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Structural Characterizations of Soft Dimodules

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ABSTRACT. This paper investigates the foundational framework of soft dimodules, a novel algebraic structure that unifies concepts from soft set theory and dimodule theory. By extending the classical notions of dimonoids and modules, soft dimodules provide enhanced flexibility for modeling uncertainty and parameterization in algebraic systems. Theoretical analysis highlights important characteristics such as soft subdimodules and soft dimodule homomorphisms. Furthermore, the study explores several specialized classes, including idempotent, commutative, and distributive soft dimodules, characterizing their structural properties and establishing their relations with classical algebraic systems. This research reveals deeper structural correspondences within soft algebraic systems and provides a more comprehensive modeling approach for situations involving uncertainty.

1. INTRODUCTION

The theory of dimonoids was originally pioneered by Jean-Louis Loday in 2001 as a sophisticated mathematical framework designed to facilitate the investigation of Leibniz algebras, which represent a non-antisymmetric generalization of Lie algebras [11]. Distinct from classical algebraic structures characterized by a single binary operation, a dimonoid is defined by two binary associative operations—commonly referred to as the left operation ($\dashv$) and the right operation ($\vdash$)—that satisfy a specific set of compatibility axioms. This structural innovation has provided new trajectories for algebraic modeling, fostering significant advancements in both theoretical and applied mathematics [10, 21, 23].

A noteworthy development within this field is the conceptualization of dimodules, which emerge from the adaptation of classical module theory to the dual-operational architecture of dimonoids. These structures extend the traditional understanding of modules over rings while maintaining the rigorous axiomatic consistency inherent in dimonoid theory. Dimodules over a dimonoid function as a vital instrument for characterizing the representations and intrinsic structural properties of complex algebraic systems [26]. Furthermore, various constructions in theoretical physics, particularly within the realm of quantum groups, can be effectively interpreted as module-like systems over specialized algebraic structures. Existing literature has extensively explored the properties of dimodules associated with specific classes of dimonoids, such as commutative, idempotent, or distributive dimonoids [5, 6, 9, 25].

However, these classical formulations are predominantly based on crisp sets, which inherently limits their capacity to model systems involving parameter-dependent uncertainty or vague data.To address these limitations, the integration of soft set theory, introduced by Molodtsov, with algebraic structures has become a prominent area of contemporary research [14-19]. Soft set theory offers a robust mathematical paradigm for

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managing uncertainty by providing a parameterized description of objects, thus bypassing the foundational difficulties encountered in traditional fuzzy or rough set theories [3, 20, 22]. Consequently, the successful application of soft set theory to groups, semigroups, and rings has paved the way for more complex hybrid structures [2, 7, 8]. This research trajectory has recently evolved toward the study of soft modules and generalized hyperstructures, highlighting the versatility of soft sets in algebraic contexts [4, 16-19].

The objective of the present paper is to introduce and formalize the concept of soft dimodules by merging dimodule theory with the parametric flexibility of soft sets. This study aims to establish the fundamental characterizations of soft dimodules and investigate the algebraic properties of their substructures. In addition, the notion of a soft dimodule homomorphism is defined, and the structural behavior of soft dimodules under homomorphic transformations—specifically regarding images and pre-images—is rigorously examined. By doing so, this work seeks to provide a comprehensive and theoretically sound basis for the analysis of algebraic systems operating within uncertain or parameterized environments.

2. Preliminaries

In this section, we recall some basic definitions and properties regarding semigroups, dimonoids, dimodules, and soft sets that are essential for the development of our main results.

Let X be an initial universe set and E be a set of parameters. Morever, $P(X)$ denotes the power set of X and $A \subseteq E$. A soft set, as proposed by Molodtsov, is defined as follows:

Definition 2.1. [13] *A pair (Θ, A) is called a soft set over X, where Θ is a mapping defined by*

$$\Theta : A \longrightarrow P(X)$$

It is worth noting that a soft set over X is, in fact, a parametrized family of subsets of the universe X, where each parameter corresponds to a particular subset. This structure facilitates the modeling of uncertainty and partial knowledge in various contexts.

Definition 2.2. [12] *Let (Θ_1, A) and (Θ_2, B) be two soft sets over the common universe X. Then, (Θ_1, A) is said to be a soft subset of (Θ_2, B) (i.e., $(\Theta_1, A)\widetilde{\subset}(\Theta_2, B)$) if the following conditions hold:*

- ***i.*** *$A \subseteq B$,*
- ***ii.*** *$\Theta_1(\varepsilon)$ and $\Theta_2(\varepsilon)$ are identical approximations for all $\varepsilon \in A$.*

Definition 2.3. [12] *The* support *of a soft set (Θ_1, A) over $\mathcal{H}$ is defined as the set:*

$$\mathrm{Supp}(\Theta_1, A) = \{\varepsilon \in A : \Theta_1(\varepsilon) \neq \emptyset\}.$$

If $\mathrm{Supp}(\Theta_1, A) \neq \emptyset$, then (Θ_1, A) is said to be non-null.

Definition 2.4. [12] *Let (Θ_1, A) and (Θ_2, B) be soft sets over $\mathcal{H}$ and let $R = A \cap B$. The* intersection *of (Θ_1, A) and (Θ_2, B), denoted by $(\Theta_1, A)\widetilde{\cap}(\Theta_2, B) = (\Theta^\star, R)$, is defined by the mapping:*

$$\Theta^\star : R \longrightarrow P(\mathcal{H})$$
$$\varepsilon \longrightarrow \Theta^\star(\varepsilon) = \Theta_1(\varepsilon) \cap \Theta_2(\varepsilon)$$

Definition 2.5. [12] *Let (Θ_1, A) and (Θ_2, B) be soft sets over $\mathcal{H}$ and let $C = A \cup B$. The* union *of (Θ_1, A) and (Θ_2, B), denoted by $(\Theta_1, A)\widetilde{\cup}(\Theta_2, B) = (\Theta^\star, C)$, is defined by the mapping:*

$$\Theta^\star(\varepsilon) = \begin{cases} \Theta_1(\varepsilon), & \varepsilon \in A - B \\ \Theta_2(\varepsilon), & \varepsilon \in B - A \\ \Theta_1(\varepsilon) \cup \Theta_2(\varepsilon), & \varepsilon \in A \cap B \end{cases}$$

Definition 2.6. [1] *Let $\mathcal{H}$ be a nonempty set and "$\cdot$" be a binary operation on $\mathcal{H}$. The algebraic structure $(\mathcal{H}, \cdot)$ is termed a semigroup if and only if*

$$d_i \cdot (d_j \cdot d_k) = (d_i \cdot d_j) \cdot d_k, \quad \forall d_i, d_j, d_k \in \mathcal{H}.$$

Definition 2.7. [1] *Let $P(\mathcal{H})$ represent the power set of the semigroup $\mathcal{H}$, and let D_i, D_j be elements of $P(\mathcal{H})$. The operation is defined as follows:*

$$D_i \cdot D_j = \begin{cases} \emptyset, & \text{if } D_i = \emptyset \text{ or } D_j = \emptyset, \\ \{\, d_i \cdot d_j \mid d_i \in D_i,\ d_j \in D_j \,\}, & \text{otherwise.} \end{cases}$$

Definition 2.8. [1] *An element $0 \in \mathcal{H}$ is referred to as a left zero element if $0 \cdot d_i = 0$ holds for every $d_i \in \mathcal{H}$. Conversely, if $d_i \cdot 0 = 0$ for all $d_i \in \mathcal{H}$, then 0 is termed a right zero element. In the case where an element $0 \in \mathcal{H}$ functions as both a left and a right zero element, it is defined as a zero element.*

Definition 2.9. [1] *A semigroup $\mathcal{H}$ is referred to as a left (resp. right) zero semigroup if every element within $\mathcal{H}$ functions as a left (resp. right) zero element. Furthermore, if there exists a specific element $0 \in \mathcal{H}$ satisfying the condition*

$$d_i \cdot d_j = 0, \quad \forall d_i, d_j \in \mathcal{H},$$

then $\mathcal{H}$ is defined as a zero semigroup.

