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An Inertial Method for Approximating Solutions of Split Equality Problems for Generalized Mixed Equilibrium and Fixed Points of Multi-Valued Mappings in Banach Spaces

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ABSTRACT. In this paper, we introduce and study an inertial method for approximating a solution of the split equality of generalized mixed equilibrium and fixed point of multi-valued quasi-Bregman nonexpansive mapping problems in reflexive real Banach spaces. We also apply our result to the split equality of monotone inclusion and generalized mixed equilibrium problems, the split equality of multi-Objective constrained optimization problems, the multiple-sets split equality feasibility problems and the multiple-sets split equality feasibility problems. In addition, we give numerical results to demonstrate the applicability and efficiency of the proposed method.

1. INTRODUCTION

One area of applied mathematics that has drawn the attention of numerous scholars over the past forty years is the theory of equilibrium problems.

Problems arising from image restoration, computer tomography, radiation therapy treatment planning, economics, optimization, etc. have been studied using an equilibrium problem theory. In some systems, solutions of equilibrium problems are also solutions of the fixed point problems of a nonlinear mapping. Many researchers looked for common solutions to the equilibrium and fixed point problems of a system. Numerous writers have investigated the presence and approximation of common solutions to equilibrium and fixed point problems using a variety of compactness assumptions and flexible monotonicity notions. To mention some, see Blum and Oettli [6], Bnouhachem [7], Byrne[11, 12], Moudafi [18], Zegeye et al. [33], and the references therein.

The split feasibility problem (SFP), introduced by Censor and Elfving [13] in 1994, is to find $x^* \in C$ such that $Ax^* \in Q$, where C and Q are nonempty subsets of real Banach spaces E_1 and E_2, respectively, and $A : E_1 \to E_2$ is a bounded linear mapping. In signal processing and medical image reconstruction, Byrne demonstrated the usefulness of SFP [11, 12]. Using the SFP, Censor et al. [14] simulated intensity-modulated radiation treatment in 2006. Moudafi [19] introduced the new split feasibility problem (SEP), also known as the Split Equality Problem (SEP), which is to find $x^* \in C$ and $y^* \in Q$ such that $Ax^* = By^*$, where C and Q are nonempty subsets of real Banach spaces E_1 and E_2, respectively, and $A : E_1 \to E$ and $B : E_2 \to E$ are bounded linear mappings with E another real Banach space. Numerous scholars presented several approximation techniques in the Hilbert space context. In the general Banach space settings, we continue to study an approximation of a shared solution of the split equality fixed point problem of multi-valued mapping and the split generalized mixed equality equilibrium problem.

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Let us start by introducing some important notions which will be used in the sequel. Consider the real Banach space E and its dual, E^*. Let C represent a nonempty, convex, and closed subset of E. The family of nonempty subsets of C is represented by $N(C)$, and the family of nonempty, closed, convex and bounded subsets of C is represented by $CB(C)$. A multi-valued mapping $T : C \to N(C)$, has a fixed point set $F(T)$, which is defined as

$$F(T) = \{p \in C : \ p \in Tp\}.$$

Let $F : C \times C \to \mathbb{R}$ be a bi-function, $\varphi : C \to \mathbb{R}$ a function, and $B : C \to E^*$ a nonlinear mapping. The *generalized mixed equilibrium problem* (GMEP) for F, φ and B is to find $x^* \in C$ such that

$$H(x^*, y) := F(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Bx^*, y - x^* \rangle \geq 0, \forall y \in C. \tag{1.1}$$

The solution set of the generalized mixed equilibrium problem (1.1) is represented by $GME(F, \varphi, B)$, which is

$$GME(F, \varphi, B) = \{p \in C : H(p, y) \geq 0, \forall y \in C\}.$$

It is well recognized that this class of problems includes problems with equilibrium, variational inequality, and other problems. Numerous problems in applied sciences like as economics, physics, and optimization can be reduced to specific instances of GMEP.

A bi-function $F : C \times C \to \mathbb{R}$ is said to satisfy **Condition A** (see Blum and Oettli [6]), if the following four characteristics are true:

1. $F(x, x) = 0, \forall x \in C$;
2. F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$;
3. $\lim_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y), \forall x, y, z \in C$;
4. for each $x \in C, y \longmapsto F(x, y)$ is convex and lower semicontinuous.

Next, we go over the concepts of Bregman distance in relation to a function f. Suppose that E is a real Banach space and that $f : E \to (-\infty, +\infty]$ is a function with domain f,

$$dom(f) = \{x \in E : f(x) < \infty\}.$$

Then, f is said to be

i. *proper* if $dom(f) \neq \emptyset$;
ii. *lower semi-continuous* if for all $r \in \mathbb{R}$ the set $\{x \in E : f(x) \leq r\}$ is closed;
iii. *convex* if $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$ for all $x, y \in E$ and $\alpha \in [0, 1]$;
iv. *uniformly convex* if there exists a continuous increasing function $\psi : [0, +\infty) \to \mathbb{R}$ such that $\psi(0) = 0$ and $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)-t(1-t)\psi(\|x-y\|)$, for all $x, y \in dom(f)$, for all $t \in [0, 1]$. The function ψ is called a *modulus of convexity* of f;
v. *strongly convex* if f is uniformly convex with the modulus of convexity $\psi(t) = ct^2$, $c > 0$;
vi. *strongly coercive* if $\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty$.

The *subdifferential* of a proper, lower semi-continuous, and convex function $f : E \to (-\infty, +\infty]$ at x is defined by

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^* \rangle, \forall y \in E\}.$$

The Fenchel conjugate of f is a function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x) : x \in E\}.$$

For every $x \in int(domf)$ and any $y \in E$, we represent by $f^0(x,y)$ the right-hand derivative of f at x in the direction of y, i.e.,

$$f^0(x,y) = \lim_{t \to 0^+} \frac{f(x+ty) - f(x)}{t}. \tag{1.2}$$

Given any $y \in E$, the function f is called *Gâteaux differentiable* at x if $\lim\limits_{t \to 0} \dfrac{f(x+ty) - f(x)}{t}$ exists. For every $y \in E$, the gradient of f at x, $\nabla f(x)$, in this instance corresponds with $f^0(x,y)$.

If it is Gâteaux differentiable at every point $x \in int(domf)$, it is called *Gâteaux differentiable*. We observe that $\partial f = \nabla f$ if the subdifferential of f is single-valued. If the limit (1.2) is attained uniformly for each $y \in E$ with $\|y\| = 1$, then the function f is classified as *Fréchet differentiable* at x. If the limit (1.2) is attained uniformly for $x \in C$ and $\|y\| = 1$, then f is *uniformly Fréchet differentiable* on a subset C of E.

If f is a uniformly convex and Gâteaux differentiable function in $(domf)$ with modulus of convexity ψ, then $\langle x - y, \nabla f(x) - \nabla f(y)\rangle \geq 2\psi(\|x - y\|), \forall x, y \in domf$, or equivalently, $f(y) \geq f(x) + \langle y - x, \nabla f(x)\rangle + \psi(\|x - y\|), \forall x, y \in domf$; If f is a strongly convex function with constant $\mu > 0$ and Gâteaux differentiable in $(domf)$, then $\langle x - y, \nabla f(x) - \nabla f(y)\rangle \geq \mu\|x - y\|^2, \forall x, y \in domf$. Alternatively, $f(y) \geq f(x) + \langle y - x, \langle f(x)\rangle + \frac{\mu}{2}\|x - y\|^2, \forall x, y \in domf$; The function $f(x) = \|x\|^2, \forall x \in E$ is strongly convex with constant $\mu \in (0, 1]$; if f has a Lipschitz gradient with parameter L, then f^* is strongly convex with parameter $\frac{1}{L}$; and if f has a Lipschitz gradient with parameter $\frac{1}{\mu}$, then f^* is strongly convex with parameter $\frac{1}{L}$.

Definition 1.1. *(Bauschke* [3] *and Bauschke et al.* [4]*) Let $f : E \to (-\infty, +\infty]$ and $f : E^* \to (-\infty, +\infty]$ be Gâteaux differentiable functions. The function f is called Legendre if $dom(\nabla f) = int(domf) \neq \emptyset$ and $dom(\nabla f^*) = int(domf^*) \neq \emptyset$.*

Example 1.1. *(Bauschke* [3] *and Bauschke et al.* [4]*) Let E be a smooth and strictly convex Banach space, and let $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < +\infty)$ with conjugate function. $f^*(x^*) = \frac{1}{q}\|x^*\|^q$ $(1 < q < +\infty)$, with $\frac{1}{p} + \frac{1}{q} = 1$. Then f is the Legendre function. In this case, the gradient of f, ∇f, coincides with the generalized duality mapping, J_p, of E, that is, $\nabla f = J_p$, where $J_p : E \to 2^{E^*}$ is defined by*

$$J_p(x) = \{y^* \in E^* : \langle x, y^*\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\}, \forall x \in E.$$

If $p = 2$, we write $J_2 = J$, which is called the normalized duality mapping. If $E = H$, a real Hilbert space, then $J = I$, with I denoting the identity mapping on H.

