

Group with double operators

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ABSTRACT. A group with double operators $(\mathcal{G}, \mathcal{M}, \mathcal{N})$ is defined as a group $\mathcal{G} = (\mathcal{G}, \cdot)$ equipped with two actions: a left action $*_1 : \mathcal{M} \times \mathcal{G} \rightarrow \mathcal{G}$ given by $(m, g) \mapsto m *_1 g$ and a right action $*_2 : \mathcal{G} \times \mathcal{N} \rightarrow \mathcal{G}$ given by $(g, n) \mapsto g *_2 n$. These actions are distributive with respect to the group operation; that is, for all $m \in \mathcal{M}, n \in \mathcal{N}$, and $g, h \in \mathcal{G}$, the following hold:

$$m *_1 (g \cdot h) = (m *_1 g) \cdot (m *_1 h) \text{ and } (g \cdot h) *_2 n = (g *_2 n) \cdot (h *_2 n).$$

The sets \mathcal{M} and \mathcal{N} are referred as the operator domains. Moreover, for each $m \in \mathcal{M}$ and $n \in \mathcal{N}$, the maps $g \mapsto m *_1 g$ and $g \mapsto g *_2 n$ are group endomorphisms of \mathcal{G} . The actions also satisfy a compatibility condition:

$$(m *_1 g) *_2 n = m *_1 (g *_2 n), \text{ for all } m \in \mathcal{M}, n \in \mathcal{N}, \text{ and } g \in \mathcal{G}.$$

This structure gives rise to the notion of an $(\mathcal{M} - \mathcal{N})$ -group. This paper formally defines $(\mathcal{M} - \mathcal{N})$ -groups, explore their algebraic properties, presents illustrative examples, and discusses potential applications.

1. INTRODUCTION

Algebraic structures enriched by additional operations or external symmetries play a central role in modern mathematics, with groups serving as one of the most fundamental frameworks for modeling symmetry and transformation. Classical group theory has been extended in various directions by incorporating operator actions, endomorphisms, and compatible mappings acting externally on group elements, leading to rich algebraic systems with broad applicability. Foundational contributions to this line of research include the theory of groups with operators developed by Jacobson [3] and the systematic treatment of algebraic structures in Bourbaki [1].

In parallel, the study of actions of semigroups and monoids on algebraic structures has gained increasing attention, motivated both by intrinsic algebraic interest and by applications in automata theory, representation theory, and categorical algebra. Seminal works by Clifford and Preston [8] and Howie [9] provide a comprehensive framework for understanding external operator actions beyond the classical group setting. More recently, contemporary research has focused on generalized action systems and compatibility conditions, such as actions of E-dense semigroups and monoid self-actions, highlighting the structural role of endomorphic behavior and distributive properties (see, for example, Renshaw [5] and Mayeux [6]).

Within this broader context, the concept of a group equipped with multiple compatible operator actions arises naturally. In particular, structures involving dual or bidirectional actions have appeared in recent literature under various forms, including generalized digroups and action-representation systems. Marín-Gaviria et al. [4] investigate actions and representations of g-digroups, emphasizing compatibility between operations and action domains, while Reis, San Martín, and Rocha [7] study reversibility and transitivity properties of semigroup actions on homogeneous spaces. These developments indicate a

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growing interest in algebraic systems governed by more than one external action.

Motivated by these ideas, we introduce in this paper the notion of a group with double operators, or an $(\mathcal{M} - \mathcal{N})$ -group. An $(\mathcal{M} - \mathcal{N})$ -group consists of a group \mathcal{G} equipped with a left action by an operator domain \mathcal{M} and a right action by an operator domain \mathcal{N} , both acting distributively over the group operation and inducing group endomorphisms. A crucial compatibility condition ensures coherent interaction between the two actions. This framework generalizes classical operator groups [3] and extends existing action-based structures by allowing two independent yet compatible operator domains.

The purpose of this paper is to initiate a systematic study of $(\mathcal{M} - \mathcal{N})$ -groups and their associated algebraic properties. We introduce and analyze $(\mathcal{M} - \mathcal{N})$ -subgroups and normal subgroups, examine their behavior under standard group-theoretic constructions, and define $(\mathcal{M} - \mathcal{N})$ -quotient groups. Furthermore, we study homomorphisms between $(\mathcal{M} - \mathcal{N})$ -groups and establish analogues of classical homomorphism theorems within this enriched setting. Through illustrative examples, we demonstrate how the theory unifies and extends known operator-based constructions, while opening new avenues for further research in generalized group actions and related algebraic frameworks.

2. $(\mathcal{M} - \mathcal{N})$ -GROUPS

In this section, we present the concept of $(\mathcal{M} - \mathcal{N})$ -groups and define $(\mathcal{M} - \mathcal{N})$ -(normal) subgroups within a group \mathcal{G} . We explore various structural properties, including the intersection, union, sum, and product of $(\mathcal{M} - \mathcal{N})$ -subgroups. To illustrate and validate these results, several examples are provided. Furthermore, we introduce the notion of an $(\mathcal{M} - \mathcal{N})$ -quotient group and discuss related theoretical developments.

Definition 2.1. Let (\mathcal{G}, \cdot) be a group. We say that \mathcal{G} is an $(\mathcal{M} - \mathcal{N})$ -group if there exists two sets \mathcal{M} and \mathcal{N} , along with the following structures:

- (1) A **left action** $*_1 : \mathcal{M} \times \mathcal{G} \rightarrow \mathcal{G}$, denoted $(m, g) \mapsto m *_1 g$ such that for each $m \in \mathcal{M}$, the map $f_m : \mathcal{G} \rightarrow \mathcal{G}$ defined by $f_m(g) = m *_1 g$ is a group endomorphism of \mathcal{G} , that is, for all $g, h \in \mathcal{G}$ we have

$$m *_1 (g \cdot h) = (m *_1 g) \cdot (m *_1 h).$$

- (2) A **right action** $*_2 : \mathcal{G} \times \mathcal{N} \rightarrow \mathcal{G}$, denoted $(g, n) \mapsto g *_2 n$ such that for each $n \in \mathcal{N}$, the map $g_n : \mathcal{G} \rightarrow \mathcal{G}$ defined by $g_n(g) = g *_2 n$ is a group endomorphism of \mathcal{G} , that is, for all $g, h \in \mathcal{G}$ we have

$$(g \cdot h) *_2 n = (g *_2 n) \cdot (h *_2 n).$$

- (3) A **compatibility condition between the two actions** For all $m \in \mathcal{M}, n \in \mathcal{N}$, and $g \in \mathcal{G}$, the following hold

$$(m *_1 g) *_2 n = m *_1 (g *_2 n) \text{ or equivalently, } (f_m \circ g_n)(g) = (g_n \circ f_m)(g).$$

In this context, the structure $(\mathcal{G}, \cdot, \mathcal{M}, \mathcal{N}, *_1, *_2)$ is referred to as an $(\mathcal{M} - \mathcal{N})$ -group.