Definition 2.10. [1] *Let $(\mathcal{H}, \cdot)$ and $(\mathcal{K}, *)$ be two semigroups. A mapping $f : \mathcal{H} \to \mathcal{K}$ is termed a* homomorphism of semigroups *if the following condition is satisfied:*

$$f(d_i \cdot d_j) = f(d_i) * f(d_j), \quad \forall d_i, d_j \in \mathcal{H}.$$

Definition 2.11. [1] *Let $\{\mathcal{H}_i \mid i \in I\}$ represent a collection of semigroups. The Cartesian product, denoted by*

$$\prod_{i \in I} \mathcal{H}_i,$$

consequently forms a semigroup itself.

Definition 2.12. [23] *Let $\mathcal{D}$ be a nonempty set endowed with two associative binary operations, denoted by $*$ and $\circ$. The triple $(\mathcal{D}, *, \circ)$ is defined as a dimonoid if the following axioms are satisfied for every $d_i, d_j, d_k \in \mathcal{D}$:*

$$(d_i * d_j) * d_k = d_i * (d_j \circ d_k)$$
$$(d_i \circ d_j) * d_k = d_i \circ (d_j * d_k)$$
$$(d_i * d_j) \circ d_k = d_i \circ (d_j \circ d_k)$$

Definition 2.13. [23] *A dimonoid $(\mathcal{D}, *, \circ)$ is referred to as an* idempotent dimonoid *(or* diband*) if the following condition is satisfied for every $d_i \in \mathcal{D}$:*

$$d_i * d_i = d_i = d_i \circ d_i.$$

Example 2.1. [25] *Let $(\mathcal{D}, *)$ represent a zero semigroup where 0 is the zero element. Choose fixed elements $d_i, d_j \in \mathcal{D}$ such that $d_i \neq d_j$ and $d_j \neq 0$. We define an additional binary operation $\circ$ on $\mathcal{D}$ as follows:*

$$d_k \circ d_l = \begin{cases} d_i, & \text{if } d_k = d_l = d_j \\ 0, & \text{otherwise} \end{cases}$$

Under these conditions, the triple $(\mathcal{D}, *, \circ)$ constitutes a dimonoid.

Definition 2.14. [25] *Let* $(\mathcal{D}_1, *_1, \circ_1)$ *and* $(\mathcal{D}_2, *_2, \circ_2)$ *be two dimonoids. A mapping*

$$f : \mathcal{D}_1 \longrightarrow \mathcal{D}_2$$

is termed a homomorphism of dimonoids *if the following conditions are satisfied for any* $d_i, d_j \in \mathcal{D}_1$*:*

$$f(d_i *_1 d_j) = f(d_i) *_2 f(d_j)$$
$$f(d_i \circ_1 d_j) = f(d_i) \circ_2 f(d_j)$$

Definition 2.15. [23] *Let* T *be a nonempty subset of a dimonoid* $\mathcal{D}$*. The set* T *is referred to as a* subdimonoid *if it is closed under both operations, meaning that for any* $d_i, d_j \in T$*, the following conditions are satisfied:*

$$d_i * d_j \in T, \qquad d_i \circ d_j \in T.$$

Definition 2.16. [25] *A dimonoid* $(\mathcal{D}, *, \circ)$ *is termed a* commutative dimonoid *if both binary operations* $*$ *and* $\circ$ *satisfy the commutative law.*

Theorem 2.1. [25] *Let* $(\mathcal{D}, *, \circ)$ *be a commutative dimonoid. For any elements* $d_i, d_j, d_k \in \mathcal{D}$*, the following series of equalities are satisfied:*

$$(d_i * d_j) * d_k = d_i * (d_j \circ d_k) = (d_i \circ d_j) * d_k$$
$$= d_i \circ (d_j * d_k) = (d_i * d_j) \circ d_k = d_i \circ (d_j \circ d_k).$$

Theorem 2.2. [25] *Every commutative idempotent dimonoid* $(\mathcal{D}, *, \circ)$ *is inherently a distributive dimonoid.*

Proof. Suppose that $(\mathcal{D}, *, \circ)$ is a commutative idempotent dimonoid. In view of Theorem 2.1, the binary operations "$*$" and "$\circ$" are identical. Consequently, for any $d_i, d_j, d_k \in \mathcal{D}$, we can write:

$$(d_i \circ d_j) * (d_i \circ d_k) = (d_i * d_j) * (d_i * d_k) = (d_i * d_i) * (d_j * d_k).$$

Given that the operation is idempotent, it follows that $d_i * d_i = d_i$. Thus, we obtain:

$$(d_i * d_i) * (d_j * d_k) = d_i * (d_j * d_k) = d_i \circ (d_j * d_k).$$

This leads to the following equality:

$$(d_i \circ d_j) * (d_i \circ d_k) = d_i \circ (d_j * d_k),$$

which confirms the property of distributivity[cite: 130, 131]. Since the dimonoid $(\mathcal{D}, *, \circ)$ is commutative, it is concluded that it constitutes a distributive dimonoid. □

Example 2.2. [1] *Consider the commutative dimonoid* $\mathcal{D} = \{d_i, d_j, d_k\}$*, where the binary operation "*$*$*" is characterized by the table below:*

$*$	d_i	d_j	d_k
d_i	d_i	d_i	d_i
d_j	d_i	d_j	d_k
d_k	d_i	d_k	d_j

In this case, the structure $(\mathcal{D}, *, *)$ *fails to satisfy the distributive law because:*

$$d_k * (d_j * d_j) \neq (d_k * d_j) * (d_k * d_j).$$

Furthermore, $(\mathcal{D}, *, *)$ *is not an idempotent dimonoid as:*

$$d_k * d_k \neq d_k.$$

Definition 2.17. [1] *Let $(\mathcal{D}, *, \circ)$ be a dimonoid. A (left) $\mathcal{D}$-dimodule consists of a semigroup $(\mathcal{H}, \cdot)$ equipped with an external operation*

$$\mathcal{D} \times \mathcal{H} \to \mathcal{H}, \quad (d_i, h_i) \mapsto d_i h_i$$

such that for any $d_i, d_j \in \mathcal{D}$ and $h_i, h_j \in \mathcal{H}$, the following axioms are satisfied:

$$d_i(h_i \cdot h_j) = d_i h_i \cdot d_i h_j$$
$$(d_i * d_j)h_i = d_i h_i \cdot d_j h_i$$
$$d_i(d_j h_i) = (d_i \circ d_j)h_i$$

Example 2.3. [1] *Let $\mathcal{D} = \{d_1, d_2\}$ and $\mathcal{H} = \{h_1, h_2\}$. Suppose that $(\mathcal{D}, *, \circ)$ is a dimonoid and $(\mathcal{H}, \cdot)$ is a semigroup, where the operations "$*, \circ, \cdot$" are defined by the following Cayley tables:*

$*$	d_1	d_2
d_1	d_1	d_1
d_2	d_1	d_1

$\circ$	d_1	d_2
d_1	d_1	d_2
d_2	d_1	d_2

$\cdot$	h_1	h_2
h_1	h_1	h_2
h_2	h_2	h_2

(i) Consider the mapping $\mathcal{D} \times \mathcal{H} \to \mathcal{H}$ defined by $(d_i, h_i) \mapsto d_i h_i = h_i$. Under this assignment, $\mathcal{H}$ constitutes a $\mathcal{D}$-dimodule.

