathcal{I}} \subsetneq c_{d_{M,G,F}}$ *for generic choices of parameters.*

Proof. We provide explicit examples distinguishing these spaces.

Example 5.5 ($c \subsetneq c_u$). *Let $Q = \{k^2 : k \in \mathbb{N}\}$ be the set of perfect squares, and define*

$$x_n = \begin{cases} 2, & \text{if } n \in Q, \\ 0, & \text{otherwise.} \end{cases}$$

Then (x_n) does not converge classically to 0, since $x_n = 2$ for infinitely many n.
For any $\varepsilon \in (0, 2]$,

$$\{n \in \mathbb{N} : |x_n - 0| \geq \varepsilon\} = Q.$$

We claim that Q has upper uniform (Banach) density zero. Indeed, in any interval $\{m+1, \ldots, m+n\}$ there are at most $2\sqrt{n}+1$ perfect squares, hence

$$\sup_{m \geq 0} \frac{|Q \cap \{m+1, \ldots, m+n\}|}{n} \leq \frac{2\sqrt{n}+1}{n} \longrightarrow 0.$$

Thus $u^*(Q) = 0$, and therefore $x_n \to 0$ uniformly statistically. Consequently $x \in c_u \setminus c$.

Example 5.6 ($c_u \subsetneq c_{st}$). *For each $k \in \mathbb{N}$ construct a block consisting of k ones followed by k^2 zeros, and define (x_n) by listing these blocks successively. Let $A = \{n \in \mathbb{N} : x_n = 1\}$.*

Up to the k-th block, the number of ones is $O(k^2)$ while the total length is $O(k^3)$; hence $\overline{\delta}(A) = 0$, and therefore $x_n \to 0$ statistically.

However, each block contains an interval of length k consisting entirely of ones. Choosing $n = k$ and placing the window at the beginning of such a block gives

$$\sup_{m \geq 0} \frac{|A \cap \{m+1, \ldots, m+n\}|}{n} = 1$$

for infinitely many n. Consequently $u^*(A) = 1$, so x_n does not converge uniformly statistically to 0.

Thus $x \in c_{st} \setminus c_u$.

Example 5.7. *(Filter vs. Generalized): For generic choices of M, G, F where the density is strictly coarser than the characteristic function of the filter, there exist sequences that are $d_{M,G,F}$-convergent but not F-convergent. These typically involve sequences that oscillate in a "dense" pattern that is captured by the generalized density but not by the filter.*

Example 5.8. *(Ideal vs. Generalized): Similarly, for parameters where $d_{M,G,F}$ is strictly coarser than the characteristic function of the ideal's dual filter, proper inclusion occurs.*

These examples demonstrate the increased flexibility of our generalized framework. □

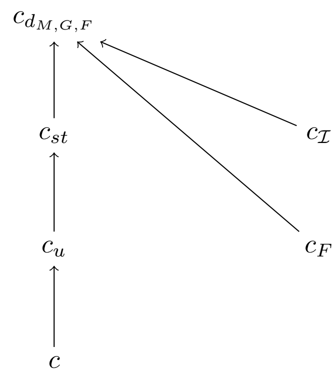

FIGURE 1. Hierarchy of convergence spaces. Arrows indicate proper inclusions.

Remark 5.5. *The hierarchy established in this section provides a comprehensive map of the relationships between various convergence concepts. This organizational structure not only clarifies the relative strengths of different convergence types but also highlights the unifying power of our $d_{M,G,F}$-framework in encompassing this diverse landscape.*

6. Summability Theory

The theory of summability methods concerns the extension of the limit functional to larger classes of sequences. Our $d_{M,G,F}$-convergence framework naturally gives rise to a comprehensive family of summability methods that generalize and unify classical approaches.

6.1. **The Generalized Summability Method.**

Definition 6.16. *($d_{M,G,F}$-Summability Method) The $d_{M,G,F}$-summability method is defined as the linear functional*

$$S : c_{d_{M,G,F}} \to \mathbb{R} \quad \textit{given by} \quad S(x) = \lim_{d_{M,G,F}} x_n$$

where $\lim_{d_{M,G,F}} x_n = L$ means that for every $\varepsilon > 0$

$$d_{M,G,F}(\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}) = 0$$

This definition extends the classical limit functional to the larger space of $d_{M,G,F}$-convergent sequences. The functional S assigns to each $d_{M,G,F}$-convergent sequence its generalized limit.

6.2. **Fundamental Properties.**

Theorem 6.11 (Basic properties of $d_{M,G,F}$-summability)**.** *Let $S : c_{d_{M,G,F}} \to \mathbb{R}$ be defined by*

$$S(x) = \lim_{d_{M,G,F}} x_n,$$

whenever the $d_{M,G,F}$-limit exists. Then the following properties hold.