Example 2.1. Let \mathcal{G} be any Abelian group. Take $\mathcal{M} = \mathcal{N} = \text{Aut}(\mathcal{G})$. Then \mathcal{G} is an $(\text{Aut}(\mathcal{G}) - \text{Aut}(\mathcal{G}))$ -group, where the left and right $\text{Aut}(\mathcal{G})$ actions on \mathcal{G} are defined as follows:

$$(\phi, g) \mapsto \phi *_1 g = \phi(g) \text{ and } (g, \psi) \mapsto g *_2 \psi = \psi(g), \text{ for all } g \in \mathcal{G}, \phi, \psi \in \text{Aut}(\mathcal{G}).$$

Also, for all $g, h \in \mathcal{G}, \phi, \psi \in \text{Aut}(\mathcal{G})$, we have

$$\phi *_1 (gh) = \phi(gh) = \phi(g)\phi(h) = (\phi *_1 g)(\phi *_1 h)$$

and

$$(gh) *_2 \psi = \psi(gh) = \psi(g)\psi(h) = (g *_2 \psi)(h *_2 \psi).$$

Furthermore, for all $g \in \mathcal{G}$, $\phi, \psi \in \text{Aut}(\mathcal{G})$, we have

$$(\phi *_1 g) *_2 \psi = \phi(g) *_2 \psi = \psi(\phi(g)) = (\psi \circ \phi)(g) = (\phi \circ \psi)(g) = \phi(\psi(g)) = \phi(\psi *_2 g) = \phi *_1 (g *_2 \psi).$$

Remark 2.1. By Example 2.1 we notice that any group \mathcal{G} whose $\text{Aut}(\mathcal{G})$ is an abelian group is an $(\text{Aut}(\mathcal{G}) - \text{Aut}(\mathcal{G}))$ -group.

Example 2.2. Consider the group $\mathcal{G} = \mathcal{S}_3 = \{i, (12), (13), (23), (123), (132)\}$, the symmetric group on 3 elements 1, 2, 3. Let $\mathcal{M} = \{m\}$ and $\mathcal{N} = \{n\}$, and $\theta = (12) \in \mathcal{S}_3$ be a fixed element. The left and right actions of \mathcal{M} and \mathcal{N} on \mathcal{G} are defined respectively, as:

$$m *_1 \sigma = \theta \sigma \theta^{-1} \text{ and } \sigma *_2 n = \theta^{-1} \sigma \theta.$$

Then each action defines an endomorphism, conjugation preserve group structure.

Moreover, for the compatibility condition: For all $m \in \mathcal{M}$, $n \in \mathcal{N}$, and $\sigma \in \mathcal{G}$, we have

$$(m *_1 \sigma) *_2 n = (\theta \sigma \theta^{-1}) *_2 n = \theta^{-1} (\theta \sigma \theta^{-1}) \theta = \sigma = m *_1 (\sigma *_2 n).$$

Thus, \mathcal{G} is an $(\mathcal{M} - \mathcal{N})$ -group.

Example 2.3. Consider $\mathcal{G} = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ be the abelian group under addition modulo 4. Let $\mathcal{M} = \mathcal{N} = \{1, 3\}$. The left and right actions of \mathcal{M} and \mathcal{N} on \mathcal{G} are defined respectively, as:

$$f_m(g) = m *_1 g = mg(\text{mod}4) \text{ and } g_n(g) = g *_2 n = gn(\text{mod}4)$$

It is easy to verify that f_1, f_3, g_1, g_3 are group endomorphisms over \mathcal{G} .

Moreover, for the compatibility condition: For all $m \in \mathcal{M}$, $n \in \mathcal{N}$, and $g \in \mathcal{G}$, we have

$$(m *_1 g) *_2 n = (mg)(\text{mod}4) *_2 n = (mgn)(\text{mod}4) = m *_1 (g *_2 n).$$

Thus, \mathcal{G} is an $(\mathcal{M} - \mathcal{N})$ -group.

Example 2.4. Consider the group $\mathcal{G} = \mathcal{D}_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$, the dihedral group of symmetries of a square, where $r^4 = e, s^2 = e, sr = r^{-1}s$.

Let $\mathcal{M} = \mathcal{N} = \{s\}$. The left and right actions of \mathcal{M} and \mathcal{N} on \mathcal{G} are defined respectively, as:

$$f_m(g) = m *_1 g = sgs \text{ and } g_n(g) = g *_2 n = sgs$$

It is easy to verify that both f_s and g_s are group endomorphisms over \mathcal{G} .

Moreover, for the compatibility condition: For all $m \in \mathcal{M}$, $n \in \mathcal{N}$, and $g \in \mathcal{G}$, we have

$$(m *_1 g) *_2 n = (sgs) *_2 n = s(sgs)s = g = m *_1 (g *_2 n).$$

Thus, \mathcal{G} is an $(\mathcal{M} - \mathcal{N})$ -group.

Example 2.5. Consider the group $\mathcal{G} = \mathcal{D}_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$, the dihedral group of symmetries of a square, where $r^4 = e, s^2 = e, sr = r^{-1}s$.

Let $\mathcal{M} = \mathcal{N} = \{0, 1\}$. The left and right actions of \mathcal{M} and \mathcal{N} on \mathcal{G} are defined respectively, as:

$$m *_1 g = \begin{cases} g, & \text{if } m = 0 \\ r^2 g r^2, & \text{if } m = 1. \end{cases}$$

and

$$g *_2 n = \begin{cases} g, & \text{if } n = 0 \\ sgs, & \text{if } n = 1. \end{cases}$$

Also, for each $m \in \mathcal{M}$, $n \in \mathcal{N}$ and $g, h \in \mathcal{G}$, we have

$$m *_1 (gh) = (m *_1 g)(m *_1 h) \text{ and } (gh) *_2 n = (g *_2 n)(h *_2 n).$$

Further, let $m \in \mathcal{M}$, $n \in \mathcal{N}$, and $g \in \mathcal{G}$ be any elements. Then:

$$(m *_1 g) *_2 n = \begin{cases} g, & \text{if } m = 0, n = 0 \\ sgs, & \text{if } m = 0, n = 1 \\ r^2gr^2, & \text{if } m = 1, n = 0 \\ (sr^2)g(sr^2s), & \text{if } m = 1, n = 1. \end{cases}$$

and

$$m *_1 (g *_2 n) = \begin{cases} g, & \text{if } m = 0, n = 0 \\ sgs, & \text{if } m = 0, n = 1 \\ r^2gr^2, & \text{if } m = 1, n = 0 \\ (r^2s)g(sr^2), & \text{if } m = 1, n = 1. \end{cases}$$

Since r^2 commutes with every element of \mathcal{G} , so $r^2s = sr^2$.