(ii) Alternatively, define the mapping $\mathcal{D} \times \mathcal{H} \to \mathcal{H}$ by $(d_i, h_i) \mapsto d_i h_i = d_i$ (where we identify d_1 with h_1 and d_2 with h_2). In this case, $\mathcal{H}$ is not a $\mathcal{D}$-dimodule because the following condition fails:

$$(d_1 * d_2)h_1 = d_1 \neq h_2 = d_1 h_1 \cdot d_2 h_1.$$

Proposition 2.1. [1] *Let $(\mathcal{D}, *, \circ)$ be a distributive dimonoid. Define an external operation as follows:*

$$\mathcal{D} \times \mathcal{D} \longrightarrow \mathcal{D}, \quad (d_i, d_j) \mapsto d_i d_j = d_i \circ d_j.$$

Then the semigroup $(\mathcal{D}, *)$ constitutes a $\mathcal{D}$-dimodule.

Proof. The proof follows directly from the distributive axioms of the dimonoid $(\mathcal{D}, *, \circ)$ and the definitions of a $\mathcal{D}$-dimodule. □

Example 2.4. *Recall the dimonoid $\mathcal{D} = \{d_i, d_j, d_k\}$ introduced in Example 2.2. It is evident that $(\mathcal{D}, *)$ fails to form a $\mathcal{D}$-dimodule, since the distributive property is not satisfied:*

$$d_k * (d_j * d_j) = d_k \neq d_j = (d_k * d_j) * (d_k * d_j).$$

Definition 2.18. [1] *Let $(\mathcal{H}, \cdot)$ be a $\mathcal{D}$-dimodule and let E be a nonempty subset of $\mathcal{H}$. Then E is referred to as a $\mathcal{D}$-subdimodule of $\mathcal{H}$ if it is closed under the semigroup operation and the external action; specifically, for any $h_i, h_j \in E$ and $d_k \in \mathcal{D}$, the following conditions are satisfied:*

$$h_i \cdot h_j \in E \quad \text{and} \quad d_k h_i \in E.$$

Proposition 2.2. [1] *Let $(\mathcal{H}, \cdot)$ be a $\mathcal{D}$-dimodule and let $\{E_i \mid i \in I\}$ be a collection of $\mathcal{D}$-subdimodules of $\mathcal{H}$. If the intersection $\bigcap_{i \in I} E_i$ is nonempty, then it constitutes a $\mathcal{D}$-subdimodule of $\mathcal{H}$.*

Proof. Suppose that $h_i, h_j \in \bigcap_{i \in I} E_i$ and $d_k \in \mathcal{D}$. By the definition of an intersection, it follows that $h_i, h_j \in E_i$ for every $i \in I$. Since each E_i is a $\mathcal{D}$-subdimodule, it is closed under the respective operations; thus, $h_i \cdot h_j \in E_i$ and $d_k h_i \in E_i$ for all $i \in I$. Consequently, we have $h_i \cdot h_j \in \bigcap_{i \in I} E_i$ and $d_k h_i \in \bigcap_{i \in I} E_i$. This confirms that the intersection $\bigcap_{i \in I} E_i$ is indeed a $\mathcal{D}$-subdimodule of $\mathcal{H}$. □

Example 2.5. [1] *Consider the semigroup $(\mathcal{H}, *)$ defined on the set $\mathcal{H} = \{d_i, d_j, d_k\}$ by the following operation table:*

$*$	d_i	d_j	d_k
d_i	d_i	d_i	d_i
d_j	d_i	d_j	d_i
d_k	d_i	d_i	d_k

If we define the external action $\mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ as the mapping $(d_m, d_n) \mapsto d_m d_n = d_n$, then $\mathcal{H}$ possesses the structure of a $\mathcal{D}$-dimodule.

Let us examine the subsets $A = \{d_j\}$ and $B = \{d_k\}$ of $\mathcal{H}$. While both A and B independently satisfy the conditions to be $\mathcal{D}$-subdimodules, their union $A \cup B = \{d_j, d_k\}$ fails to be a $\mathcal{D}$-subdimodule. This is due to the fact that the closure property under the semigroup operation is violated:

$$d_j * d_k = d_i \notin A \cup B.$$

Definition 2.19. [1] *Let $(\mathcal{D}, *, \circ)$ be a commutative dimonoid and $(\mathcal{H}, \cdot)$ be a $\mathcal{D}$-dimodule. The structure $\mathcal{H}$ is said to be a commutative $\mathcal{D}$-dimodule if the following conditions are satisfied:*

(1) $h_i \cdot h_j = h_j \cdot h_i$ for all $h_i, h_j \in \mathcal{H}$ (i.e., $(\mathcal{H}, \cdot)$ is a commutative semigroup),
(2) The dimodule actions are compatible with the commutativity of $\mathcal{D}$, such that $d_k h_i = h_i d_k$ (if the right action is defined) for all $d_k \in \mathcal{D}$ and $h_i \in \mathcal{H}$.

Definition 2.20. [1] *Let $(\mathcal{D}, *, \circ)$ be a dimonoid and $(\mathcal{H}, \cdot)$ be a $\mathcal{D}$-dimodule. The structure $\mathcal{H}$ is characterized as follows:*

(1) $\mathcal{H}$ is called an idempotent $\mathcal{D}$-dimodule if $h \cdot h = h$ for every $h \in \mathcal{H}$.

Definition 2.21. [1] *Let $(\mathcal{D}, *, \circ)$ be a distributive dimonoid and $(\mathcal{H}, \cdot)$ be a $\mathcal{D}$-dimodule. The structure $\mathcal{H}$ is called a distributive $\mathcal{D}$-dimodule if the following distributive laws are satisfied for all $u, v \in \mathcal{D}$ and $h_1, h_2 \in \mathcal{H}$:*

(1) $u(h_1 \cdot h_2) = (uh_1) \cdot (uh_2)$
(2) $(u * v)h_1 = (uh_1) \cdot (vh_1)$
(3) $(u \circ v)h_1 = (uh_1) \cdot (vh_1)$

Proposition 2.3. [1] *Let $(\mathcal{H}, \cdot)$ be a $\mathcal{D}$-dimodule and A be a subset of $\mathcal{H}$.*

(i) *Suppose $h_0 \in A$ is an idempotent element and define the set*

$$(A :_{\mathcal{D}} \mathcal{H})_{h_0} = \{d_k \in \mathcal{D} \mid d_k h_i = h_0 \text{ for all } h_i \in A\}.$$

If $(A :_{\mathcal{D}} \mathcal{H})_{h_0}$ is nonempty, it constitutes a subdimonoid of $\mathcal{D}$.

(ii) *Let A be a subsemigroup of $\mathcal{H}$ and consider the set*

$$(A :_{\mathcal{D}} \mathcal{H}) = \{d_k \in \mathcal{D} \mid d_k \mathcal{H} \subseteq A\}.$$

If $(A :_{\mathcal{D}} \mathcal{H})$ is nonempty, then it forms a subdimonoid of $\mathcal{D}$.