(i) ***Linearity.*** *The space $c_{d_{M,G,F}}$ is a linear subspace of ω, and for all $x, y \in c_{d_{M,G,F}}$ and $\alpha, \beta \in \mathbb{R}$,*

$$S(\alpha x + \beta y) = \alpha S(x) + \beta S(y).$$

(ii) ***Regularity.*** *If $x_n \to L$ classically, then $x \in c_{d_{M,G,F}}$ and*

$$S(x) = L.$$

In particular, $c \subseteq c_{d_{M,G,F}}$.

(iii) ***Positivity.*** *Assume that $d_{M,G,F}(\mathbb{N}) = 1$. If $x_n \geq 0$ for all n and $x \in c_{d_{M,G,F}}$, then*

$$S(x) \geq 0.$$

(iv) ***Solidity under additional assumptions.*** *Suppose that the convergence space $c_{d_{M,G,F}}$ is solid, i.e., whenever $x \in c_{d_{M,G,F}}$ and $|y_n| \leq |x_n|$ for all n, then $y \in c_{d_{M,G,F}}$. Then the null space*

$$c^0_{d_{M,G,F}} = \{x \in c_{d_{M,G,F}} : S(x) = 0\}$$

is also solid, and the method is of R-type in the sense of Freedman and Sember [13].

Proof. **(i)** Linearity of $c_{d_{M,G,F}}$ follows from Theorem 3.1. If $S(x) = L_1$ and $S(y) = L_2$, then by the same theorem,

$$\lim_{d_{M,G,F}} (\alpha x_n + \beta y_n) = \alpha L_1 + \beta L_2,$$

hence $S(\alpha x + \beta y) = \alpha S(x) + \beta S(y)$.

(ii) By Theorem 3.2, classical convergence implies $d_{M,G,F}$-convergence with the same limit. Thus $c \subseteq c_{d_{M,G,F}}$ and $S(x) = L$ whenever $x_n \to L$ in the classical sense.

(iii) Suppose $x_n \geq 0$ for all n and $S(x) = L < 0$. Let $\varepsilon = -L/2 > 0$. Since $x_n \geq 0$ and $L < 0$, we have for every n,

$$|x_n - L| \geq -L = 2\varepsilon.$$

Hence

$$\{n : |x_n - L| \geq \varepsilon\} = \mathbb{N}.$$

By the definition of $d_{M,G,F}$-convergence, the set $\{n : |x_n - L| \geq \varepsilon\}$ must have $d_{M,G,F}$-density zero. But $d_{M,G,F}(\mathbb{N}) = 1$, yielding a contradiction. Therefore $L \geq 0$.

(iv) Assume the space $c_{d_{M,G,F}}$ is solid. If $x \in c^0_{d_{M,G,F}}$ and $|y_n| \leq |x_n|$ for all n, then by solidity $y \in c_{d_{M,G,F}}$. Moreover, since $|y_n| \leq |x_n|$ and x converges $d_{M,G,F}$-to 0, it follows from the definition of $d_{M,G,F}$-convergence that y also converges $d_{M,G,F}$-to 0. Thus $y \in c^0_{d_{M,G,F}}$, so the method is of R-type [13, 8].

□

Example 6.9 (Numerical Illustration of $d_{M,G,F}$-Summability)**.** *Consider the divergent sequence $y_n = (-1)^n$, which oscillates between -1 and 1 and hence does not converge classically. Let us take $M_m = C$ (the Cesàro matrix) for all m, $G = \{\mathbb{N}\}$, and $F = \mathcal{F}_0$. For this choice one has $d_{M,G,F} = d_C$, i.e. the associated density is the usual (lower) asymptotic density, and therefore $d_{M,G,F}$-convergence coincides with statistical convergence.*

In particular, y is not *$d_{M,G,F}$-convergent (hence not $d_{M,G,F}$-summable, since the generalized limit $S(y) = \lim_{d_{M,G,F}} y_n$ does not exist). Indeed, for $L = 0$ and $\varepsilon = \frac{1}{2}$,*

$$\{n \in \mathbb{N} : \ |y_n - L| \geq \varepsilon\} = \mathbb{N},$$

so $d_{M,G,F}(\mathbb{N}) = 1 \neq 0$, and $y_n \not\to 0$ in the $d_{M,G,F}$-sense.

Nevertheless, y is Cesàro summable *to* 0*. Indeed, the Cesàro transform is*

$$(Cy)_n = \frac{y_1 + \cdots + y_n}{n},$$

and a direct computation gives $(Cy)_{2k} = 0$ and $(Cy)_{2k+1} = -\frac{1}{2k+1} \to 0$. Hence $(Cy)_n \to 0$. This highlights the distinction between $d_{M,G,F}$-summability (defined via $d_{M,G,F}$-limits of the original sequence) and classical matrix summability (here, Cesàro summability via the transform Cy).