Thus $(m *_1 g) *_2 n = m *_1 (g *_2 n)$. Hence, \mathcal{G} is an $(\mathcal{M} - \mathcal{N})$ -group.

Definition 2.2. Let \mathcal{G} be an $(\mathcal{M} - \mathcal{N})$ -group. A subset \mathcal{H} of \mathcal{G} is called an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G} if \mathcal{H} is also an $(\mathcal{M} - \mathcal{N})$ -group under the same action of \mathcal{M} and \mathcal{N} on \mathcal{H} .

Definition 2.3. Let \mathcal{G} be an $(\mathcal{M} - \mathcal{N})$ -group. A subgroup \mathcal{H} of \mathcal{G} is called an $(\mathcal{M} - \mathcal{N})$ -normal subgroup of \mathcal{G} if:

- (1) \mathcal{H} is a normal subgroup of \mathcal{G} and
- (2) \mathcal{H} is invariant under the actions of \mathcal{M} and \mathcal{N} , meaning:

$$m *_1 h \in \mathcal{H} \text{ and } h *_2 n \in \mathcal{H}, \text{ for all } m \in \mathcal{M}, n \in \mathcal{N}, h \in \mathcal{H}.$$

Example 2.6. Consider $\mathcal{G} = \mathbb{Z}_4 \times \mathbb{Z}_2 = \{(0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (1, 1), (2, 1), (3, 1)\}$ be an Abelian group under the operation componentwise addition modulo 4 and modulo 2.

Let $\mathcal{M} = \{\alpha\}$, $\mathcal{N} = \{\beta\}$. The left and right actions of \mathcal{M} and \mathcal{N} on \mathcal{G} are defined respectively, as:

$$\alpha *_1 (a, b) = (2a(\text{mod}4), 0) \text{ and } (a, b) *_2 \beta = (0, b)$$

Now,

$$\begin{aligned} \alpha *_1 ((a, b) + (c, d)) &= \alpha *_1 ((a + c)(\text{mod}4), (b + d)(\text{mod}2)) \\ &= (2(a + c)(\text{mod}4), 0) \\ &= (2a(\text{mod}4), 0) + (2c(\text{mod}4), 0) \\ &= \alpha *_1 (a, b) + \alpha *_1 (c, d). \end{aligned}$$

and

$$\begin{aligned} ((a, b) + (c, d)) *_2 \beta &= ((a + c)(\text{mod}4), (b + d)(\text{mod}2)) *_2 \beta \\ &= (0, (b + d)(\text{mod}2)) \\ &= (0, b(\text{mod}2)) + (0, d(\text{mod}2)) \\ &= (a, b) *_2 \beta + (c, d) *_2 \beta. \end{aligned}$$

Thus, $*_1$ and $*_2$ are group homomorphisms on \mathcal{G} . Further, for compatibility condition, we have $(\alpha *_1 (a, b)) *_2 \beta = (2a(\text{mod}4), 0) *_2 \beta = (0, 0)$ and $\alpha *_1 ((a, b) *_2 \beta) = \alpha *_1 (0, b) = (0, 0)$.

Thus,

$$(\alpha *_1 (a, b)) *_2 \beta = \alpha *_1 ((a, b) *_2 \beta).$$

Hence, \mathcal{G} is an $(\mathcal{M} - \mathcal{N})$ -group.

Next, consider the subset $\mathcal{H} = \{(0, 0), (2, 0), (0, 1), (2, 1)\}$ of \mathcal{G} . It is easy to verify that \mathcal{H} is a normal subgroup of \mathcal{G} and it is invariant under the actions of \mathcal{M} and \mathcal{N} , for $\alpha *_1 (a, b) = (0, 0) \in \mathcal{H}$ and $(a, b) *_2 \beta \in \{(0, 0), (0, 1)\} \subseteq \mathcal{H}$, for all $\alpha \in \mathcal{M}, \beta \in \mathcal{N}$, and $(a, b) \in \mathcal{H}$. Hence \mathcal{H} is an $(\mathcal{M} - \mathcal{N})$ -normal subgroup of \mathcal{G} .

Remark 2.2. Clearly, there exist $(\mathcal{M}-\mathcal{N})$ -subgroups of \mathcal{G} that are not $(\mathcal{M}-\mathcal{N})$ normal subgroup of \mathcal{G} . For instance, if \mathcal{G} is any non-abelian $(\mathcal{M}-\mathcal{N})$ -group and \mathcal{H} is an $(\mathcal{M}-\mathcal{N})$ -subgroup of \mathcal{G} that is not invariant under the actions of \mathcal{M} and \mathcal{N} , then \mathcal{H} is not an $(\mathcal{M}-\mathcal{N})$ normal subgroup of \mathcal{G} .

Theorem 2.1. If \mathcal{H}_1 and \mathcal{H}_2 are two $(\mathcal{M}-\mathcal{N})$ subgroups of \mathcal{G} , then $\mathcal{H}_1 \cap \mathcal{H}_2$ is also an $(\mathcal{M}-\mathcal{N})$ -subgroup of \mathcal{G} .

Proof. It is a known fact that $\mathcal{H}_1 \cap \mathcal{H}_2$ is a subgroup of \mathcal{G} (see Theorem 2.37 of [2]). Next, we show that $\mathcal{H}_1 \cap \mathcal{H}_2$ is also an $(\mathcal{M}-\mathcal{N})$ subgroup of \mathcal{G} .

Let $m \in \mathcal{M}, n \in \mathcal{N}$, and $h \in \mathcal{H}_1 \cap \mathcal{H}_2$. Then $h \in \mathcal{H}_1$ and $h \in \mathcal{H}_2$.

Thus, $m *_1 h \in \mathcal{H}_1$ and $m *_1 h \in \mathcal{H}_2$ implies $m *_1 h \in \mathcal{H}_1 \cap \mathcal{H}_2$.

Similarly, we can show that $h *_2 n \in \mathcal{H}_1 \cap \mathcal{H}_2$. Hence, $\mathcal{H}_1 \cap \mathcal{H}_2$ is an $(\mathcal{M}-\mathcal{N})$ -subgroup of \mathcal{G} . \square

Now, the following result is immediate to follows:

Proposition 2.1. If $\{\mathcal{H}_i : i \in J\}$ is an arbitrary collection of $(\mathcal{M}-\mathcal{N})$ -subgroups of \mathcal{G} , then $\bigcap_{i \in J} \mathcal{H}_i$ is also an $(\mathcal{M}-\mathcal{N})$ -subgroup of \mathcal{G} .

However, the union of two $(\mathcal{M}-\mathcal{N})$ -subgroups of \mathcal{G} need not be an $(\mathcal{M}-\mathcal{N})$ -subgroup of \mathcal{G} . See the following Example:

Example 2.7. Consider the group $\mathcal{G} = \mathbb{Z}$, the group of integers under addition. Then \mathcal{G} can be regarded as an $(\text{Aut}(\mathcal{G}) - \text{Aut}(\mathcal{G}))$ -group (by Remark 2.1).