Proposition 2.4. [1] *Let $\{\mathcal{H}_i \mid i \in I\}$ be a collection of $\mathcal{D}$-dimodules. The Cartesian product $\prod_{i \in I} \mathcal{H}_i$ consequently forms a $\mathcal{D}$-dimodule, which is referred to as the direct product of the collection $\{\mathcal{H}_i \mid i \in I\}$.*

Proof. Let us define the external action as the mapping

$$\mathcal{D} \times \prod_{i \in I} \mathcal{H}_i \longrightarrow \prod_{i \in I} \mathcal{H}_i, \quad (d_k, (h_i)_{i \in I}) \mapsto d_k (h_i)_{i \in I} = (d_k h_i)_{i \in I}.$$

It follows from the component-wise operations that $\prod_{i \in I} \mathcal{H}_i$ satisfies the required axioms, thus constituting a $\mathcal{D}$-dimodule. □

Definition 2.22. [1] *Let $(\mathcal{H}_1, \cdot)$ and $(\mathcal{H}_2, \cdot)$ be two $\mathcal{D}$-dimodules. A mapping $f : \mathcal{H}_1 \to \mathcal{H}_2$ is termed a* homomorphism of $\mathcal{D}$-dimodules *if the following conditions hold for all $h_i, h_j \in \mathcal{H}_1$ and $d_k \in \mathcal{D}$:*

$$f(h_i \cdot h_j) = f(h_i) \cdot f(h_j) \quad \textit{and} \quad f(d_k h_i) = d_k f(h_i).$$

Theorem 2.3. [1] *Let $(\mathcal{H}, \cdot)$ and $(\mathcal{K}, \cdot)$ be $\mathcal{D}$-dimodules, and let $f : \mathcal{H} \to \mathcal{K}$ represent a homomorphism of $\mathcal{D}$-dimodules. If E constitutes a subdimodule of $\mathcal{H}$, then its image $f(E)$ is a subdimodule of $\mathcal{K}$.*

Proof. Given that E is a subdimodule of $\mathcal{H}$, the image $f(E)$ is a nonempty subset of $\mathcal{K}$. Consider $d_k \in \mathcal{D}$ and elements $k_1, k_2 \in f(E)$. By definition, there exist $h_i, h_j \in E$ such that $k_1 = f(h_i)$ and $k_2 = f(h_j)$. Utilizing the homomorphism property of f, we obtain:

$$k_1 \cdot k_2 = f(h_i) \cdot f(h_j) = f(h_i \cdot h_j), \quad d_k k_1 = d_k f(h_i) = f(d_k h_i).$$

Since E is a subdimodule, it is closed under the respective operations, implying $h_i \cdot h_j \in E$ and $d_k h_i \in E$. Consequently, it follows that $k_1 \cdot k_2 \in f(E)$ and $d_k k_1 \in f(E)$. Therefore, $f(E)$ is indeed a subdimodule of $\mathcal{K}$. □

Theorem 2.4. [1] *Let $(\mathcal{H}, \cdot)$ and $(\mathcal{K}, \cdot)$ be $\mathcal{D}$-dimodules, and let $f : \mathcal{H} \to \mathcal{K}$ be a homomorphism of $\mathcal{D}$-dimodules. If X is a subdimodule of $\mathcal{K}$, then its preimage $f^{-1}(X)$ constitutes a subdimodule of $\mathcal{H}$, provided that $f^{-1}(X)$ is nonempty.*

Proof. Assume that $f^{-1}(X)$ is a nonempty subset of $\mathcal{H}$. Let $d_k \in \mathcal{D}$ and consider elements $h_i, h_j \in f^{-1}(X)$. By the definition of the preimage, we have $f(h_i), f(h_j) \in X$. Since f is a homomorphism of $\mathcal{D}$-dimodules, the following relations hold:

$$f(h_i \cdot h_j) = f(h_i) \cdot f(h_j) \in X, \quad f(d_k h_i) = d_k f(h_i) \in X.$$

The inclusion in X is guaranteed because X is a subdimodule of $\mathcal{K}$, which is closed under both the semigroup operation and the $\mathcal{D}$-action. Consequently, it follows that $h_i \cdot h_j \in f^{-1}(X)$ and $d_k h_i \in f^{-1}(X)$. This confirms that $f^{-1}(X)$ is indeed a subdimodule of $\mathcal{H}$. □

Example 2.6. [1] *Consider the set $\mathcal{D} = \mathbb{Z}_5$ equipped with two binary operations defined by:*

$$\bar{d}_i * \bar{d}_j = \begin{cases} \bar{2}, & \textit{if } \bar{d}_i = \bar{d}_j = \bar{1} \\ \bar{0}, & \textit{otherwise} \end{cases} \quad \textit{and} \quad \bar{d}_i \circ \bar{d}_j = \begin{cases} \bar{4}, & \textit{if } \bar{d}_i = \bar{d}_j = \bar{3} \\ \bar{0}, & \textit{otherwise} \end{cases}$$

Under these operations, $(\mathcal{D}, *, \circ)$ forms a dimonoid. Now, let us define two $\mathcal{D}$-dimodules as follows:

- *(i) The semigroup $\mathcal{H}_1 = (\mathbb{Z}_2, \cdot)$ acts as a $\mathbb{Z}_5$-dimodule under the external action $\mathcal{D} \times \mathcal{H}_1 \to \mathcal{H}_1$ defined by $(\bar{d}_k, \bar{h}_i) \mapsto \bar{d}_k \bar{h}_i = \bar{1}$.*
- *(ii) The semigroup $\mathcal{H}_2 = (\mathbb{Z}_4, +)$ acts as a $\mathbb{Z}_5$-dimodule under the external action $\mathcal{D} \times \mathcal{H}_2 \to \mathcal{H}_2$ defined by $(\bar{d}_k, \bar{h}_j) \mapsto \bar{d}_k \bar{h}_j = \bar{0}$.*

Furthermore, the mapping $f : \mathcal{H}_2 \to \mathcal{H}_1$ defined by $f(\bar{h}_j) = \bar{1}$ for all $\bar{h}_j \in \mathbb{Z}_4$ satisfies the conditions of a $\mathbb{Z}_5$-dimodule homomorphism.

Theorem 2.5. [1] *Let $(\mathcal{H}, \cdot)$ be a $\mathcal{D}$-dimodule and $h_0 \in \mathcal{H}$. Define the set $\mathcal{D}h_0 = \{d_k h_0 \mid d_k \in \mathcal{D}\}$. Then $\mathcal{D}h_0$ constitutes a subdimodule of $\mathcal{H}$.*

Proof. It is evident that $\mathcal{D}h_0$ is a nonempty subset of $\mathcal{H}$. To verify the subdimodule properties, let $h_i, h_j \in \mathcal{D}h_0$ and $d_m \in \mathcal{D}$. By the definition of the set $\mathcal{D}h_0$, there exist elements $d_1, d_2 \in \mathcal{D}$ such that $h_i = d_1 h_0$ and $h_j = d_2 h_0$. Considering the dimodule axioms and the closure of $\mathcal{D}$ under its binary operations, we obtain:

- $h_i \cdot h_j = (d_1 h_0) \cdot (d_2 h_0) = (d_1 * d_2) h_0$. Since $d_1 * d_2 \in \mathcal{D}$, it follows that $h_i \cdot h_j \in \mathcal{D}h_0$.
- $d_m h_i = d_m(d_1 h_0) = (d_m \circ d_1) h_0$. Given that $d_m \circ d_1 \in \mathcal{D}$, we have $d_m h_i \in \mathcal{D}h_0$.