Remark 1.1. *(Bonnans and Shapiro* [8]*) If E is a reflexive Banach space and f is a Legendre function, then $\nabla f^* = (\nabla f)^{-1}$.*

Assume that $f : E \to (-\infty, +\infty]$ is a Gâteaux differentiable convex function. The function $D_f : domf \times int(domf) \to [0, +\infty)$ defined by

$$D_f(y,x) = f(y) - f(x) - \langle y - x, \nabla f(x)\rangle, \forall x, y \in E. \tag{1.3}$$

is known as the *Bregman distance* with regard to f (see Bregman [9]).

The Bregman distance has two significant identities (see Reich [23]), known as the *three-point identity*: For any $x \in domf$ and $y, z \in int(domf)$,

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle x - y, \nabla f(z) - \nabla f(y)\rangle, \tag{1.4}$$

The four-point identity states that for any $y, w \in domf$ and $x, z \in int(domf)$,

$$D_f(y,x) - D_f(y,z) - D_f(w,x) + D_f(w,z) = \langle y - w, \nabla f(z) - \nabla f(x)\rangle. \tag{1.5}$$

If E is a smooth and strictly convex Banach space and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then $\nabla f = J$, where J is the normalized duality mapping from E into 2^{E^*} and the Bregman distance with respect to f, D_f, reduces to the Lyapunov functional $\phi : E \times E \to [0, +\infty)$ defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \forall x, y \in E. \tag{1.6}$$

Assume $f : E \to (-\infty, +\infty)$ is a convex function that is differentiable on G. The total convexity modulus of f at $x \in int(domf)$ is defined by the function $\nu_f(x, .) : [0, +\infty) \to [0, \infty)$, where

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in dom(f), ||y - x|| = t\}.$$

When $t > 0$, the function f is said to be totally convex at x if $\nu_f(x, t) > 0$. Totally convex means that f is totally convex at all points $x \in int(domf)$. If $\nu_f(B, t) > 0$ for each nonempty bounded subset B of E and $t > 0$, the function is said to be totally convex on bounded subsets.

Let $\mathcal{H}$ be the Pompeiu-Hausdorff metric(Berinde and Păcurar [5]) on $CB(C)$ defined by

$$\mathcal{H}(W, Y) = \max\{\sup_{a \in W} d(a, Y), \sup_{b \in Y} d(b, W)\}, \tag{1.7}$$

for all $W, Y \in CB(C)$, where $d(a, Y) = \inf\{D_f(a, b) : b \in Y\}$ is the distance from the point a to the subset Y. Consider the mapping $T : C \to CB(C)$. Let a sequence $\{x_n\}$ in C converges weakly to p such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. then a point $p \in C$ is called an asymptotic fixed point of T. The symbol $\widetilde{F}(T)$ will represent the set of asymptotic fixed points of T.

Definition 1.2. *A multi-valued mapping $T : C \to CB(C)$ is said to be*

(a) *Bregman nonexpansive if $\mathcal{H}(Tx, Ty) \leq D_f(x, y)$, for all $x, y \in C$;*
(b) *Bregman relatively nonexpansive if*
  (i) *$F(T) \neq \emptyset$;*
  (ii) *$\mathcal{H}(Tp, Tx) \leq D_f(x, p)$ for all $p \in F(T)$, $x \in C$;*
  (iii) *$F(T) = \widetilde{F}(T)$;*
(c) *quasi-Bregman nonexpansive if $F(T) \neq \emptyset$ and $\mathcal{H}(Tp, Tx) \leq D_f(x, p)$ for all $p \in F(T)$, $x \in C$.*

Remark 1.2. *The class of multi-valued quasi-Bregman nonexpansive mappings is a more general class of mappings since we see that all Bregman nonexpansive with fixed point nonempty and all Bregman relatively nonexpansive mappings are quasi-Bregman nonexpansive mappings.*

The presence and approximation of fixed points for multi-valued mappings in different spaces under various assumptions have been explored in recent years by a number of writers (see [2], [16], [29], [31]). In order to approximate the common fixed point problem of a finite family of Bregman weak relatively nonexpansive mappings in reflexive real Banach spaces, Zegeye and Shahzad [30] devised a one step iteration. They demonstrated the following outcome.

Theorem 1.1. [30] *Let C be a nonempty, closed, and convex subset of $int(domf)$. Let $f : E \to \mathbb{R}$ be a highly coercive Legendre function that is bounded, uniformly Fréchet differentiable, and fully convex on bounded subsets of E. Consider a finite family of Bregman weak relatively nonexpansive mappings $T_i : C \to E$ such that $F := \bigcap_{i=1}^{N} F(T_i)$ is nonempty for $i = 1, 2, \cdots, N$. If the* *sequence* $\{x_n\}$ *is produced by*

$$\begin{cases} x_0 \in C, \\ x_{n+1} = P_C^f \nabla f^* \left(\beta_0 \nabla f(x_n) + \sum_{i=1}^N \beta_i \nabla f(T_i x_n) \right), \ n \geq 0, \end{cases} \tag{1.8}$$

where $\{\beta_i\}_{i=1}^N \subset [c, d] \subset (0, 1)$ *and* $\sum_{i=1}^N \beta_i = 1$, *then* $\{x_n\}$ *converges strongly to an element of* F.

The following theorem, which approximates the solution to a finite family of Bregman substantially nonexpansive mappings in reflexive real Banach spaces, was established by Shahzad and Zegeye [27] in 2014.

Theorem 1.2. *Let* C *be a nonempty, closed, and convex subset of* $int(domf)$. *Let* $f : E \to \mathbb{R}$ *be a highly coercive Legendre function that is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of* E. *Consider a finite family of Bregman relatively nonexpansive mappings* $T_i : C \to E$ *such that* $F := \bigcap_{i=1}^N F(T_i)$ *is nonempty for* $i = 1, 2, \cdots, N$. *Let* $T_i : C \to CB(C)$. *If the sequence* $\{x_n\}$ *is produced by*

$$\begin{cases} x_0, w \in C, \\ y_n = P_C^f \nabla f \left(\alpha_n \nabla f(w) + (1 - \alpha_n) \nabla f(x_n) \right), \\ x_{n+1} = \nabla f^{-1} \left(\beta_{n,0} \nabla f(x_n) + \sum_{i=1}^N \beta_{n,i} \nabla f(u_{n,i}) \right), u_{n,i} \in T_i y_n, n \geq 0, \end{cases} \tag{1.9}$$

where $\{\alpha_n\} \subseteq (0, 1)$ *and* $\{\beta_{n,i}\} \subset [a, b] \subset (0, 1)$ *, for* $i = 1, 2, \cdots, N$, *satisfy certain conditions, then* $\{x_n\}$ *converges strongly to an element of* F.

Alghamdi et al. [1] demonstrated the following convergence theorem in 2016 for the fixed point problem of Bregman relatively nonexpansive mapping in a smooth, strictly convex, and reflexive real Banach space E as well as a common point of variational inequality problem. The following theorem was proven by them.

Theorem 1.3. *Suppose that* C *is a nonempty, closed, and convex subset of* E, *a smooth, reflexive, and strictly convex real Banach space. The Legendre function* $f : E \to \mathbb{R}$ *is bounded, uniformly Fréchet differentiable, and fully convex on bounded subsets of* E. *It is also highly coercive. Consider the Bregman relatively nonexpansive mapping* $T : C \to E$ *and the continuous monotone mapping* $B : C \to E^*$. *Assume that* $F := F(T) \bigcap VI(C, B)$ *is nonempty. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_0, w \in C, \\ u_n \in C \text{ such that} \\ \langle Bu_n, y - u_n \rangle + \frac{1}{r_n} \langle \nabla f(u_n) - \nabla f(x_n), y - u_n \rangle \geq 0, \forall y \in C, \\ y_n = \nabla f^* \left(\alpha_n \nabla f(x_n) + \beta_n \nabla f(u_n) + \gamma_n \nabla f(T(x_n)) \right), \\ x_{n+1} = P_C^f \nabla f^* \left(a_n \nabla f(w) + (1 - a_n) \nabla f(y_n) \right), n \geq 0, \end{cases} \tag{1.10}$$

where $\alpha_n, \beta_n, \gamma_n \in [c, d] \subseteq (0, 1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\{a_n\} \subseteq (0, 1)$ *satisfies* $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^\infty a_n = \infty$. *Then,* $\{x_n\}$ *converges strongly to* $P_F^f(w)$.