Consider the subsets $\mathcal{H}_1 = 3\mathbb{Z}$ and $\mathcal{H}_2 = 2\mathbb{Z}$ of \mathcal{G} . Now, it is easy to verify that \mathcal{H}_1 and \mathcal{H}_2 are subgroups of \mathcal{G} , and hence are $(\text{Aut}(\mathcal{G}) - \text{Aut}(\mathcal{G}))$ -subgroups of \mathcal{G} . However, the union $\mathcal{H}_1 \cup \mathcal{H}_2$ is not even a subgroup of \mathcal{G} but to talk about $(\text{Aut}(\mathcal{G}) - \text{Aut}(\mathcal{G}))$ -subgroup of \mathcal{G} . Hence, $\mathcal{H}_1 \cup \mathcal{H}_2$ is not an $(\text{Aut}(\mathcal{G}) - \text{Aut}(\mathcal{G}))$ -subgroup of \mathcal{G} .

Theorem 2.2. Let \mathcal{H}_1 and \mathcal{H}_2 be two $(\mathcal{M}-\mathcal{N})$ -subgroups of an $(\mathcal{M}-\mathcal{N})$ -group \mathcal{G} . Then $\mathcal{H}_1 \circ \mathcal{H}_2$ is also an $(\mathcal{M}-\mathcal{N})$ -subgroup of \mathcal{G} , provided $\mathcal{H}_1 \circ \mathcal{H}_2 = \mathcal{H}_2 \circ \mathcal{H}_1$.

Proof. By Theorem 2.39 of [2], $\mathcal{H}_1 \circ \mathcal{H}_2$ is a subgroup of \mathcal{G} if and only if $\mathcal{H}_1 \circ \mathcal{H}_2 = \mathcal{H}_2 \circ \mathcal{H}_1$. Next, let $m \in \mathcal{M}, n \in \mathcal{N}$, and $h \in \mathcal{H}_1 \circ \mathcal{H}_2$ be any elements.

Suppose $h = h_1 h_2$, where $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$, then

$$\begin{aligned} (m *_1 h) *_2 n &= (m *_1 h_1 h_2) *_2 n \\ &= [(m *_1 h_1)(m *_1 h_2)] *_2 n \\ &= [(m *_1 h_1) *_2 n][(m *_1 h_2) *_2 n] \\ &= [m *_1 (h_1 *_2 n)][m *_1 (h_2 *_2 n)] \\ &= m *_1 [(h_1 *_2 n)(h_2 *_2 n)] \\ &= m *_1 [(h_1 h_2) *_2 n] \\ &= m *_1 (h *_2 n). \end{aligned}$$

Hence, $\mathcal{H}_1 \circ \mathcal{H}_2$ is an $(\mathcal{M}-\mathcal{N})$ -subgroup of \mathcal{G} . \square

Example 2.8. Consider the group $\mathcal{G} = \mathcal{D}_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$, the dihedral group of symmetries of a square, where $r^4 = e, s^2 = e, sr = r^{-1}s$.

Let $\mathcal{M} = \mathcal{N} = \{0, 1\}$. The left and right actions of \mathcal{M} and \mathcal{N} on \mathcal{G} are defined respectively, as:

$$m *_1 g = \begin{cases} g, & \text{if } m = 0 \\ r^2 g r^2, & \text{if } m = 1. \end{cases}$$

and

$$g *_2 n = \begin{cases} g, & \text{if } n = 0 \\ sgs, & \text{if } n = 1. \end{cases}$$

Let $m \in \mathcal{M}, n \in \mathcal{N}$, and $g \in \mathcal{G}$ be any elements. Then:

$$(m *_1 g) *_2 n = m *_1 (g *_2 n)$$

Thus, \mathcal{G} is an $(\mathcal{M} - \mathcal{N})$ -group. Take $\mathcal{H}_1 = \{e, r^2\}$ and $\mathcal{H}_2 = \{e, s\}$. Now, it can be easily verified that both \mathcal{H}_1 and \mathcal{H}_2 are $(\mathcal{M} - \mathcal{N})$ -subgroups of \mathcal{G} and that $*_1$ and $*_2$ are group homomorphisms. i.e., for each $m \in \mathcal{M}, n \in \mathcal{N}$ and $g, h \in \mathcal{G}$, we have

$$m *_1 (gh) = (m *_1 g)(m *_1 h) \text{ and } (gh) *_2 n = (g *_2 n)(h *_2 n).$$

Moreover, $\mathcal{H}_1 \circ \mathcal{H}_2 = \{e, r^2, s, r^2s\}$ is an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G} such that $\mathcal{H}_1 \circ \mathcal{H}_2 = \mathcal{H}_2 \circ \mathcal{H}_1$.

Theorem 2.3. Let \mathcal{H}_1 and \mathcal{H}_2 be two $(\mathcal{M} - \mathcal{N})$ -subgroups of an $(\mathcal{M} - \mathcal{N})$ -group \mathcal{G} . Then the subgroup $\mathcal{H}_1 + \mathcal{H}_2 := \langle \mathcal{H}_1 \cup \mathcal{H}_2 \rangle$, i.e., the subgroup generated by $\mathcal{H}_1 \cup \mathcal{H}_2$, is the smallest $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G} that contain both \mathcal{H}_1 and \mathcal{H}_2 .

Proof. Let $\mathcal{H} = \langle \mathcal{H}_1 \cup \mathcal{H}_2 \rangle$ be the subgroup generated by $\mathcal{H}_1 \cup \mathcal{H}_2$.

Suppose $h \in \mathcal{H}$ and let $h = h_1 h_2^{-1} \dots h_k$, where $h_i \in \mathcal{H}_1 \cup \mathcal{H}_2$.

Then, for each $m \in \mathcal{M}, n \in \mathcal{N}$, we have

$$m *_1 h = m *_1 (h_1 h_2^{-1} \dots h_k) = (m *_1 h_1)(m *_1 h_2^{-1}) \dots (m *_1 h_k)$$

and

$$h *_2 n = (h_1 h_2^{-1} \dots h_k) *_2 n = (h_1 *_2 n)(h_2^{-1} *_2 n) \dots (h_k *_2 n).$$

Now, since \mathcal{H}_1 and \mathcal{H}_2 are $(\mathcal{M} - \mathcal{N})$ -subgroups, they are closed under both the actions. Hence $m *_1 h_i, h_i *_2 n \in \mathcal{H}$ and the inverses are also in \mathcal{H} . So, $m *_1 h, h *_2 n \in \mathcal{H}$.

Also, for each $m \in \mathcal{M}, n \in \mathcal{N}$, and $h \in \mathcal{H}$, we have $(m *_1 h) *_2 n = m *_1 (h *_2 n)$.