Consequently, $\mathcal{D}h_0$ is closed under both the semigroup operation and the $\mathcal{D}$-action, thus confirming it is a subdimodule of $\mathcal{H}$. □

3. Soft Dimodules

This section introduces the concept of soft dimodules and presents several fundamental characterizations. By integrating soft set theory with the algebraic structure of dimodules, we establish a framework to study parameterized collections of sub-structures. Furthermore, by defining the notions of soft subdimodules and soft dimodule homomorphisms, this section explores their associated structural properties and provides illustrative examples. In what follows, for the sake of brevity, we shall use the term dimodule to refer to a $\mathcal{D}$-dimodule unless otherwise specified.

Definition 3.23. *Let $\mathcal{H}$ be a $\mathcal{D}$-dimodule and let (Θ, A) be a non-null soft set over $\mathcal{H}$. The pair (Θ, A) is called a* soft $\mathcal{D}$-dimodule *over $\mathcal{H}$ if*

$$\Theta(\alpha) \text{ is a subdimodule of } \mathcal{H}, \quad \forall \alpha \in Supp(\Theta, A).$$

Example 3.7. *Let $\mathcal{R}$ be a ring and consider $\mathcal{H} = \mathcal{R} \times \mathcal{R}$ as a dimodule over $\mathcal{R}$, where the left and right actions are defined componentwise by*

$$a \cdot (r_1, r_2) = (ar_1, ar_2), \qquad (r_1, r_2) \cdot a = (r_1 a, r_2 a), \quad \forall a \in \mathcal{R},\ (r_1, r_2) \in \mathcal{H}.$$

Define the soft set (Θ, A) over $\mathcal{H}$ with the parameter set $A = \{\alpha_1, \alpha_2\}$ as follows:

$$\Theta(\alpha_1) = \mathcal{R} \times \{0\}, \qquad \Theta(\alpha_2) = \{0\} \times \mathcal{R}.$$

Since both $\Theta(\alpha_1)$ and $\Theta(\alpha_2)$ are subdimodules of $\mathcal{H}$, the soft set (Θ, A) is a soft $\mathcal{D}$-dimodule *over $\mathcal{H}$.*

Definition 3.24. *Let $(\mathcal{D}, *, \circ)$ be a distributive dimonoid and $\mathcal{H}$ be a $\mathcal{D}$-dimodule. A soft $\mathcal{D}$-dimodule (Θ, A) over $\mathcal{H}$ is called a soft distributive $\mathcal{D}$-dimodule if, for each $\alpha \in Supp(\Theta, A)$, the subdimodule $\Theta(\alpha)$ satisfies the following distributive properties:*

(1) $u(h_1 \cdot h_2) = (uh_1) \cdot (uh_2)$
(2) $(u * v)h_1 = (uh_1) \cdot (vh_1)$
(3) $(u \circ v)h_1 = (uh_1) \cdot (vh_1)$

for all $u, v \in \mathcal{D}$ and $h_1, h_2 \in \Theta(\alpha)$.

Definition 3.25. *Let $(\mathcal{D}, *, \circ)$ be a commutative dimonoid and $\mathcal{H}$ be a $\mathcal{D}$-dimodule. A soft $\mathcal{D}$-dimodule (Θ, A) over $\mathcal{H}$ is called a soft commutative $\mathcal{D}$-dimodule if, for each $\alpha \in Supp(\Theta, A)$, the subdimodule $\Theta(\alpha)$ satisfies the following conditions:*

(1) $h_1 \cdot h_2 = h_2 \cdot h_1$ for all $h_1, h_2 \in \Theta(\alpha)$ (i.e., $\Theta(\alpha)$ is a commutative subsemigroup);
(2) $uh_1 = h_1 u$ (if the right action is defined) or more generally, the dimodule actions are compatible with the commutativity of $\mathcal{D}$, for all $u \in \mathcal{D}$ and $h_1 \in \Theta(\alpha)$.

Definition 3.26. *Let $\mathcal{D}$ be a dimonoid and $\mathcal{H}$ be a $\mathcal{D}$-dimodule. A soft $\mathcal{D}$-dimodule (Θ, A) over $\mathcal{H}$ is called a soft idempotent $\mathcal{D}$-dimodule if, for each $\alpha \in Supp(\Theta, A)$, the subdimodule $\Theta(\alpha)$ is idempotent, i.e.,*

$$h \cdot h = h, \quad \forall h \in \Theta(\alpha).$$

Definition 3.27. *Let (Θ_1, A_1) and (Θ_2, A_2) be two soft $\mathcal{D}$-dimodules over $\mathcal{H}_1$ and $\mathcal{H}_2$, resp. The* product *of these soft $\mathcal{D}$-dimodules, denoted by $(\Theta_1, A_1) \times (\Theta_2, A_2)$, is defined as the pair $(\Theta, A_1 \times A_2)$, where the mapping $\Theta : A_1 \times A_2 \to P(\mathcal{H}_1 \times \mathcal{H}_2)$ is given by*

$$\Theta(\alpha_1, \alpha_2) = \Theta_1(\alpha_1) \times \Theta_2(\alpha_2)$$

for all $(\alpha_1, \alpha_2) \in A_1 \times A_2$.

Theorem 3.6. *Let (Θ_1, A_1) and (Θ_2, A_2) be soft $\mathcal{D}$-dimodules over $\mathcal{D}$-dimodules $\mathcal{H}_1$ and $\mathcal{H}_2$, resp. Then their product $(\Theta, A_1 \times A_2)$ is a soft $\mathcal{D}$-dimodule over $\mathcal{H}_1 \times \mathcal{H}_2$.*

Proof. To prove the theorem, it is sufficient to show that $\Theta(\alpha_1, \alpha_2) = \Theta_1(\alpha_1) \times \Theta_2(\alpha_2)$ is a subdimodule of $\mathcal{H}_1 \times \mathcal{H}_2$ for all $(\alpha_1, \alpha_2) \in \mathrm{Supp}(\Theta, A_1 \times A_2)$.

(1) Since (Θ_1, A_1) and (Θ_2, A_2) are soft $\mathcal{D}$-dimodules, $\Theta_1(\alpha_1) \neq \emptyset$ and $\Theta_2(\alpha_2) \neq \emptyset$ for all $\alpha_1 \in \mathrm{Supp}(\Theta_1, A_1)$ and $\alpha_2 \in \mathrm{Supp}(\Theta_2, A_2)$. Thus, $\Theta(\alpha_1, \alpha_2) \neq \emptyset$.

(2) For any $(h_1, h_2), (h_1', h_2') \in \Theta(\alpha_1, \alpha_2)$, we have $h_1, h_1' \in \Theta_1(\alpha_1)$ and $h_2, h_2' \in \Theta_2(\alpha_2)$. Since each $\Theta_i(\alpha_i)$ is a subdimodule, they are closed under the semigroup operation:

$$(h_1, h_2) \cdot (h_1', h_2') = (h_1 \cdot h_1', h_2 \cdot h_2') \in \Theta_1(\alpha_1) \times \Theta_2(\alpha_2) = \Theta(\alpha_1, \alpha_2).$$

(3) Let $d_k \in \mathcal{D}$. Since $\Theta_1(\alpha_1)$ and $\Theta_2(\alpha_2)$ are subdimodules, $d_k h_1 \in \Theta_1(\alpha_1)$ and $d_k h_2 \in \Theta_2(\alpha_2)$. Consequently,

$$d_k(h_1, h_2) = (d_k h_1, d_k h_2) \in \Theta_1(\alpha_1) \times \Theta_2(\alpha_2) = \Theta(\alpha_1, \alpha_2).$$

Thus, $\Theta(\alpha_1, \alpha_2)$ is a subdimodule of $\mathcal{H}_1 \times \mathcal{H}_2$ for each $(\alpha_1, \alpha_2) \in A_1 \times A_2$. Hence, $(\Theta, A_1 \times A_2)$ is a soft $\mathcal{D}$-dimodule over $\mathcal{H}_1 \times \mathcal{H}_2$. □

Definition 3.28. *Let (Θ_1, A) be a soft $\mathcal{D}$-dimodule over $\mathcal{H}_1$ and (Θ_2, B) a soft $\mathcal{D}$-dimodule over $\mathcal{H}_2$. Let $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ and $\rho : A \to B$ be two mappings. The pair (φ, ρ) is called a soft $\mathcal{D}$-dimodule homomorphism if the following conditions hold:*

(1) φ is a $\mathcal{D}$-dimodule homomorphism.
(2) $\varphi(\Theta_1(\alpha)) = \Theta_2(\rho(\alpha))$ for all $\alpha \in A$.