Consider reflexive real Banach spaces E_1, E_2, and E, and their respective duals E_1^*, E_2^*, and E^*. Assume that C and D are nonempty, closed, and convex subsets of E_1 and E_2, separately. Let's say we have the mappings listed below.

(1) $T_1 : C \to N(C)$ and $T_2 : D \to N(D)$ are multi-valued mappings;
(2) $F_1 : C \times C \to \mathbb{R}$ and $F_2 : D \times D \to \mathbb{R}$ are bi-functions;
(3) $\varphi_1 : C \to \mathbb{R}$ and $\varphi_2 : D \to \mathbb{R}$ are real valued functions;
(4) $B_1 : C \to E_1^*$ and $B_2 : D \to E_2^*$ are nonlinear mappings; and

(5) $S : E_1 \to E$ and $K : E_2 \to E$ are bounded linear operators with adjoints S^* and K^*, respectively.

The Split Equality of Generalized Mixed Equilibrium and Fixed Point Problems (SEGMEFPP) is defined as finding a point $(p, q) \in C \times D$ such that

$$(p, q) \in [GMEP(F_1, \varphi_1, B_1) \cap F(T_1)] \times [GMEP(F_2, \varphi_2, B_2) \cap F(T_2)] \text{ and } S(p) = K(q). \tag{1.11}$$

In 2020, Nnakwe [20] constructed Krasnoselskii-type algorithm given by

$$\begin{cases} (x_1, y_1) \in E_1 \times E_2,\ C_1 = E_1,\ D_1 = E_2,\ e_n \in J_E(Su_n - Kv_n), \\ u_n = Q_r x_n,\ v_n = Q_r y_n, \\ \theta_n = J_{E_1}^{-1}(J_{E_1} u_n - \mu S^* e_n),\ \delta_n = J_{E_2}^{-1}(J_{E_2} v_n + \mu K^* e_n), \\ z_n = J_{E_1}^{-1}(\beta J_{E_1} x_n + (1-\beta) J_{E_1} T_1 \theta_n),\ w_n = J_{E_2}^{-1}(\beta J_{E_2} y_n + (1-\beta) J_{E_2} T_2 \delta_n), \\ C_{n+1} = \{p \in C_n : \phi(p, z_n) \le \phi(p, x_n)\},\ D_{n+1} = \{q \in D_n : \phi(q, w_n) \le \phi(q, y_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1,\ y_{n+1} = \Pi_{D_{n+1}} y_1, \end{cases} \tag{1.12}$$

where ϕ is the Lyapunov functional, to solve the SEGMEP and proved a strong convergence theorem to a common solution of split equality generalized mixed equality equilibrium problems and fixed point problem for quasi-ϕ-nonexpansive mappings in 2-uniformly convex and uniformly smooth real Banach space.

The following question naturally arises.

Question 1. *Is it possible to construct a method that strongly converges to a common solution for split equality generalized mixed equilibrium and fixed point problems for multi-valued quasi-Bregman nonexpansive mappings in Banach spaces beyond 2-uniformly convex and uniformly smooth real Banach spaces?*

In this paper, we introduce and study an inertial method for approximating a common solution of the split equality of generalized mixed equilibrium problem and fixed point of multi-valued quasi-Bregman nonexpansive problem in reflexive real Banach spaces, inspired by the works of Zegeye and Shahzad [30], Shahzad and Zegeye [27], Alghamdia et al. [1], and Nnakwe [20]. As a result, we present an inertial approximation approach for a shared fixed point in a finite family of multi-valued quasi-Bregman nonexpansive mappings. In addition, we provide numerical data to demonstrate the applicability and efficiency of the suggested strategy.

2. PRELIMINARIES

Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ be a Legendre function. The function $V_f : E \times E^* \to \mathbb{R}$ associated with f is given by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \text{ for all } x \in E \text{ and } x^* \in E^*.$$

We note that V_f is a nonnegative function which satisfies (see, Senakka and Cholamjiak [26])

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \text{ for all } x \in E \text{ and } x^* \in E^*, \tag{2.1}$$

and

$$V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \le V_f(x, x^* + y^*), \text{ for all } x \in E \text{ and } x^*, y^* \in E^*. \tag{2.2}$$

We shall use the following lemmas in the proof of our main results.

Lemma 2.1. *(Wega and Zegeye* [28]*) Let f be a strongly convex function with constant $\mu > 0$. Then, for all $y \in domf$ and $x \in int(domf)$,*

$$D_f(y,x) \geq \frac{\mu}{2}\|x-y\|^2,$$

where $D_f(y,x)$ is a Bregman distance with respect to f .

Lemma 2.2. *(Phelps* [22]*) If $f : E \to (-\infty,+\infty]$ is a proper, convex, lower semi-continuous function, then $f^* : E^* \to (-\infty,+\infty]$ is a proper, weak lower semi-continuous, convex function, and for each $w \in E$, $\{\theta_n\}_{n=1}^N \subseteq (0,1)$ and $\{z_n\}_{n=1}^N \subseteq E$. The following is true with $\sum_{n=1}^N \alpha_n = 1$:*

$$D_f(w, \nabla f^*\left(\sum_{n=1}^N \theta_n \nabla f(z_n)\right)) \leq \sum_{n=1}^N \theta_n D_f(w,z_n). \tag{2.3}$$

The Bregman projection of $x \in int(domf)$ onto the nonempty, closed, and convex set $C \subset domf$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x),x) = \inf\{D_f(y,x) : y \in C\}.$$

The well-known Bregman projection qualities include:

Lemma 2.3. *(Bunariu and Resmerita* [10]*) Assume f is a totally convex and Gâteaux differentiable function on $int(domf)$ and $x \in int(domf)$. Assume C is a nonempty, closed, and convex subset of $int(domf)$. Then,*

(i) *$z = P_C^f(x)$ if and only if $\langle y-z, \nabla f(x) - \nabla f(z)\rangle \leq 0, \forall y \in C$;*
(ii) *$D_f(y, P_C^f(x)) + D_f(P_C^f(x),x) \leq D_f(y,x), \forall y \in C$.*

Lemma 2.4. *(Reich and Sabach* [24]*) Assume that $f : E \to \mathbb{R}$ is a totally convex function that is Gâteaux differentiable. For $x \in E$, if the sequence $D_f(x_n,x)$ is bounded then $\{x_n\}$ is bounded.*

Lemma 2.5. *(Reich and Sabach* [24]*) Consider $f : E \to \mathbb{R}$, a Gâteaux differentiable function that is uniformly convex on a limited subset of E. If $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E, then $\lim_{n\to\infty} D_f(x_n,y_n) = 0$ if and only if $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.6 (Darvish [15]). *Let $f : E \to (-\infty,+\infty]$ be a coercive and Gâteaux differentiable function. Let C be a closed and convex subset of a real reflexive Banach space E. Assume that $B : C \to E^*$ is a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ is a lower semi-continuous and convex function and let $F : C \times C \to \mathbb{R}$ be a bi-function satisfying **Condition A**. For $r > 0$ and $x \in E$, define a mapping $T_H^{f,r} : E \to C$ as follows:*

$$T_H^{f,r}x = \{z \in C : H(z,y) + \frac{1}{r}\langle y-z, \nabla f(z) - \nabla f(x)\rangle \geq 0, \forall y \in C\}, \tag{2.4}$$

where $H(z,y) := F(z,y) + \varphi(y) - \varphi(z) + \langle y-z, Bz\rangle$. Then, $T_H^{f,r}(x) \neq \emptyset$, and the following hold:

(1) $T_H^{f,r}$ is single-valued;
(2) $F(T_H^{f,r}) = GMEP(F,\varphi,B)$;
(3) $GMEP(F,\varphi,B)$ is closed and convex;
(4) $T_H^{f,r}$ is quasi-Bregman nonexpansive;
(5) $D_f(p,T_H^{f,r}x) + D_f(T_H^{f,r}x,x) \leq D_f(p,x), \forall p \in F(T_H^{f,r})$.