This implies that \mathcal{H} is an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G} . Obviously, $\mathcal{H} \supseteq \mathcal{H}_1$ and $\mathcal{H} \supseteq \mathcal{H}_2$.

Next, let \mathcal{K} be any $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G} containing both \mathcal{H}_1 and \mathcal{H}_2 , then we claim $\mathcal{H} \subseteq \mathcal{K}$. Since \mathcal{K} is an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G} containing $\mathcal{H}_1 \cup \mathcal{H}_2$, it must also contain all finite product of elements and their inverses from $\mathcal{H}_1 \cup \mathcal{H}_2$, i.e., $\mathcal{K} \supseteq \langle \mathcal{H}_1 \cup \mathcal{H}_2 \rangle = \mathcal{H}$.

Thus, $\mathcal{H}_1 + \mathcal{H}_2 = \langle \mathcal{H}_1 \cup \mathcal{H}_2 \rangle$ is the smallest $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G} containing both $\mathcal{H}_1, \mathcal{H}_2$. \square

Example 2.9. Consider the $(\mathcal{M} - \mathcal{N})$ -group \mathcal{G} same as in Example 2.7.

Also, let $\mathcal{H}_1 = \{e, r^2\}$ and $\mathcal{H}_2 = \{e, s\}$ are both $(\mathcal{M} - \mathcal{N})$ -subgroups of \mathcal{G} . Then

$$\mathcal{H}_1 + \mathcal{H}_2 = \langle \mathcal{H}_1 \cup \mathcal{H}_2 \rangle = \langle \{e, r^2, s\} \rangle = \{e, r^2, s, sr^2\}.$$

It is now easy to verify that $\mathcal{H}_1 + \mathcal{H}_2$ is an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G} .

3. $(\mathcal{M} - \mathcal{N})$ -HOMOMORPHISM OF $(\mathcal{M} - \mathcal{N})$ -GROUPS

In this section, we introduce the notion of an $(\mathcal{M} - \mathcal{N})$ -homomorphism of $(\mathcal{M} - \mathcal{N})$ -groups. We investigate the image and preimage of $(\mathcal{M} - \mathcal{N})$ -groups under such homomorphisms. We also establish homomorphism theorems related to these $(\mathcal{M} - \mathcal{N})$ -homomorphisms.

Definition 3.4. Let \mathcal{G}_1 and \mathcal{G}_2 be two $(\mathcal{M} - \mathcal{N})$ groups, and let $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be a group homomorphism. Then f is called an $(\mathcal{M} - \mathcal{N})$ -homomorphism if:

$$f(m *_1 g) = m *_1 f(g) \text{ and } f(g *_2 n) = f(g) *_2 n$$

for all $m \in \mathcal{M}, n \in \mathcal{N}$, and $g \in \mathcal{G}_1$.

Example 3.10. (i) Let $\mathcal{G}_1 = \mathcal{G}_2 = \mathbb{Z}$, the group of integers under addition.

Take $\mathcal{M} = \mathcal{N} = \text{Aut}(\mathbb{Z}) = \{i, \phi\}$, where i is the identity homomorphism, and ϕ is the inverse homomorphism i.e., $\phi(x) = -x, \forall x \in \mathbb{Z}$. Define left and right actions of \mathcal{M}, \mathcal{N} respectively, as:

$$m *_1 g = m(g) \text{ and } g *_2 n = n(g),$$

for all $g \in \mathbb{Z}, m \in \mathcal{M}, n \in \mathcal{N}$. Then, \mathcal{G}_1 and \mathcal{G}_2 are $(\mathcal{M} - \mathcal{N})$ groups. (By Remark 2.1)

The map $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ defined by $f(g) = 2g$ for all $g \in \mathcal{G}_1$, is an $(\mathcal{M} - \mathcal{N})$ -homomorphism.

(ii) Let $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{D}_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$, the dihedral group of symmetries of a square, where $r^4 = e, s^2 = e, sr = r^{-1}s$.

Let $\mathcal{M} = \mathcal{N} = \{s\}$. The left and right actions of \mathcal{M} and \mathcal{N} on \mathcal{G}_1 (or \mathcal{G}_2) are defined respectively, as:

$$m *_1 g = sgs \text{ and } g *_2 n = sgs, \text{ for all } m \in \mathcal{M}, n \in \mathcal{N}, \text{ and } g \in \mathcal{G}_1.$$

Clearly, \mathcal{G}_1 and \mathcal{G}_2 are $(\mathcal{M} - \mathcal{N})$ groups. (See Example 2.4)

Then the map $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ defined by

$$f(g) = \begin{cases} e, & \text{if } g \in \{e, r^2, s, sr^2\} \\ r^2, & \text{if } g \in \{r, r^3, sr, sr^3\}. \end{cases}$$

Now, it is easy to verify that f is an $(\mathcal{M} - \mathcal{N})$ -homomorphism.

Theorem 3.4. Let $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be an $(\mathcal{M} - \mathcal{N})$ injective homomorphism, and let \mathcal{H}' be an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G}_2 . Then the preimage $f^{-1}(\mathcal{H}')$ is an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G}_1 .

Proof. It is already known that $f^{-1}(\mathcal{H}')$ is a subgroup of \mathcal{G}_1 (see Problem 7(ii) of [2]).

Furthermore, let $h \in f^{-1}(\mathcal{H}'), m \in \mathcal{M}$, and $n \in \mathcal{N}$ be any elements. Then $f(h) \in \mathcal{H}'$.

As \mathcal{H}' is an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G}_2 . So, $(m *_1 f(h)) *_2 n = m *_1 (f(h) *_2 n)$

$$\Rightarrow f(m *_1 h) *_2 n = m *_1 f(h *_2 n) \Rightarrow f((m *_1 h) *_2 n) = f(m *_1 (h *_2 n))$$

$$\Rightarrow (m *_1 h) *_2 n = m *_1 (h *_2 n) \text{ [As } f \text{ is injective]}$$

Next, let $h_1, h_2 \in f^{-1}(\mathcal{H}')$, so $f(h_1), f(h_2) \in \mathcal{H}'$. Then

$$\begin{aligned} f(m *_1 (h_1 h_2)) &= m *_1 f(h_1 h_2) \\ &= m *_1 (f(h_1) f(h_2)) \\ &= (m *_1 f(h_1))(m *_1 f(h_2)) \\ &= f(m *_1 h_1) f(m *_1 h_2) \\ &= f((m *_1 h_1)(m *_1 h_2)). \end{aligned}$$

Since f is injective, so $m *_1 (h_1 h_2) = (m *_1 h_1)(m *_1 h_2)$.