In this case, (Θ_1, A) and (Θ_2, B) are said to be soft homomorphic, denoted by $(\Theta_1, A) \sim (\Theta_2, B)$. Furthermore:

- *If φ is a monomorphism and ρ is an injective mapping, then (φ, ρ) is called a soft monomorphism.*
- *If φ is an epimorphism and ρ is a surjective mapping, then (φ, ρ) is called a soft epimorphism.*
- *If φ is an isomorphism and ρ is a bijective mapping, then (φ, ρ) is called a soft isomorphism. In this case, we denote $(\Theta_1, A) \cong (\Theta_2, B)$.*

Example 3.8. *Consider the $\mathbb{Z}_5$-dimodules $\mathcal{H}_1 = (\mathbb{Z}_4, +)$ and $\mathcal{H}_2 = (\mathbb{Z}_2, \cdot)$ as described in Example 2.7. Let $A = \{e_1, e_2\}$ be the set of parameters. We define the soft $\mathbb{Z}_5$-dimodules (F, A) over $\mathcal{H}_1$ and (G, A) over $\mathcal{H}_2$ by the following set-valued mappings:*

- $F(e_1) = \{\bar{0}, \bar{2}\}, \quad F(e_2) = \mathbb{Z}_4,$
- $G(e_1) = \{\bar{1}\}, \quad G(e_2) = \{\bar{1}\}.$

Let $f : \mathcal{H}_1 \to \mathcal{H}_2$ be a $\mathcal{D}$-dimodule homomorphism defined by $f(\bar{h}) = \bar{1}$ for all $\bar{h} \in \mathbb{Z}_4$ and let $\rho = I_A$ be the identity mapping on the parameter set A. Since $f(F(e_i)) \subseteq G(e_i)$ for each $e_i \in A$ and f is a dimodule homomorphism, the pair (f, ρ) constitutes a soft $\mathbb{Z}_5$-dimodule homomorphism from (F, A) to (G, A).

Definition 3.29. *Let (Θ, A) be a soft $\mathcal{D}$-dimodule over $\mathcal{H}_1$, and let $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ be a $\mathcal{D}$-dimodule homomorphism. The soft image of (Θ, A) under φ is defined as the pair*

$$\varphi(\Theta, A) = (\Theta_\varphi, A),$$

where the set-valued mapping $\Theta_\varphi : A \to P(\mathcal{H}_2)$ is given by

$$\Theta_\varphi(\alpha) = \varphi(\Theta(\alpha)), \quad \forall \alpha \in A.$$

Theorem 3.7. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two $\mathcal{D}$-dimodules and $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ be a $\mathcal{D}$-dimodule homomorphism. If (Θ, A) is a soft $\mathcal{D}$-dimodule over $\mathcal{H}_1$, then $\varphi(\Theta, A)$ is a soft $\mathcal{D}$-dimodule over $\mathcal{H}_2$.*

Proof. Let (Θ, A) be a soft $\mathcal{D}$-dimodule over $\mathcal{H}_1$. By definition, for each $\alpha \in Supp(\Theta, A)$, the set $\Theta(\alpha)$ is a subdimodule of $\mathcal{H}_1$. To show that $\varphi(\Theta, A) = (\Theta_\varphi, A)$ is a soft $\mathcal{D}$-dimodule over $\mathcal{H}_2$, we must verify that $\Theta_\varphi(\alpha) = \varphi(\Theta(\alpha))$ is a subdimodule of $\mathcal{H}_2$ for all $\alpha \in A$.

Since the image of a subdimodule under a $\mathcal{D}$-dimodule homomorphism is a subdimodule, $\Theta_\varphi(\alpha)$ is a subdimodule of $\mathcal{H}_2$. Specifically, for any $k_1, k_2 \in \Theta_\varphi(\alpha)$ and $d \in \mathcal{D}$, there exist $h_1, h_2 \in \Theta(\alpha)$ such that $k_1 = \varphi(h_1)$ and $k_2 = \varphi(h_2)$. Then:

(1) $k_1 \cdot k_2 = \varphi(h_1) \cdot \varphi(h_2) = \varphi(h_1 \cdot h_2) \in \varphi(\Theta(\alpha)) = \Theta_\varphi(\alpha)$
(2) $dk_1 = d\varphi(h_1) = \varphi(dh_1) \in \varphi(\Theta(\alpha)) = \Theta_\varphi(\alpha)$ and $k_1 d = \varphi(h_1 d) \in \varphi(\Theta(\alpha)) = \Theta_\varphi(\alpha)$

The closure follows from the fact that $\Theta(\alpha)$ is a subdimodule of $\mathcal{H}_1$ and f is a $\mathcal{D}$-dimodule homomorphism. Thus, $\varphi(\Theta, A)$ is a soft $\mathcal{D}$-dimodule over $\mathcal{H}_2$. □

Theorem 3.8. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two $\mathcal{D}$-dimodules and $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ be an epimorphism of $\mathcal{D}$-dimodules. If (Θ, A) is a soft distributive $\mathcal{D}$-dimodule over $\mathcal{H}_1$, then its soft image $\varphi(\Theta, A)$ is a soft distributive $\mathcal{D}$-dimodule over $\mathcal{H}_2$.*

Proof. Let (Θ, A) be a soft distributive $\mathcal{D}$-dimodule over $\mathcal{H}_1$. We denote the soft image as $\varphi(\Theta, A) = (\Theta_\varphi, A)$. To show that (Θ_φ, A) is a soft distributive $\mathcal{D}$-dimodule over $\mathcal{H}_2$, we must verify the distributive properties for any $\alpha \in Supp(\Theta_\varphi, A)$.

Let $u, v \in \mathcal{D}$ and $k_1, k_2 \in \Theta_\varphi(\alpha)$. Since $k_1, k_2 \in \varphi(\Theta(\alpha))$, there exist $h_1, h_2 \in \Theta(\alpha)$ such that $\varphi(h_1) = k_1$ and $\varphi(h_2) = k_2$.