Lemma 2.7 (Jailoka et al. [17]). *Let E be a reflexive real Banach space. Consider $T : E \to CB(E)$, a multivalued Bregman quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Then, $F(T)$ is closed and convex.*

Lemma 2.8. *(Saejung and Yotkaew* [25]*) Let $\{\xi_n\} \subset \mathbb{R}$ and $\{\tau_n\} \subset (0,1)$ be sequences such that $\sum_{n=1}^{\infty} \zeta_n = \infty$ and*

$$\tau_{n+1} \leq (1-\zeta_n)\tau_n + \zeta_n \xi_n, n \geq 1.$$

If for every subsequence $\{\tau_{n_k}\}$ of $\{\tau_n\}$ and $\{\xi_{n_k}\}$ of $\{\xi_n\}$ which satisfies $\liminf_{k\to\infty}(\tau_{n_k+1} - \tau_{n_k}) \geq 0$ and $\limsup_{k\to\infty} \xi_{n_k} \leq 0$, then $\lim_{n\to\infty} \tau_n = 0$.

Consider E_1 and E_2 as reflexive real Banach spaces with dual E_1^* and E_2^*, respectively. If $E = E_1 \times E_2$ and its dual $E^* = E_1^* \times E_2^*$, then the duality pairing is provided by

$$\langle x, y^* \rangle = \langle x_1, y_1^* \rangle + \langle x_2, y_2^* \rangle,$$

where $x = (x_1, x_2) \in E$ and $y^* = (y_1^*, y_2^*) \in E^*$. Let $h : E = E_1 \times E_2 \to (-\infty, +\infty]$ Define $h(x_1, x_2) = f(x_1) + g(x_2)$, $\forall (x_1, x_2) \in E_1 \times E_2$, and $f : E_1 \to (-\infty, +\infty]$ and $g : E_2 \to [-\infty, +\infty]$ denotes proper, lower semi-continuous, and convex functions. The subdifferential of h at $x = (x_1, x_2)$ corresponds to the convex set given by

$$\begin{aligned} \partial h(x) &= \{x^* \in E^* : h(y) - h(x) \geq \langle y - x, x^* \rangle, \forall y \in E\} \\ &= \{(x_1^*, x_2^*) \in E_1^* \times E_2^* : x_1^* \in \partial f(x_1) \text{ and } x_2^* \in \partial g(x_2)\}. \end{aligned}$$

Assuming $f : E_1 \to (-\infty, +\infty), (-\infty, +\infty)$ and $g : E_2 \to (-\infty, +\infty), (-\infty, +\infty)$ are Gâteaux differentiable convex functions, then h is a Gâteaux differentiable convex function, and $\forall (x_1, x_2) \in E_1 \times E_2, \nabla h(x_1, x_2) = (\nabla f(x_1), \nabla g(x_2))$.

3. MAIN RESULT

This section introduces an inertial algorithm to solve the split equality of generalized mixed equilibrium and fixed point of multi-valued quasi-Bregman nonexpansive problems in reflexive real Banach spaces. The following assumptions will be made in the sequel.

Assumptions

- (A1) Let E_1, E_2 and E be reflexive real Banach spaces with their dual spaces E_1^*, E_2^* and E^*, respectively. Let C and D be nonempty, closed and convex subsets of E_1 and E_2, respectively.
- (A2) Let $f : E_1 \to \mathbb{R}$ and $g : E_2 \to \mathbb{R}$ be strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre functions on bounded subsets with strongly convex conjugates f^* and g^*, respectively. Assume that μ_1 and μ_2 are the strong convexity constants of f and g, respectively, and let $\mu = \min\{\mu_1, \mu_2\}$;
- (A3) Let $T_i : C \to CB(C)$ and $G_i : D \to CB(D)$, $i = 1, 2, \dots, N$, be quasi-Bregman nonexpansive mappings with $T_i(p) = \{p\}, \forall p \in F(T_i)$ and $G_i(p) = \{p\}, \forall p \in F(G_i)$ such that $I - T_i$ and $I - G_i$ are demiclosed at zero;
- (A4) Let $F_{1,i} : C \times C \to \mathbb{R}$ and $F_{2,i} : D \times D \to \mathbb{R}$, $i = 1, 2, \dots, N$, be bi-functions satisfying Condition **A**;
- (A5) Let $B_{1,i} : C \to E_1^*$ and $B_{2,i} : D \to E_2^*$, $i = 1, 2, \dots, N$, be monotone mappings;
- (A6) Let $\varphi_{1,i} : C \to \mathbb{R}$ and $\varphi_{2,i} : D \to \mathbb{R}$, $i = 1, 2, \dots, N$, be real valued functions; for each $i = 1, 2, \dots, N$ we denote by

$$\begin{aligned} &H_{1,i}(z, y) := F_{1,i}(z, y) + \varphi_{1,i}(y) - \varphi_{1,i}(z) + \langle y - z, B_{1,i} z \rangle \text{ and} \\ &H_{2,i}(z, y) := F_{2,i}(z, y) + \varphi_{2,i}(y) - \varphi_{2,i}(z) + \langle y - z, B_{2,i} z \rangle. \end{aligned}$$

- (A6) Let $S : E_1 \to E$ and $K : E_2 \to E$ be bounded linear mappings with adjoints $S^* : E^* \to E_1^*$ and $K^* : E^* \to E_2^*$, respectively;

(A7) Assume that

$$\Omega = \{(p,q) \in \Omega_1 \times \Omega_2 : S(p) = K(q)\} \neq \emptyset,$$

where $\Omega_1 = \bigcap_{j=1}^{N}(GMEP(H_{1,j}) \cap F(T_j))$ and $\Omega_1 = \bigcap_{j=1}^{N}(GMEP(H_{2,j}) \cap F(G_j))$.

(A8) For each $i = 0,1,2,\dots,N$, let $\{\alpha_{i,n}\}$ be a sequence in $[a,b] \subset (0,1)$ such that $\sum_{i=0}^{N} \alpha_{i,n} = 1$ for $n = 1,2,\cdots$. For each $i = 1,2,\dots,N$, let $\beta_{1,i}$ and $\beta_{2,i}$ be numbers in $[0,1]$, such that $\sum_{i=1}^{N} \beta_{1,i} = 1$ and $\sum_{i=1}^{N} \beta_{2,i} = 1$.

(A9) Let $\{\lambda_n\} \subset (0,1)$ be such that $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$; choose a sequence $\{\chi_n\}$ in $(0, \frac{\mu}{2})$ such that $\lim_{n\to\infty} \frac{\chi_n}{\lambda_n} = 0$; choose a sequence $\{r_n\}$ in (c,∞) for some $c > 0$.

(A10) Let J_E be normalized duality mapping on E.

Algorithm 3.1

Initialization: Choose $(w,u),(x_0,y_0),(x_1,y_1) \in C \times D$, $0 < \sigma, \rho$. Define the algorithm as follows:

Step 0: Choose σ_n such that $0 \leq \sigma_n \leq \overline{\sigma}_n$ where

$$\overline{\sigma}_n = \begin{cases} \min\left\{\frac{\chi_n}{\|\nabla f(x_n)-\nabla f(x_{n-1})\|+\|\nabla g(y_n)-\nabla g(y_{n-1})\|}, \sigma\right\}, & \text{if } x_n \neq x_{n-1} \;\&\; y_n \neq y_{n-1} \\ \sigma, & \text{otherwise} \end{cases} \tag{3.1}$$

Step 1: Compute

$$\begin{aligned} a_n &= \nabla f^*(\nabla f(x_n) + \sigma_n(\nabla f(x_n) - \nabla f(x_{n-1}))), \\ b_n &= \nabla g^*(\nabla g(y_n) + \sigma_n(\nabla g(y_n) - \nabla g(y_{n-1}))). \end{aligned} \tag{3.2}$$

Step 2: Choose γ_n such that $\rho \leq \gamma_n \leq \rho_n$ for $S(a_n) \neq K(b_n)$ otherwise $\gamma_n = \rho$, for some $\rho > 0$, where

$$\rho_n = \min\left\{\rho + 1, \frac{\mu\|S(a_n) - K(b_n)\|^2}{2\left[\|S^*J_E(S(a_n)) - K(b_n)\|^2 + \|K^*J_E(K(b_n)) - S(a_n)\|^2\right]}\right\}. \tag{3.3}$$

Step 3: Compute

$$\begin{aligned} d_n &= \nabla f^*(\nabla f(a_n) - \gamma_n S^*J_E(S(a_n) - K(b_n))), \\ e_n &= \nabla g^*(\nabla g(b_n) - \gamma_n K^*J_E(K(b_n) - S(a_n))), \end{aligned} \tag{3.4}$$