6.3. **Special Cases and Applications.**

Theorem 6.12 (Recovery of classical convergence types)**.** *The $d_{M,G,F}$-framework recovers several well-known convergence notions:*

(i) If $M_m = C$ (Cesàro matrix) for all m, $G = \{\mathbb{N}\}$ and $F = \mathcal{F}_0$, then $d_{M,G,F}$-convergence is the (lower) statistical convergence, since

$$d_{C,\{\mathbb{N}\},\mathcal{F}_0}(A) = \liminf_{n\to\infty} \frac{|A \cap \{1,\ldots,n\}|}{n}.$$

(ii) If $M = \{CN^m\}_{m\geq 0}$, $G = \{\mathbb{N}\}$ and $F = \mathcal{F}_0$, then $d_{M,G,F}$-convergence coincides with uniform statistical (Buck) convergence. Indeed, since C is the Cesàro matrix and N is the forward shift, we have

$$(CN^m x)_n = (C(N^m x))_n = \frac{1}{n}\sum_{k=1}^{n}(N^m x)_k = \frac{1}{n}\sum_{k=1}^{n} x_{m+k}.$$

In particular, for $x = X_A$,

$$(CN^m X_A)_n = \frac{|A \cap \{m+1,\ldots,m+n\}|}{n}.$$

Hence

$$d_{M,G,F}(A) = \liminf_{n\to\infty} \inf_{m\geq 0} \frac{|A \cap \{m+1,\ldots,m+n\}|}{n} =: u_*(A).$$

Therefore $d_{M,G,F}(\{n : |x_n - L| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$ is equivalent to $u_*(\{n : |x_n - L| \geq \varepsilon\}) = 0$ for all $\varepsilon > 0$, i.e. uniform statistical (Buck) convergence.

(iii) If $M_m = A$ (a fixed non-negative regular matrix) for all m, $G = \{\mathbb{N}\}$ and $F = \mathcal{F}_0$, then $d_{M,G,F}$-convergence coincides with A-statistical convergence.

(iv) If $M_m = I$, $G = \{\mathbb{N}\}$ and F is an ultrafilter, then $d_{M,G,F}$-convergence reduces to ultrafilter convergence. (If, in addition, one imposes shift-invariance, one obtains Banach limits [7]*.)*

(v) If $M_m = I$, $G = \{\mathbb{N}\}$ and $F = \{A^c : A \in \mathcal{I}\}$ for an ideal $\mathcal{I}$, then $d_{M,G,F}$-convergence coincides with $\mathcal{I}$-convergence (ideal convergence).

Proof. Each case follows directly from the corresponding specialization in Section 4 and the definition of the summability functional S. For Banach limits, the use of an ultrafilter F ensures that the functional is defined on all bounded sequences and satisfies the shift-invariance and positivity properties characteristic of Banach limits [7]. For matrix methods, the regularity of the matrix A guarantees that the resulting summability method preserves classical limits. □

Remark 6.6. *The unified framework of $d_{M,G,F}$-summability provides several significant advantages:*

- *It allows for the systematic study of convergence-preserving functionals on sequence spaces within a single theoretical framework.*

- *The choice of parameters M, G, and F enables the construction of summability methods tailored to specific applications or theoretical needs.*
- *It reveals the fundamental connections between seemingly disparate summability methods, showing how they arise from different notions of "large" sets and density.*
- *It facilitates the creation of novel summability methods by exploring previously unconsidered combinations of matrices and filters.*

7. Conclusion and Future Work

We have developed a comprehensive unified framework for sequence convergence based on Chun's generalized density $d_{M,G,F}$. Our main contributions include:

- Defining $d_{M,G,F}$-convergence as a master concept that encompasses classical, statistical, filter, ideal, and matrix convergence
- Systematically recovering known convergence types through appropriate parameter choices
- Establishing a hierarchy of convergence spaces with proper inclusions
- Developing the associated summability theory with linearity, regularity, and continuity properties
- Introducing new convergence types not previously studied

The flexibility in choosing M, G, and F allows both generality and specificity within a coherent theoretical framework. Promising directions for future research include:

Investigate completeness and topological structure of $c_{d_{M,G,F}}$ spaces.
Study dual spaces and connections with Banach limits.
Extend to double sequences, non-Archimedean fields, and variable parameters.
Explore uses in ergodic theory, approximation theory, and harmonic analysis.
Develop algorithms for determining $d_{M,G,F}$-convergence.

The $d_{M,G,F}$-approach provides a powerful platform for both theoretical investigations and practical applications, opening new avenues in generalized convergence theory.

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