(1) For the first property:

$$u(k_1 \cdot k_2) = u(\varphi(h_1) \cdot \varphi(h_2)) = u(\varphi(h_1 \cdot h_2)) = \varphi(u(h_1 \cdot h_2))$$

Since $\Theta(\alpha)$ is distributive:

$$\varphi(u(h_1 \cdot h_2)) = \varphi((uh_1) \cdot (uh_2)) = \varphi(uh_1) \cdot \varphi(uh_2) = (uk_1) \cdot (uk_2)$$

(2) For the second property $(u * v)k_1$:

$$(u * v)k_1 = (u * v)\varphi(h_1) = \varphi((u * v)h_1) = \varphi((uh_1) \cdot (vh_1))$$

Using the fact that φ is a homomorphism:

$$\varphi((uh_1) \cdot (vh_1)) = \varphi(uh_1) \cdot \varphi(vh_1) = (uk_1) \cdot (vk_1)$$

The third property $u(vk_1) = (u \circ v)k_1$ is verified analogously. Thus, (Θ_φ, A) satisfies the distributive conditions for each parameter, making it a soft distributive $\mathcal{D}$-dimodule over $\mathcal{H}_2$. □

Definition 3.30. *Let (Θ, A) be a soft $\mathcal{D}$-dimodule over $\mathcal{H}_2$, and let $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ be a $\mathcal{D}$-dimodule homomorphism. The* soft preimage *of (Θ, A) under φ is defined as the pair*

$$\varphi^{-1}(\Theta, A) = (\Theta_{\varphi^{-1}}, A),$$

where the set-valued mapping $\Theta_{\varphi^{-1}} : A \to P(\mathcal{H}_1)$ is given by

$$\Theta_{\varphi^{-1}}(\alpha) = \varphi^{-1}(\Theta(\alpha)), \quad \forall \alpha \in A.$$

Theorem 3.9. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two $\mathcal{D}$-dimodules and $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ be a $\mathcal{D}$-dimodule homomorphism. If (Θ, A) is a soft $\mathcal{D}$-dimodule over $\mathcal{H}_2$ such that $\varphi^{-1}(\Theta(\alpha)) \neq \emptyset$ for all $\alpha \in A$, then the preimage $\varphi^{-1}(\Theta, A)$ is a soft $\mathcal{D}$-dimodule over $\mathcal{H}_1$.*

Proof. Let (Θ, A) be a soft $\mathcal{D}$-dimodule over $\mathcal{H}_2$. By definition, for each $\alpha \in Supp(\Theta, A)$, the set $\Theta(\alpha)$ is a subdimodule of $\mathcal{H}_2$. To show that $\varphi^{-1}(\Theta, A) = (\Theta_{\varphi^{-1}}, A)$ is a soft $\mathcal{D}$-dimodule over $\mathcal{H}_1$, we must verify that $\Theta_{\varphi^{-1}}(\alpha) = \varphi^{-1}(\Theta(\alpha))$ is a subdimodule of $\mathcal{H}_1$ for each $\alpha \in A$.

(1) By hypothesis, $\varphi^{-1}(\Theta(\alpha))$ is non-empty for all $\alpha \in A$.

(2) Let $h_1, h_2 \in \varphi^{-1}(\Theta(\alpha))$ and $d \in \mathcal{D}$. By the definition of preimage, we have $\varphi(h_1), \varphi(h_2) \in \Theta(\alpha)$. Since $\Theta(\alpha)$ is a subdimodule of $\mathcal{H}_2$ and φ is a $\mathcal{D}$-dimodule homomorphism:

$$\varphi(h_1 \cdot h_2) = \varphi(h_1) \cdot \varphi(h_2) \in \Theta(\alpha)$$

$$\varphi(dh_1) = d\varphi(h_1) \in \Theta(\alpha) \quad \text{and} \quad \varphi(h_1 d) = \varphi(h_1)d \in \Theta(\alpha)$$

(3) Since the images of $h_1 \cdot h_2$, dh_1, and $h_1 d$ under φ are contained in $\Theta(\alpha)$, it follows by the definition of preimage that:

$$h_1 \cdot h_2 \in \varphi^{-1}(\Theta(\alpha)), \quad dh_1 \in \varphi^{-1}(\Theta(\alpha)), \quad h_1 d \in \varphi^{-1}(\Theta(\alpha))$$

Thus, for each $\alpha \in A$, $\varphi^{-1}(\Theta(\alpha))$ is a subdimodule of $\mathcal{H}_1$. Consequently, $\varphi^{-1}(\Theta, A)$ is a soft $\mathcal{D}$-dimodule over $\mathcal{H}_1$. □

3.1. **Soft Subdimodules.**

Definition 3.31. *Let (Θ, A) be a soft $\mathcal{D}$-dimodule over $\mathcal{H}$. A soft set (Φ, B) over $\mathcal{H}$ is called a soft subdimodule of (Θ, A), denoted by $(\Phi, B) \sqsubseteq (\Theta, A)$, if the following conditions hold:*

(1) $B \subseteq A$;
(2) For each $\beta \in B$, $\Phi(\beta)$ is a subdimodule of $\Theta(\beta)$.

Example 3.9. *Let $\mathcal{D} = \{e, a\}$ be a dimonoid and $\mathcal{H} = (\mathbb{Z}, +)$ be a $\mathcal{D}$-dimodule where the actions are defined as $eh = h$ and $ah = 0$ for all $h \in \mathbb{Z}$.*

Let $A = \{\alpha_1, \alpha_2, \alpha_3\}$ be the parameter set and define the soft $\mathcal{D}$-dimodule (Θ, A) over $\mathcal{H}$ as follows:

- $\Theta(\alpha_1) = 2\mathbb{Z}$,
- $\Theta(\alpha_2) = 4\mathbb{Z}$,
- $\Theta(\alpha_3) = \mathbb{Z}$.

Now, consider a subset of parameters $B = \{\alpha_1, \alpha_2\} \subseteq A$ and define the soft set (Φ, B) over $\mathcal{H}$ by:

- $\Phi(\alpha_1) = 4\mathbb{Z}$,
- $\Phi(\alpha_2) = 8\mathbb{Z}$.

Since $B \subseteq A$ and for each $\beta \in B$, $\Phi(\beta)$ is a subdimodule of $\Theta(\beta)$, the soft set (Φ, B) is a soft subdimodule of (Θ, A).

Theorem 3.10. *Let $\{(\Theta_i, A) \mid i \in I\}$ be a family of soft subdimodules of (Θ, A) over the same parameter set A. Then their soft intersection $(\Omega, A) = \bigcap_{i\in I}(\Theta_i, A)$, defined by the mapping $\Omega(a) = \bigcap_{i\in I} \Theta_i(a)$ for each $a \in A$, is a soft subdimodule of (Θ, A) provided that $\Omega(a) \neq \emptyset$ for all $a \in A$.*

Proof. To prove that (Ω, A) is a soft subdimodule, it is sufficient to verify that for each $a \in A$, the set $\Omega(a)$ is a subdimodule of $\mathcal{H}$. Let $a \in A$ and assume $\Omega(a) = \bigcap_{i\in I} \Theta_i(a) \neq \emptyset$.

By hypothesis, each (Θ_i, A) is a soft subdimodule, which implies that $\Theta_i(a)$ is a subdimodule of $\mathcal{H}$ for every $i \in I$. Since the intersection of any family of subdimodules of $\mathcal{H}$ is itself a subdimodule (Proposition 2.3), it follows that $\Omega(a)$ is a subdimodule of $\mathcal{H}$ for each $a \in A$. Consequently, (Ω, A) is a soft subdimodule of (Θ, A) over A. □

Theorem 3.11. *Let $\mathcal{H}$ be a $\mathcal{D}$-dimodule and $A \subseteq \mathcal{H}$ be a set of parameters. If a soft set (Θ, A) over $\mathcal{H}$ is defined by $\Theta(a) = \mathcal{D}a = \{da \mid d \in \mathcal{D}\}$ for each $a \in A$, then (Θ, A) is a soft subdimodule of $\mathcal{H}$ over A.*

Proof. To prove that (Θ, A) is a soft subdimodule, it is sufficient to show that for each parameter $a \in A$, the set $\Theta(a)$ is a subdimodule of $\mathcal{H}$. Let $a \in A$. By definition, $\Theta(a) = \mathcal{D}a$. The set $\mathcal{D}a$ is a subdimodule of $\mathcal{H}$ as it satisfies the following conditions:

(1) $\Theta(a) = \mathcal{D}a$ is a non-empty subset of $\mathcal{H}$.
(2) For any $h_1, h_2 \in \mathcal{D}a$, there exist $d_1, d_2 \in \mathcal{D}$ such that $h_1 = d_1 a$ and $h_2 = d_2 a$. Then, $h_1 \cdot h_2 = (d_1 a) \cdot (d_2 a) = (d_1 * d_2)a$. Since $d_1 * d_2 \in \mathcal{D}$, it follows that $h_1 \cdot h_2 \in \mathcal{D}a$.
(3) For any $d_k \in \mathcal{D}$ and $h_1 \in \mathcal{D}a$, where $h_1 = d_1 a$, the action is given by $d_k h_1 = d_k(d_1 a) = (d_k \circ d_1)a$. Since $d_k \circ d_1 \in \mathcal{D}$, we have $d_k h_1 \in \mathcal{D}a$.