Step 4: Compute

$$h_n = \nabla f^*\left(\sum_{i=1}^{N} \beta_{1,i}\nabla f(T_{H_{1,i}}^{f,r_n} d_n)\right), z_n = \nabla g^*\left(\sum_{i=1}^{N} \beta_{2,i}\nabla g(T_{H_{2,i}}^{g,r_n} e_n)\right), \tag{3.5}$$

Step 5: Compute

$$\begin{aligned} u_n &= P_C^f \nabla f^*(\lambda_n \nabla f(w) + (1-\lambda_n)\nabla f(h_n)), \\ v_n &= P_D^g \nabla g^*(\lambda_n \nabla g(u) + (1-\lambda_n)\nabla g(z_n)), \end{aligned} \tag{3.6}$$

Step 6: Choose $w_{i,n} \in T_i(u_n),\ l_{i,n} \in G_i(v_n)$, and Compute

$$\begin{aligned} x_{n+1} &= \nabla f^*(\alpha_{0,n}\nabla f(h_n) + \sum_{i=1}^{N} \alpha_{i,n}\nabla f(w_{i,n})), \\ y_{n+1} &= \nabla g^*(\alpha_{0,n}\nabla g(z_n) + \sum_{i=1}^{N} \alpha_{i,n}\nabla g(l_{i,n})). \end{aligned} \tag{3.7}$$

Set $n := n + 1$ and go to **Step 0**.

Lemma 3.1. *Suppose that the assumptions (A1)- (A10) hold. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 3.1 are bounded.*

Proof. Let $(p, q) \in \Omega$. From (3.6), Lemma 2.6, Lemma 2.2 and Lemma 2.3, we get that

$$\begin{aligned} D_f(p, u_n) &= D_f(p, P_C^f \nabla f^*(\lambda_n \nabla f(w) + (1-\lambda_n)\nabla f(h_n))) \\ &\leq D_f(p, \nabla f^*(\lambda_n \nabla f(w) + (1-\lambda_n)\nabla f(h_n))) \\ &\leq \lambda_n D_f(p, w) + (1-\lambda_n) D_f(p, h_n) \\ &= \lambda_n D_f(p, w) + (1-\lambda_n) D_f(p, \nabla f^*(\sum_{i=1}^{N} \beta_{1,i}\nabla f(T_{H_{1,i}}^{f,r_n} d_n))) \\ &\leq \lambda_n D_f(p, w) + (1-\lambda_n) \sum_{i=1}^{N} \beta_{1,i} D_f(p, T_{H_{1,i}}^{f,r_n} d_n) \\ &\leq \lambda_n D_f(p, w) + (1-\lambda_n) \sum_{i=1}^{N} \beta_{1,i} D_f(p, d_n) \\ &\leq \lambda_n D_f(p, w) + (1-\lambda_n) D_f(p, d_n). \end{aligned} \tag{3.8}$$

Now, from (3.7), (3.8), Lemma 2.2, Lemma 2.3 and 2.6, and the quasi-Bregman nonexpansiveness of T_i, we have

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\alpha_{0,n}\nabla f(h_n) + \sum_{i=1}^{N} \alpha_{i,n}\nabla f(w_{i,n}))) \\ &\leq \alpha_{0,n} D_f(p, h_n) + \sum_{i=1}^{N} \alpha_{i,n} D_f(p, w_{i,n}) \\ &\leq \alpha_{0,n} D_f(p, \nabla f^*(\sum_{i=1}^{N} \beta_{1,i}\nabla f(T_{H_{1,i}}^{f,r_n} d_n))) + \sum_{i=1}^{N} \alpha_{i,n} \mathcal{H}(T_i p, T_i u_n) \\ &\leq \alpha_{0,n} \sum_{i=1}^{N} \beta_{1,i} D_f(p, T_{H_{1,i}}^{f,r_n} d_n) + \sum_{i=1}^{N} \alpha_{i,n} D_f(p, u_n) \\ &\leq \alpha_{0,n} \sum_{i=1}^{N} \beta_{1,i} D_f(p, d_n) + \sum_{i=1}^{N} \alpha_{i,n} D_f(p, u_n) \\ &\leq \alpha_{0,n} D_f(p, d_n) + \sum_{i=1}^{N} \alpha_{i,n} D_f(p, u_n) \end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_{0,n}D_f(p,d_n)+\lambda_n\sum_{i=1}^{N}\alpha_{i,n}D_f(p,w)+(1-\lambda_n)\sum_{i=1}^{N}\alpha_{i,n}D_f(p,d_n)\\
&= \alpha_{0,n}D_f(p,d_n)+\lambda_n(1-\alpha_{0,n})D_f(p,w)+(1-\lambda_n)(1-\alpha_{0,n})D_f(p,d_n)\\
&\leq \lambda_n^*D_f(p,w)+(1-\lambda_n^*)D_f(p,d_n), \qquad (3.9)
\end{aligned}
$$

where $\lambda_n^* := \lambda_n(1-\alpha_{0,n})$.

Furthermore, from (2.1) and (2.2), we obtain

$$
\begin{aligned}
D_f(p,d_n) &= D_f(p,\nabla f^*(\nabla f(a_n)-\gamma_n S^*J_E(S(a_n)-K(b_n))))\\
&= V_f(p,\nabla f(a_n)-\gamma_n S^*J_E(S(a_n)-K(b_n)))\\
&\leq V_f(p,\nabla f(a_n))-\gamma_n\left\langle d_n-p,S^*J_E(S(a_n)-K(b_n))\right\rangle\\
&= D_f(p,a_n)-\gamma_n\left\langle S(d_n)-S(p),J_E(S(a_n)-K(b_n))\right\rangle. \qquad (3.10)
\end{aligned}
$$

From (1.4), we have

$$
D_f(p,a_n)=D_f(p,x_n)-D_f(a_n,x_n)+\left\langle p-a_n,\nabla f(x_n)-\nabla f(a_n)\right\rangle. \qquad (3.11)
$$

Now, from (3.1), (3.2) and Lemma 2.1 we obtain that

$$
\begin{aligned}
\left\langle p-a_n,\nabla f(x_n)-\nabla f(a_n)\right\rangle &\leq \|\nabla f(x_n)-\nabla f(a_n)\|\|p-a_n\|\\
&= \sigma_n\|\nabla f(x_n)-\nabla f(x_{n-1})\|\|p-a_n\|\\
&\leq \frac{\sigma_n}{2}\|\nabla f(x_n)-\nabla f(x_{n-1})\|\left[\|p-a_n\|^2+1\right]\\
&\leq \sigma_n\|\nabla f(x_n)-\nabla f(x_{n-1})\|\left[\|p-x_n\|^2+\|x_n-a_n\|^2\right]\\
&\quad +\frac{\sigma_n}{2}\|\nabla f(x_n)-\nabla f(x_{n-1})\|\\
&\leq \frac{2\chi_n}{\mu}D_f(p,x_n)+\frac{2\chi_n}{\mu}D_f(a_n,x_n)+\frac{\chi_n}{2}. \qquad (3.12)
\end{aligned}
$$

Combining (3.9), (3.10), (3.11) and (3.12), we obtain that

$$
\begin{aligned}
D_f(p,x_{n+1}) &\leq \lambda_n^*D_f(p,w)+(1-\lambda_n^*)\left(1+\frac{2\chi_n}{\mu}\right)D_f(p,x_n)\\
&\quad -(1-\lambda_n^*)\left(1-\frac{2\chi_n}{\mu}\right)D_f(a_n,x_n)\\
&\quad -(1-\lambda_n^*)\gamma_n\left\langle S(d_n)-S(p),J_E(S(a_n)-K(b_n))\right\rangle+\frac{\chi_n}{2}(1-\lambda_n^*). \qquad (3.13)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
D_g(q,y_{n+1}) &\leq \lambda_n^*D_g(q,u)+(1-\lambda_n^*)\left(1+\frac{2\chi_n}{\mu}\right)D_g(q,y_n)\\
&\quad -(1-\lambda_n^*)\left(1-\frac{2\chi_n}{\mu}\right)D_g(b_n,y_n)\\
&\quad -(1-\lambda_n^*)\gamma_n\left\langle K(e_n)-K(q),J_E(K(b_n)-S(a_n))\right\rangle+\frac{\chi_n}{2}(1-\lambda_n^*). \qquad (3.14)
\end{aligned}
$$