Similarly, we can show that $(h_1 h_2) *_2 n = (h_1 *_2 n)(h_2 *_2 n)$. Therefore, $*_1$ and $*_2$ are group homomorphism on $f^{-1}(\mathcal{H}')$. Hence, $f^{-1}(\mathcal{H}')$ is an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G}_1 . \square

Theorem 3.5. Let $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be an $(\mathcal{M} - \mathcal{N})$ injective homomorphism, and let \mathcal{H}' be an $(\mathcal{M} - \mathcal{N})$ normal subgroup of \mathcal{G}_2 . Then the preimage $f^{-1}(\mathcal{H}')$ is an $(\mathcal{M} - \mathcal{N})$ normal subgroup of \mathcal{G}_1 .

Proof. It is already known that $f^{-1}(\mathcal{H}')$ is a normal subgroup of \mathcal{G}_1 (see Problem 7(ii) of [2]). Furthermore, let $h \in f^{-1}(\mathcal{H}'), m \in \mathcal{M}$, and $n \in \mathcal{N}$ be any elements. Then $f(h) \in \mathcal{H}'$.

As \mathcal{H}' is an $(\mathcal{M} - \mathcal{N})$ normal subgroup of \mathcal{G}_2 . So, $m *_1 f(h) \in \mathcal{H}'$ and $f(h) *_2 n \in \mathcal{H}'$

$$\Rightarrow f(m *_1 h) \in \mathcal{H}' \text{ and } f(h *_2 n) \in \mathcal{H}' \Rightarrow m *_1 h \in f^{-1}(\mathcal{H}') \text{ and } h *_2 n \in f^{-1}(\mathcal{H}').$$

Also, by Theorem 3.4, $*_1$ and $*_2$ are group homomorphism on $f^{-1}(\mathcal{H}')$. Hence, $f^{-1}(\mathcal{H}')$ is an $(\mathcal{M} - \mathcal{N})$ normal subgroup of \mathcal{G}_1 . \square

Theorem 3.6. Let $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be a surjective $(\mathcal{M} - \mathcal{N})$ -homomorphism, and let \mathcal{H} be an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G}_1 . Then $f(\mathcal{H})$ is an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G}_2 .

Proof. It is already known that $f(\mathcal{H})$ is a subgroup of \mathcal{G}_2 (see Problem 7(i) of [2]). Furthermore, let $h' \in f(\mathcal{H})$, $m \in \mathcal{M}$, and $n \in \mathcal{N}$ be any elements. Then, there exists $h \in \mathcal{H}$ such that $f(h) = h'$. As \mathcal{H} is an $(\mathcal{M} - \mathcal{N})$ subgroup of \mathcal{G}_1 . So, $(m *_1 h) *_2 n = m *_1 (h *_2 n) \Rightarrow f[(m *_1 h) *_2 n] = f[m *_1 (h *_2 n)] \Rightarrow f(m *_1 h) *_2 n = m *_1 f(h *_2 n) \Rightarrow (m *_1 f(h)) *_2 n = m *_1 (f(h) *_2 n) \Rightarrow (m *_1 h') *_2 n = m *_1 (h' *_2 n)$.

Next, let $h'_1, h'_2 \in f(\mathcal{H})$, then \exists 's $h_1, h_2 \in \mathcal{H}$ such that $f(h_1) = h'_1$ and $f(h_2) = h'_2$. Then

$$\begin{aligned} m *_1 (h'_1 h'_2) &= m *_1 (f(h_1)f(h_2)) \\ &= (m *_1 f(h_1))(m *_1 f(h_2)) \\ &= (m *_1 h'_1)(m *_1 h'_2). \end{aligned}$$

Similarly, we can show that $(h'_1 h'_2) *_2 n = (h'_1 *_2 n)(h'_2 *_2 n)$. Therefore, $*_1$ and $*_2$ are group homomorphism on $f(\mathcal{H})$. Hence, $f(\mathcal{H})$ is an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G}_2 . \square

Theorem 3.7. Let $f : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ be a surjective $(\mathcal{M} - \mathcal{N})$ -homomorphism, and let \mathcal{H} be an $(\mathcal{M} - \mathcal{N})$ normal subgroup of \mathcal{G}_1 . Then $f(\mathcal{H})$ is an $(\mathcal{M} - \mathcal{N})$ normal subgroup of \mathcal{G}_2 .

Proof. It is already known that $f(\mathcal{H})$ is a normal subgroup of \mathcal{G}_2 (see Problem 7(iii) of [2]). Furthermore, let $h' \in f(\mathcal{H})$, $m \in \mathcal{M}$, and $n \in \mathcal{N}$ be any elements. Then, there exists $h \in \mathcal{H}$ such that $f(h) = h'$. As \mathcal{H} is an $(\mathcal{M} - \mathcal{N})$ normal subgroup of \mathcal{G}_1 . So, $m *_1 h \in \mathcal{H}$ and $h *_2 n \in \mathcal{H} \Rightarrow f(m *_1 h) \in f(\mathcal{H})$ and $f(h *_2 n) \in f(\mathcal{H}) \Rightarrow m *_1 f(h) \in f(\mathcal{H})$ and $f(h) *_2 n \in f(\mathcal{H}) \Rightarrow m *_1 h' \in f(\mathcal{H})$ and $h' *_2 n \in f(\mathcal{H})$.

Also, by Theorem 3.6, $*_1$ and $*_2$ are group homomorphism on $f(\mathcal{H})$. Hence, $f(\mathcal{H})$ is an $(\mathcal{M} - \mathcal{N})$ normal subgroup of \mathcal{G}_2 . \square

Definition 3.5. Let \mathcal{G} be an $(\mathcal{M} - \mathcal{N})$ group, and let \mathcal{H} be an $(\mathcal{M} - \mathcal{N})$ -subgroup of \mathcal{G} . Let $x \in \mathcal{G}$ be a fixed element. Then the sets

$$x\mathcal{H} = \{xy : y \in \mathcal{H}\} \text{ and } \mathcal{H}x = \{yx : y \in \mathcal{H}\}$$

called respectively the left coset and the right coset of \mathcal{H} with respect to x in \mathcal{G} .

Theorem 3.8. Let \mathcal{G} be an $(\mathcal{M} - \mathcal{N})$ group, and let \mathcal{H} be an $(\mathcal{M} - \mathcal{N})$ normal subgroup of \mathcal{G} . Then the collection of left cosets $\mathcal{G}/\mathcal{H} = \{x\mathcal{H} : x \in \mathcal{G}\}$ under the operation \circ defined by:

$$(x\mathcal{H}) \circ (y\mathcal{H}) = (xy)\mathcal{H}, \forall x, y \in \mathcal{G}.$$

is an $(\mathcal{M} - \mathcal{N})$ -group.

Proof. It is a known fact that $(\mathcal{G}/\mathcal{H}, \circ)$ is a group (see Theorem 3.1 of [2]).