Since $\Theta(a)$ is a subdimodule for every $a \in A$, (Θ, A) is a soft subdimodule of $\mathcal{H}$ over A. □

Theorem 3.12. *Let $\{(\Theta_i, A_i) \mid i \in I\}$ be a family of soft D-dimodules over H. If each (Θ_i, A_i) is a soft idempotent (resp. soft commutative, soft distributive) D-dimodule, then their soft intersection $(\Theta, A) = \tilde{\cap}_{i\in I}(\Theta_i, A_i)$ is also a soft idempotent (resp. soft commutative, soft distributive) D-dimodule.*

Proof. Let $(\Theta, A) = \tilde{\cap}_{i\in I}(\Theta_i, A_i)$, where the mapping $\Theta : A \to P(H)$ is defined by $\Theta(a) = \cap_{i\in I}\Theta_i(a)$ for each $a \in A$. Since the intersection of any family of subdimodules is a subdimodule, (Θ, A) is a soft D-dimodule.

(1) Suppose each (Θ_i, A_i) is soft idempotent. For any $a \in A$ and any $h \in \Theta(a)$, we have $h \in \Theta_i(a)$ for every $i \in I$. Since each $\Theta_i(a)$ is an idempotent subdimodule, the condition $h \cdot h = h$ holds in every $\Theta_i(a)$. Consequently, $h \cdot h = h$ holds in the intersection $\Theta(a)$, confirming it is soft idempotent.

(2) Suppose each (Θ_i, A_i) is soft commutative. For $h_1, h_2 \in \Theta(a)$, it follows that $h_1, h_2 \in \Theta_i(a)$ for all $i \in I$. Since each $\Theta_i(a)$ is commutative, $h_1 \cdot h_2 = h_2 \cdot h_1$ holds for all $i \in I$. Similarly, the compatibility of the dimonoid actions $uh_1 = h_1 u$ is preserved in $\Theta(a)$.

(3) Suppose each (Θ_i, A_i) is soft distributive. For $h_1, h_2 \in \Theta(a)$ and $u, v \in \mathcal{D}$, the distributive axioms hold in each $\Theta_i(a)$ for all $i \in I$. Thus, the equalities:

- $u(h_1 \cdot h_2) = (uh_1) \cdot (uh_2)$
- $(u * v)h_1 = (uh_1) \cdot (vh_1)$
- $(u \circ v)h_1 = (uh_1) \cdot (vh_1)$

remain valid in the intersection $\Theta(a)$. Therefore, (Θ, A) is a soft distributive D-dimodule. □

Example 3.10. *Consider the dimonoid $(\mathcal{D}, *, \circ)$ and the $\mathcal{D}$-dimodule $\mathcal{H} = \{h_1, h_2\}$ defined in Example 2.3-(i), where the action is given by $(d, h) \mapsto dh = h$ for all $d \in \mathcal{D}$ and $h \in \mathcal{H}$.*

Let $A = \{e_1, e_2\}$ be the set of parameters, where we associate e_1 with h_1 and e_2 with h_2. We define a soft set (Θ, A) over $\mathcal{H}$ by utilizing the orbit structure for each parameter as follows:

- $\Theta(e_1) = \mathcal{D}h_1 = \{dh_1 \mid d \in \mathcal{D}\} = \{h_1\}$,
- $\Theta(e_2) = \mathcal{D}h_2 = \{dh_2 \mid d \in \mathcal{D}\} = \{h_2\}$.

According to Theorem 3.7, both $\Theta(e_1)$ and $\Theta(e_2)$ are subdimodules of $\mathcal{H}$. Consequently, the soft set (Θ, A) is a soft subdimodule of $\mathcal{H}$ over the parameter set A. This example demonstrates how a collection of individual orbits can be structured as a single soft algebraic entity.

Example 3.11. *Let $\mathcal{D} = (\mathbb{Z}, \cdot)$ be a dimonoid and $\mathcal{H} = \mathbb{Z} \times \mathbb{Z}$ be a $\mathcal{D}$-dimodule with the action defined by $d(h_1, h_2) = (dh_1, dh_2)$ for all $d \in \mathcal{D}$ and $(h_1, h_2) \in \mathcal{H}$. Let $A = \{e_1, e_2\}$ be the set of parameters. We define two soft subdimodules (Θ, A) and (Φ, A) over $\mathcal{H}$ as follows:*

- *For (Θ, A): $\Theta(e_1) = \{(h_1, 0) : h_1 \in \mathbb{Z}\}$ and $\Theta(e_2) = \{(h_1, h_1) : h_1 \in \mathbb{Z}\}$.*
- *For (Φ, A): $\Phi(e_1) = \{(0, h_2) : h_2 \in \mathbb{Z}\}$ and $\Phi(e_2) = \{(h_1, h_1) : h_1 \in \mathbb{Z}\}$.*

By Theorem 3.7, all component sets $\Theta(e_i)$ and $\Phi(e_i)$ are subdimodules of $\mathcal{H}$. Now, let $(\Omega, A) = (\Theta \cap \Phi, A)$ be the soft intersection defined by $\Omega(e_i) = \Theta(e_i) \cap \Phi(e_i)$. The resulting soft set is given by:

- $\Omega(e_1) = \Theta(e_1) \cap \Phi(e_1) = \{(0, 0)\}$,
- $\Omega(e_2) = \Theta(e_2) \cap \Phi(e_2) = \{(h_1, h_1) : h_1 \in \mathbb{Z}\}$.

Since both $\{(0,0)\}$ *and* $\{(h_1,h_1) : h_1 \in \mathbb{Z}\}$ *are subdimodules of* $\mathcal{H}$*, it follows that* (Ω, A) *is a soft subdimodule of* $\mathcal{H}$*. This example validates that the soft intersection of subdimodules preserves the dimodule structure for each parameter.*

4. Conclusions

In this paper, the relationship between soft set theory and dimodule structures has been investigated, and the concept of soft dimodules has been introduced. Initially, the fundamental algebraic properties of soft dimodules were defined, and the construction of these structures as parameterized collections of sub-structures was illustrated through concrete examples.Furthermore, specific algebraic conditions such as distributivity, commutativity, and idempotency were established for soft dimodules, and it was theoretically proven that these properties are preserved under soft intersection operations. Additionally, the structural preservation of soft dimodules was analyzed through dimodule homomorphisms, specifically focusing on the properties of their soft images and preimages. The results obtained demonstrate that the soft set approach is an effective tool for modeling uncertain structures within dimodule theory. This study provides a solid theoretical foundation for future research into soft ideals and soft kernel structures.

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