Then, from (3.13) and (3.14), we obtain

$$\begin{aligned} D_f(p,x_{n+1})+D_g(q,y_{n+1}) &\leq \lambda_n^*\left[D_f(p,w)+D_g(q,u)\right] \\ &\quad +(1-\lambda_n^*)\left(1+\frac{2\chi_n}{\mu}\right)\left[D_f(p,x_n)+D_g(q,y_n)\right] \\ &\quad -(1-\lambda_n^*)\left(1-\frac{2\chi_n}{\mu}\right)\left[D_f(a_n,x_n)+D_g(b_n,y_n)\right] \\ &\quad -(1-\lambda_n^*)\gamma_n\left\langle S(d_n)-S(p),J_E(S(a_n)-K(b_n))\right\rangle \\ &\quad -(1-\lambda_n^*)\gamma_n\left\langle K(e_n)-K(p),J_E(K(b_n)-S(a_n))\right\rangle \\ &\quad +\chi_n(1-\lambda_n^*) \\ &= \lambda_n^*\left[D_f(p,w)+D_g(q,u)\right] \\ &\quad +(1-\lambda_n^*)\left(1+\frac{2\chi_n}{\mu}\right)\left[D_f(p,x_n)+D_g(q,y_n)\right] \\ &\quad -(1-\lambda_n^*)\left(1-\frac{2\chi_n}{\mu}\right)\left[D_f(a_n,x_n)+D_g(b_n,y_n)\right] \\ &\quad -(1-\lambda_n^*)\gamma_n\left\langle K(e_n)-S(d_n),J_E(K(b_n)-S(a_n))\right\rangle \\ &\quad +\chi_n(1-\lambda_n^*). \end{aligned} \tag{3.15}$$

Next, we show that

$$\begin{aligned} -\left\langle K(e_n)-S(d_n),J_E(K(b_n)-S(a_n))\right\rangle &= -\left\langle K(b_n)-S(a_n),J_E(K(b_n)-S(a_n))\right\rangle \\ &\quad -\left\langle K(e_n)-K(b_n),J_E(K(b_n)-S(a_n))\right\rangle \\ &\quad -\left\langle S(a_n)-S(d_n),J_E(K(b_n)-S(a_n))\right\rangle \\ &= -\|K(b_n)-S(a_n)\|^2 \\ &\quad -\left\langle e_n-b_n,K^*J_E(K(b_n)-S(a_n))\right\rangle \\ &\quad -\left\langle a_n-d_n,S^*J_E(K(b_n)-S(a_n))\right\rangle \\ &\leq -\|K(b_n)-S(a_n)\|^2 \\ &\quad +\|e_n-b_n\|\|K^*J_E(K(b_n)-S(a_n))\| \\ &\quad +\|a_n-d_n\|\|S^*J_E(K(b_n)-S(a_n))\|. \end{aligned} \tag{3.16}$$

From the fact that ∇g^* is a Lipschitz mapping with constant $\frac{1}{\mu}$ and the definition of e_n, we obtain that

$$\begin{aligned} \|e_n-b_n\| &= \|\nabla g^*(\nabla g(b_n)-\gamma_nK^*J_E(K(b_n)-S(a_n)))-b_n\| \\ &\leq \frac{\gamma_n}{\mu}\|K^*J_E(K(b_n)-S(a_n))\|. \end{aligned} \tag{3.17}$$

Similarly, by the Lipschitz property of ∇f^* and the definition of d_n gives

$$\|d_n-a_n\| \leq \frac{\gamma_n}{\mu}\|S^*J_E(S(a_n)-K(b_n))\|. \tag{3.18}$$

Then, from (3.16), (3.17) and (3.18), we get

$$\begin{aligned}
-\gamma_n \langle K(e_n) - S(d_n), J_E(K(b_n) - S(a_n))\rangle &\leq -\gamma_n \|K(b_n) - S(a_n)\|^2 \\
&\quad + \frac{\gamma_n^2}{\mu}\|K^* J_E(K(b_n) - S(a_n))\|^2 \\
&\quad + \frac{\gamma_n^2}{\mu}\|S^* J_E(S(a_n) - K(b_n))\|^2 \\
&\leq -\gamma_n \|K(b_n) - S(a_n)\|^2 \\
&\quad + \frac{\gamma_n}{2}\|K(e_n) - S(d_n)\|^2 \\
&= -\frac{\gamma_n}{2}\|K(b_n) - S(a_n)\|^2. \qquad (3.19)
\end{aligned}$$

Take $\epsilon \in (0, \frac{\mu}{2})$. Then, from (A9), there exists $N \in \mathbb{N}$ such that

$$\frac{2\chi_n}{\mu} < \lambda_n(1-b)\epsilon \leq \lambda_n^*\epsilon, \forall n \geq N. \qquad (3.20)$$

Thus, from (A8), (3.15), (3.19) and (3.20), we obtain for all $n \geq N$

$$\begin{aligned}
D_f(p, x_{n+1}) + D_g(q, y_{n+1}) &\leq \lambda_n^* \left[D_f(p, w) + D_g(q, u)\right] \\
&\quad + (1 - \lambda_n^*)\left(1 + \lambda_n^*\epsilon\right)\left[D_f(p, x_n) + D_g(q, y_n)\right] \\
&\quad - (1 - \lambda_n^*)\left(1 - \lambda_n^*\epsilon\right)\left[D_f(a_n, x_n) + D_g(b_n, y_n)\right] \\
&\quad - (1 - \lambda_n^*)\gamma_n \langle K(e_n) - S(d_n), J_E(K(b_n) - S(a_n))\rangle \\
&\quad + \frac{\mu\epsilon}{2}\lambda_n^*(1 - \lambda_n^*) \\
&\leq \lambda_n^* \left[D_f(p, w) + D_g(q, u)\right] \\
&\quad + (1 - \lambda_n^*)\left[D_f(p, x_n) + D_g(q, y_n)\right] \\
&\quad + \lambda_n^*\epsilon(D_f(p, x_n) + D_g(q, y_n)) \\
&\quad - \frac{\gamma_n}{2}(1 - \lambda_n^*)\|K(b_n) - S(a_n)\|^2 + \lambda_n^*\epsilon\frac{\mu}{2} \\
&\leq \lambda_n^*(D_f(p, w) + D_g(q, u)) \\
&\quad + (1 - \lambda_n^*(1 - \epsilon))(D_f(p, x_n) + D_g(q, y_n)) + \lambda_n^*\epsilon\frac{\mu}{2} \\
&\leq (1 - \lambda_n^*(1 - \epsilon))(D_f(p, x_n) + D_g(q, w_n)) \\
&\quad + \lambda_n^*(1 - \epsilon)\left[\frac{1}{1-\epsilon}(D_f(p, w) + D_g(q, u)) + \frac{\mu\epsilon}{2(1-\epsilon)}\right] \\
&\leq \max\left\{D_f(p, x_n) + D_g(q, w_n), \mathfrak{L}\right\}, \qquad (3.21)
\end{aligned}$$

where $\mathfrak{L} = \frac{1}{1-\epsilon}(D_f(p, x) + D_g(q, w)) + \frac{\mu\epsilon}{2(1-\epsilon)}$. Therefore, by induction, for all $n \geq N$, we have that

$$D_f(p, x_n) + D_g(q, y_n) \leq \max\left\{D_f(p, x_N) + D_g(q, y_N), \mathfrak{L}\right\},$$

and hence $\{D_f(p, x_n) + D_g(q, y_n)\}$ is bounded which implies that the sequences $\{D_f(p, x_n)\}$ and $\{D_g(q, y_n)\}$ are bounded. Furthermore, by Lemma 2.4, we have $\{x_n\}$ and $\{y_n\}$ are bounded. □

Theorem 3.1. *Suppose that assumption (A1)- (A10) are satisfied. Then, the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm 3.1 converges strongly to* (p, q) *in* Ω, *where* $(p, q) = P_\Omega^\psi(w, u)$ *and* $\psi : E_1 \times E_2 \to \mathbb{R}$ *is given by* $\psi(x, y) = f(x) + g(y)$.