Next, we define left and right actions of the sets \mathcal{M} and \mathcal{N} , respectively on the group \mathcal{G}/\mathcal{H}

$$\mathcal{M} \times \mathcal{G}/\mathcal{H} \rightarrow \mathcal{G}/\mathcal{H} \text{ defined as } (m, g\mathcal{H}) = m *_1 g\mathcal{H} = (m *_1 g)\mathcal{H}$$

and

$$\mathcal{G}/\mathcal{H} \times \mathcal{N} \rightarrow \mathcal{G}/\mathcal{H} \text{ defined as } (g\mathcal{H}, n) = g\mathcal{H} *_2 n = (g *_2 n)\mathcal{H}, \text{ for all } m \in \mathcal{M}, n \in \mathcal{N}, g \in \mathcal{G}.$$

Now,

$$\begin{aligned} (m *_1 (g\mathcal{H})) *_2 n &= [(m *_1 g)\mathcal{H}] *_2 n = [(m *_1 g) *_2 n]\mathcal{H} = [m *_1 (g *_2 n)]\mathcal{H} = \\ &= m *_1 (g *_2 n)\mathcal{H} = m *_1 (g\mathcal{H} *_2 n). \end{aligned}$$

Also, for any $g_1\mathcal{H}, g_2\mathcal{H} \in \mathcal{G}/\mathcal{H}$, $m \in \mathcal{M}, n \in \mathcal{N}$, we have

$$\begin{aligned} m *_1 ((g_1\mathcal{H})(g_2\mathcal{H})) &= m *_1 ((g_1g_2)\mathcal{H}) = [m *_1 (g_1g_2)]\mathcal{H} \\ &= [(m *_1 g_1)(m *_1 g_2)]\mathcal{H} \\ &= [(m *_1 g_1)\mathcal{H}][(m *_1 g_2)\mathcal{H}] \\ &= (m *_1 g_1\mathcal{H})(m *_1 g_2\mathcal{H}). \end{aligned}$$

Similarly, we can show that $((g_1\mathcal{H})(g_2\mathcal{H})) *_2 n = (g_1\mathcal{H} *_2 n)(g_2\mathcal{H} *_2 n)$. Therefore, $*_1$ and $*_2$ are group homomorphism on \mathcal{G}/\mathcal{H} . Hence $(\mathcal{G}/\mathcal{H}, \circ)$ is an $(\mathcal{M} - \mathcal{N})$ -group. \square

Remark 3.3. In this case, \mathcal{G}/\mathcal{H} is called the $(\mathcal{M} - \mathcal{N})$ quotient group induced by \mathcal{H} .

Theorem 3.9. Let \mathcal{G}/\mathcal{H} be an $(\mathcal{M} - \mathcal{N})$ group, and let \mathcal{K} be an $(\mathcal{M} - \mathcal{N})$ subgroup of \mathcal{G} that contains the $(\mathcal{M} - \mathcal{N})$ normal subgroup \mathcal{H} . Then \mathcal{K}/\mathcal{H} is an $(\mathcal{M} - \mathcal{N})$ subgroup of \mathcal{G}/\mathcal{H} .

Proof. It is a known fact that \mathcal{K}/\mathcal{H} is a subgroup of \mathcal{G}/\mathcal{H} . (see Theorem 3.12 of [2]). Since \mathcal{K} is an $(\mathcal{M} - \mathcal{N})$ subgroup of \mathcal{G} . So, we have

$$(m *_1 k) \in \mathcal{K} \text{ and } (k *_2 n) \in \mathcal{K}, \text{ for each } m \in \mathcal{M}, n \in \mathcal{N}, \text{ and } k \in \mathcal{K}.$$

As a result we can notice that

$$m *_1 k\mathcal{H} = (m *_1 k)\mathcal{H} \in \mathcal{K}/\mathcal{H} \text{ and } k\mathcal{H} *_2 n = (k *_2 n)\mathcal{H} \in \mathcal{K}/\mathcal{H}.$$

Thus, \mathcal{K}/\mathcal{H} is an $(\mathcal{M} - \mathcal{N})$ subgroup of \mathcal{G}/\mathcal{H} . \square

Theorem 3.10. Let \mathcal{G}/\mathcal{H} be an $(\mathcal{M} - \mathcal{N})$ group, $\mathcal{K}_1, \mathcal{K}_2$ be two $(\mathcal{M} - \mathcal{N})$ subgroups of \mathcal{G} , where $\mathcal{K}_1 \subseteq \mathcal{K}_2$. Then

- (i) $\mathcal{K}_2/\mathcal{H} \subseteq \mathcal{K}_1/\mathcal{H}$;
- (ii) $(\mathcal{K}_1/\mathcal{H}) \cap (\mathcal{K}_2/\mathcal{H}) = (\mathcal{K}_1 \cap \mathcal{K}_2)/\mathcal{H}$.

Proof. (i) By Theorem 3.9 we have $\mathcal{K}_1/\mathcal{H}$ and $\mathcal{K}_2/\mathcal{H}$ both are $(\mathcal{M} - \mathcal{N})$ subgroups of \mathcal{G}/\mathcal{H} . Next, let $k\mathcal{H} \in \mathcal{K}_1/\mathcal{H}$, where $k \in \mathcal{K}_1$. Since $\mathcal{K}_1 \subseteq \mathcal{K}_2$, so $k \in \mathcal{K}_2$. Therefore, $k\mathcal{H} \in \mathcal{K}_2/\mathcal{H}$. Hence, $\mathcal{K}_2/\mathcal{H} \subseteq \mathcal{K}_1/\mathcal{H}$.

(ii) Since both $\mathcal{K}_1, \mathcal{K}_2$ are $(\mathcal{M} - \mathcal{N})$ subgroups of \mathcal{G} . So, $\mathcal{K}_1 \cap \mathcal{K}_2$ is also an $(\mathcal{M} - \mathcal{N})$ subgroups of \mathcal{G} . By Theorem 3.9, $\mathcal{K}_1/\mathcal{H}, \mathcal{K}_2/\mathcal{H}$ and $(\mathcal{K}_1 \cap \mathcal{K}_2)/\mathcal{H}$ are $(\mathcal{M} - \mathcal{N})$ subgroups of \mathcal{G}/\mathcal{H} .

Again, let $k\mathcal{H} \in (\mathcal{K}_1/\mathcal{H}) \cap (\mathcal{K}_2/\mathcal{H}) \Leftrightarrow k\mathcal{H} \in \mathcal{K}_1/\mathcal{H}$ and $k\mathcal{H} \in \mathcal{K}_2/\mathcal{H}$ where $k \in \mathcal{K}_1$ and $k \in \mathcal{K}_2$, so, $k \in \mathcal{K}_1 \cap \mathcal{K}_2$. Thus, we have $k\mathcal{H} \in (\mathcal{K}_1 \cap \mathcal{K}_2)/\mathcal{H}$. Hence, $(\mathcal{K}_1/\mathcal{H}) \cap (\mathcal{K}_2/\mathcal{H}) = (\mathcal{K}_1 \cap \mathcal{K}_2)/\mathcal{H}$. \square

Theorem 3.11. Let \mathcal{G} be an $(\mathcal{M} - \mathcal{N})$ group, and let \mathcal{H} be an $(\mathcal{M} - \mathcal{N})$ normal subgroup of \mathcal{G} . Then there exists a natural $(\mathcal{M} - \mathcal{N})$ -homomorphism $\sigma : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}$ defined by

$$\sigma(x) = x\mathcal{H}, \forall x \in \mathcal{G}.$$

Proof. By Theorem 4.8 of [1] there exists a surjective natural homomorphism $\sigma : \mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}$ defined by $\sigma(x) = x\mathcal{H}, \forall x \in \mathcal{G}$. Since, $\sigma(xy) = (xy)\mathcal{H} = (x\mathcal{H})(y\mathcal{H}) = \sigma(x)\sigma(y)$.