Proof. Let $(p,q)=P_{\Omega}^{\psi}(w,u)$. Then, from Algorithm 3.1, (3.6), (2.1), and (2.2), Lemma 2.2 and 2.3, we obtain that

$$\begin{aligned}
D_f(p,u_n) &= D_f(p,P_C^f\nabla f^*(\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)))\\
&\le D_f(p,\nabla f^*(\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)))\\
&\le V_f(p,\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n))\\
&\le V_f(p,\lambda_n[\nabla f(p)-\nabla f(w)]+\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n))\\
&\quad -\langle\nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p,\lambda_n[\nabla f(p)-\nabla f(w)]\rangle\\
&= V_f(p,\lambda_n\nabla f(p)+(1-\lambda_n)\nabla f(h_n))\\
&\quad -\langle\nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p,\lambda_n[\nabla f(p)-\nabla f(w)]\rangle\\
&= D_f(p,\nabla f^*(\lambda_n\nabla f(p)+(1-\lambda_n)\nabla f(h_n)))\\
&\quad -\lambda_n\left\langle\nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p,\nabla f(p)-\nabla f(w)\right\rangle\\
&\le \lambda_n D_f(p,p)+(1-\lambda_n)D_f(p,h_n)\\
&\quad -\lambda_n\left\langle\nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p,\nabla f(p)-\nabla f(w)\right\rangle\\
&= (1-\lambda_n)D_f(p,\nabla f^*(\sum_{i=1}^{N}\beta_{1,i}\nabla f(T_{H_{1,i}}^{f,r_n}d_n)))\\
&\quad -\lambda_n\left\langle\nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p,\nabla f(p)-\nabla f(w)\right\rangle\\
&\le (1-\lambda_n)\sum_{i=1}^{N}\beta_{1,i}D_f(p,T_{H_{1,i}}^{f,r_n}d_n)\\
&\quad -\lambda_n\left\langle\nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p,\nabla f(p)-\nabla f(w)\right\rangle\\
&\le (1-\lambda_n)\sum_{i=1}^{N}\beta_{1,i}D_f(p,d_n)\\
&\quad -\lambda_n\left\langle\nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p,\nabla f(p)-\nabla f(w)\right\rangle\\
\text{(3.22)}\qquad &\le (1-\lambda_n)D_f(p,d_n)\\
&\quad -\lambda_n\left\langle\nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p,\nabla f(p)-\nabla f(w)\right\rangle.
\end{aligned}$$

Now, from the definition of V_f, uniform convexity of f^* with the modulus of convexity ϕ of f, the quasi-Bregman nonexpansiveness of T_i, Lemma 2.2, (2.1), (2.2), (3.10), (3.11) and (3.22), we get

$$\begin{aligned}
D_f(p,x_{n+1}) &= D_f(p,\nabla f^*(\alpha_{0,n}\nabla f(h_n)+\sum_{i=1}^{N}\alpha_{i,n}\nabla f(w_{i,n})))\\
&= V_f(p,\alpha_{0,n}\nabla f(h_n)+\sum_{i=1}^{N}\alpha_{i,n}\nabla f(w_{i,n}))\\
&\le \alpha_{0,n}V_f(p,\nabla f(h_n))+\sum_{i=1}^{N}\alpha_{i,n}V_f(p,\nabla f(w_{i,n}))\\
&\quad -\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(h_n)-\nabla f(w_{i,n})\|)\\
&\le \alpha_{0,n}V_f(p,\sum_{i=1}^{N}\beta_{1,i}\nabla f(T_{H_{1,i}}^{f,r_n}d_n))+\sum_{i=1}^{N}\alpha_{i,n}D_f(p,w_{i,n})\\
&\quad -\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(h_n)-\nabla f(w_{i,n})\|)
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(p,T^{f,r_n}_{H_{1,i}}d_n)-\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T^{f,r_n}_{H_{1,i}}d_n)-\nabla f(T^{f,r_n}_{H_{1,k}}d_n)\|)\\
&\quad+\sum_{i=1}^{N}\alpha_{i,n}\mathcal{H}(T_ip,T_iu_n)-\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(h_n)-\nabla f(w_{i,n})\|)\\
&\leq \alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(p,d_n)-\alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T^{f,r_n}_{H_{1,i}}d_n,d_n)\\
&\quad-\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T^{f,r_n}_{H_{1,i}}d_n)-\nabla f(T^{f,r_n}_{H_{1,k}}d_n)\|)\\
&\quad+\sum_{i=1}^{N}\alpha_{i,n}D_f(p,u_n)-\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(h_n)-\nabla f(w_{i,n})\|)\\
&\leq \alpha_{0,n}D_f(p,d_n)+\sum_{i=1}^{N}\alpha_{i,n}D_f(p,u_n)-\alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T^{f,r_n}_{H_{1,i}}d_n,d_n)\\
&\quad-\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T^{f,r_n}_{H_{1,i}}d_n)-\nabla f(T^{f,r_n}_{H_{1,k}}d_n)\|)\\
&\quad-\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(h_n)-\nabla f(w_{i,n})\|)\\
&= \alpha_{0,n}D_f(p,d_n)+(1-\alpha_{0,n})D_f(p,u_n)-\alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T^{f,r_n}_{H_{1,i}}d_n,d_n)\\
&\quad-\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T^{f,r_n}_{H_{1,i}}d_n)-\nabla f(T^{f,r_n}_{H_{1,k}}d_n)\|)\\
&\quad-\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(u_n)-\nabla f(w_{i,n})\|)\\
&\leq \alpha_{0,n}D_f(p,d_n)+(1-\alpha_{0,n})(1-\lambda_n)D_f(p,d_n)-\alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T^{f,r_n}_{H_{1,i}}d_n,d_n)\\
&\quad-(1-\alpha_{0,n})\lambda_n\left\langle\nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p,\nabla f(p)-\nabla f(w)\right\rangle\\
&\quad-\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T^{f,r_n}_{H_{1,i}}d_n)-\nabla f(T^{f,r_n}_{H_{1,k}}d_n)\|)\\
&\quad-\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(u_n)-\nabla f(w_{i,n})\|)\\
&= [1-(1-\alpha_{0,n})\lambda_n]D_f(p,d_n)-\alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T^{f,r_n}_{H_{1,i}}d_n,d_n)\\
&\quad-(1-\alpha_{0,n})\lambda_n\left\langle\nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p,\nabla f(p)-\nabla f(w)\right\rangle\\
&\quad-\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T^{f,r_n}_{H_{1,i}}d_n)-\nabla f(T^{f,r_n}_{H_{1,k}}d_n)\|)\\
&\quad-\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(u_n)-\nabla f(w_{i,n})\|).
\end{aligned}
$$

Then, from (3.10), (3.11), (3.12), (3.20), (3.23) and by setting $\lambda_n^*=(1-\alpha_{0,n})\lambda_n$, we obtain that for $n\geq N$

$$
\begin{aligned}
D_f(p,x_{n+1}) \;\leq\;& [1-\lambda_n^*]D_f(p,d_n)-\alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T^{f,r_n}_{H_{1,i}}d_n,d_n)\\
&-\lambda_n^*\left\langle\nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p,\nabla f(p)-\nabla f(w)\right\rangle\\
&-\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T^{f,r_n}_{H_{1,i}}d_n)-\nabla f(T^{f,r_n}_{H_{1,k}}d_n)\|)\\
&-\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(u_n)-\nabla f(w_{i,n})\|)
\end{aligned}
$$