Also, for each $z\mathcal{H} \in \mathcal{G}/\mathcal{H}$, we have $z \in \mathcal{G}$ such that $\sigma(z) = z\mathcal{H}$.

Next, let $m \in \mathcal{M}, n \in \mathcal{N}$ be any elements. Then

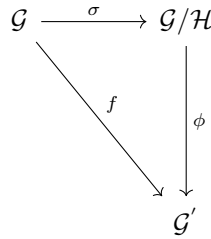
$$\begin{aligned} \sigma(m *_1 g) &= (m *_1 g)\mathcal{H} = m *_1 g\mathcal{H} = m *_1 \sigma(g) \text{ and} \\ \sigma(g *_2 n) &= (g *_2 n)\mathcal{H} = (g\mathcal{H}) *_2 n = \sigma(g) *_2 n. \end{aligned}$$

Hence σ is a $(\mathcal{M} - \mathcal{N})$ -surjective homomorphism. \square

Theorem 3.12. *If \mathcal{G} is an $(\mathcal{M} - \mathcal{N})$ -group, and let \mathcal{H} be an $(\mathcal{M} - \mathcal{N})$ -normal subgroup of \mathcal{G} , then \mathcal{G}/\mathcal{H} is a $(\mathcal{M} - \mathcal{N})$ -homomorphic image of \mathcal{G} . Conversely if any $(\mathcal{M} - \mathcal{N})$ -group \mathcal{G}' is a $(\mathcal{M} - \mathcal{N})$ -homomorphic image of \mathcal{G} , then \mathcal{G} is $(\mathcal{M} - \mathcal{N})$ -isomorphic to some $(\mathcal{M} - \mathcal{N})$ -quotient group of \mathcal{G} . In fact, if f is a $(\mathcal{M} - \mathcal{N})$ -homomorphism of \mathcal{G} onto \mathcal{G}' , then $\mathcal{G}' \cong \mathcal{G}/Ker f$.*

Proof. Direct part follows from Theorem 3.11.

Conversely, if \mathcal{G}' is an $(\mathcal{M} - \mathcal{N})$ -homomorphic image of \mathcal{G} , then there exists a $(\mathcal{M} - \mathcal{N})$ -surjective homomorphism $f : \mathcal{G} \rightarrow \mathcal{G}'$. Let $\mathcal{H} = Ker f$. Since \mathcal{H} is an $(\mathcal{M} - \mathcal{N})$ -normal subgroup of \mathcal{G} . Consider the quotient group \mathcal{G}/\mathcal{H} and define a map $\phi : \mathcal{G}/\mathcal{H} \rightarrow \mathcal{G}'$ by $\phi(x\mathcal{H}) = f(x), \forall x \in \mathcal{G}$. Consider the following diagram:



Let $x \in \mathcal{G}$. Then $f(x) = \phi(x\mathcal{H}) = \phi(\sigma(x)) = (\phi \circ \sigma)(x)$.

Thus, the above diagram commutes, i.e., $f = \phi \circ \sigma$.

Firstly, we show that the map ϕ is well-defined. For this, let $x, y \in \mathcal{G}$ with $x\mathcal{H} = y\mathcal{H}$.

This gives $xy^{-1} \in \mathcal{H} \Rightarrow f(xy^{-1}) = e' \Rightarrow f(x)(f(y))^{-1} = e' \Rightarrow f(x) = f(y) \Rightarrow \phi(x\mathcal{H}) = \phi(y\mathcal{H})$. Consequently, ϕ is well-defined. The reverse steps shows that ϕ is injective.

Moreover, since f is an $(\mathcal{M} - \mathcal{N})$ -epimorphism, so is ϕ .

Next, for any $x\mathcal{H}, y\mathcal{H} \in \mathcal{G}/\mathcal{H}$, we have

$$\phi((x\mathcal{H})(y\mathcal{H})) = \phi((xy)\mathcal{H}) = f(xy) = f(x)f(y) = \phi(x\mathcal{H})\phi(y\mathcal{H}).$$

For any $m \in \mathcal{M}, n \in \mathcal{N}$, and $x\mathcal{H} \in \mathcal{G}/\mathcal{H}$, we have

$$\phi(m *_1 x\mathcal{H}) = \phi((m *_1 x)\mathcal{H}) = f(m *_1 x) = m *_1 f(x) = m *_1 \phi(x\mathcal{H})$$

and

$$\phi(x\mathcal{H} *_2 n) = \phi((x *_2 n)\mathcal{H}) = f(x *_2 n) = f(x) *_2 n = \phi(x\mathcal{H}) *_2 n.$$

Therefore, ϕ is an $(\mathcal{M} - \mathcal{N})$ -isomorphism as such we have $\mathcal{G}/\mathcal{H} \cong \mathcal{G}'$, i.e., $\mathcal{G}/Ker f \cong \mathcal{G}'$. □

4. CONCLUSIONS

In this paper, we introduced and developed the theory of $(\mathcal{M} - \mathcal{N})$ -groups equipped with compatible left and right actions by external operator domains \mathcal{M} and \mathcal{N} , respectively. In the first part of the study, we formally defined $(\mathcal{M} - \mathcal{N})$ -groups and $(\mathcal{M} - \mathcal{N})$ - (normal) subgroups, and explored their algebraic structure through operations such as intersection, union, sum, and product. Several illustrative examples were provided to demonstrate the behavior of these subgroups. Additionally, we introduced the concept of $(\mathcal{M} - \mathcal{N})$ -quotient groups and discussed foundational results surrounding their construction and properties. In the second part, we extended the theory by defining $(\mathcal{M} - \mathcal{N})$ -homomorphisms between $(\mathcal{M} - \mathcal{N})$ -groups. We examined the behavior of these homomorphisms with respect to the image and preimage of $(\mathcal{M} - \mathcal{N})$ -subgroups and established analogues of classical homomorphism theorems in this new setting.

Together, these investigations lay the groundwork for a broader theory of $(\mathcal{M} - \mathcal{N})$ -group structures and open the door for further research into their algebraic properties, classification, and applications in related mathematical frameworks.

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