$$\begin{aligned}
&\leq [1-\lambda_n^*]D_f(p,a_n) - \alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T_{H_{1,i}}^{f,r_n}d_n,d_n)\\
&\quad -\gamma_n[1-\lambda_n^*]\left\langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\right\rangle\\
&\quad -\lambda_n^*\left\langle \nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p, \nabla f(p)-\nabla f(w)\right\rangle\\
&\quad -\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T_{H_{1,i}}^{f,r_n}d_n)-\nabla f(T_{H_{1,k}}^{f,r_n}d_n)\|)\\
&\quad -\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(u_n)-\nabla f(w_{i,n})\|)\\
&\leq [1-\lambda_n^*]D_f(p,x_n) - [1-\lambda_n^*]D_f(a_n,x_n)\\
&\quad +[1-\lambda_n^*]\left\langle p-a_n, \nabla f(x_n)-\nabla f(a_n)\right\rangle\\
&\quad -\alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T_{H_{1,i}}^{f,r_n}d_n,d_n)\\
&\quad -\gamma_n[1-\lambda_n^*]\left\langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\right\rangle\\
&\quad -\lambda_n^*\left\langle \nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p, \nabla f(p)-\nabla f(w)\right\rangle\\
&\quad -\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T_{H_{1,i}}^{f,r_n}d_n)-\nabla f(T_{H_{1,k}}^{f,r_n}d_n)\|)\\
&\quad -\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(u_n)-\nabla f(w_{i,n})\|)\\
&\leq [1-\lambda_n^*]D_f(p,x_n) - [1-\lambda_n^*]D_f(a_n,x_n) + [1-\lambda_n^*]\frac{2\chi_n}{\mu}D_f(p,x_n)\\
&\quad +[1-\lambda_n^*]\frac{2\chi_n}{\mu}D_f(a_n,x_n) + [1-\lambda_n^*]\frac{\chi_n}{2}\\
&\quad -\alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T_{H_{1,i}}^{f,r_n}d_n,d_n)\\
&\quad -\gamma_n[1-\lambda_n^*]\left\langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\right\rangle\\
&\quad -\lambda_n^*\left\langle \nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p, \nabla f(p)-\nabla f(w)\right\rangle\\
&\quad -\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T_{H_{1,i}}^{f,r_n}d_n)-\nabla f(T_{H_{1,k}}^{f,r_n}d_n)\|)\\
&\quad -\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(u_n)-\nabla f(w_{i,n})\|)\\
&\leq [1-\lambda_n^*]D_f(p,x_n) - [1-\lambda_n^*]D_f(a_n,x_n) + \lambda_n^*\epsilon D_f(p,x_n)\\
&\quad +\lambda_n^*\epsilon D_f(a_n,x_n) + \frac{\chi_n}{2} - \alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T_{H_{1,i}}^{f,r_n}d_n,d_n)\\
&\quad -\gamma_n[1-\lambda_n^*]\left\langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\right\rangle\\
&\quad -\lambda_n^*\left\langle \nabla f^*[\lambda_n\nabla f(w)+(1-\lambda_n)\nabla f(h_n)]-p, \nabla f(p)-\nabla f(w)\right\rangle\\
&\quad -\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T_{H_{1,i}}^{f,r_n}d_n)-\nabla f(T_{H_{1,k}}^{f,r_n}d_n)\|)\\
&\quad -\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(u_n)-\nabla f(w_{i,n})\|)
\end{aligned}$$

$$\begin{aligned}
&\leq [1-\lambda_n^*(1-\epsilon)]D_f(p,x_n) - [1-\lambda_n^*(1-\epsilon)]D_f(a_n,x_n)\\
&\quad +\frac{\chi_n}{2} - \alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T_{H_{1,i}}^{f,r_n}d_n,d_n)\\
&\quad -\gamma_n[1-\lambda_n^*]\left\langle S(d_n)-S(p), J_{E_3}(S(a_n)-K(b_n))\right\rangle\\
&\quad -\lambda_n^*\left\langle \Theta_n-p, \nabla f(p)-\nabla f(w)\right\rangle\\
&\quad -\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T_{H_{1,i}}^{f,r_n}d_n)-\nabla f(T_{H_{1,k}}^{f,r_n}d_n)\|)\\
&\quad -\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(u_n)-\nabla f(w_{i,n})\|),
\end{aligned}\tag{3.23}$$

where $\Theta_n = \nabla f^*[\lambda_n \nabla f(w) + (1-\lambda_n)\nabla f(h_n)]$. Similarly, we have

$$\begin{aligned}
D_g(q, y_{n+1}) \quad \leq \quad & [1-\lambda_n^*(1-\epsilon)]D_g(q, y_n) - [1-\lambda_n^*(1-\epsilon)]D_g(b_n, y_n) + \frac{\chi_n}{2} \\
& -\alpha_{0,n}\sum_{i=1}^{N}\beta_{2,i}D_g(T^{g,r_n}_{H_{2,i}}e_n, e_n) \\
& -\gamma_n[1-\lambda_n]\left\langle K(e_n) - K(q), J_{E_3}(K(b_n) - S(a_n))\right\rangle \\
& -\lambda_n\left\langle \Phi_n - q, \nabla g(q) - \nabla g(u)\right\rangle \\
& -\beta_{2,i}\beta_{2,k}\psi(\|\nabla g(T^{g,r_n}_{H_{2,i}}e_n) - \nabla g(T^{g,r_n}_{H_{2,k}}e_n)\|) \\
& -\alpha_{0,n}\alpha_{i,n}\psi(\|\nabla g(v_n) - \nabla g(l_{i,n})\|),
\end{aligned} \tag{3.24}$$

where $\Phi_n = \nabla g^*[\lambda_n \nabla g(u) + (1-\lambda_n)\nabla g(z_n)]$ and ψ is the modulus of convexity of g. Then, by combining (**??**) and (3.24), we obtain

$$\begin{aligned}
D_f(p, x_{n+1}) + D_g(q, y_{n+1}) \quad \leq \quad & (1-\lambda_n^*(1-\epsilon))[D_f(p, x_n) + D_g(q, y_n)] \\
& -(1-\lambda_n^*(1-\epsilon))[D_f(a_n, x_n) + D_g(b_n, y_n)] + \chi_n \\
& -\alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T^{f,r_n}_{H_{1,i}}d_n, d_n) \\
& -\alpha_{0,n}\sum_{i=1}^{N}\beta_{2,i}D_f(T^{g,r_n}_{H_{2,i}}e_n, e_n) \\
& -\gamma_n(1-\lambda_n^*)\left\langle S(d_n) - S(p), J_{E_3}(S(a_n) - K(b_n))\right\rangle \\
& -\gamma_n(1-\lambda_n^*)\left\langle K(e_n) - K(q), J_{E_3}(K(b_n) - S(a_n))\right\rangle \\
& -\lambda_n^*\left\langle \Theta_n - p, \nabla f(p) - \nabla f(w)\right\rangle \\
& -\lambda_n^*\left\langle \Phi_n - q, \nabla g(q) - \nabla g(u)\right\rangle \\
& -\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T^{f,r_n}_{H_{1,i}}d_n) - \nabla f(T^{f,r_n}_{H_{1,k}}d_n)\|) \\
& -\beta_{1,i}\beta_{2,k}\psi(\|\nabla g(T^{g,r_n}_{H_{2,i}}e_n) - \nabla g(T^{g,r_n}_{H_{2,k}}e_n)\|) \\
& -\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(u_n) - \nabla f(w_{i,n})\|) \\
& -\alpha_{0,n}\alpha_{i,n}\psi(\|\nabla g(v_n) - \nabla g(l_{i,n})\|) \\
\leq \quad & (1-\lambda_n^*(1-\epsilon))[D_f(p, x_n) + D_g(q, y_n)] + \chi_n \\
& -(1-\lambda_n^*(1-\epsilon))[D_f(a_n, x_n) + D_g(b_n, y_n)] \\
& -\alpha_{0,n}\sum_{i=1}^{N}\beta_{1,i}D_f(T^{f,r_n}_{H_{1,i}}d_n, d_n) \\
& -\alpha_{0,n}\sum_{i=1}^{N}\beta_{2,i}D_g(T^{g,r_n}_{H_{2,i}}e_n, e_n) \\
& -\gamma_n(1-\lambda_n^*)\left\langle K(e_n) - S(d_n), J_{E_3}(K(b_n) - S(a_n))\right\rangle \\
& -\lambda_n^*\left\langle \Theta_n - p, \nabla f(p) - \nabla f(w)\right\rangle \\
& -\lambda_n^*\left\langle \Phi_n - q, \nabla g(q) - \nabla g(u)\right\rangle \\
& -\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T^{f,r_n}_{H_{1,i}}d_n) - \nabla f(T^{f,r_n}_{H_{1,k}}d_n)\|) \\
& -\beta_{1,i}\beta_{1,k}\psi(\|\nabla g(T^{g,r_n}_{H_{2,i}}e_n) - \nabla g(T^{g,r_n}_{H_{2,k}}e_n)\|) \\
& -\alpha_{0,n}\alpha_{i,n}\phi(\|\nabla f(u_n) - \nabla f(w_{i,n})\|) \\
& -\alpha_{0,n}\alpha_{i,n}\psi(\|\nabla g(v_n) - \nabla g(l_{i,n})\|)
\end{aligned}$$

$$\begin{aligned}
&\le (1-\lambda_n^*(1-\epsilon))[D_f(p,x_n)+D_g(q,y_n)]+\chi_n\\
&\quad -(1-\lambda_n^*(1-\epsilon))[D_f(a_n,x_n)+D_g(b_n,y_n)]\\
&\quad -\alpha_{0,n}[\sum_{i=1}^{N}\beta_{1,i}D_f(T_{H_{1,i}}^{f,r_n}d_n,d_n)+\sum_{i=1}^{N}\beta_{2,i}D_g(T_{H_{2,i}}^{g,r_n}e_n,e_n)]\\
&\quad -(1-\lambda_n^*)\frac{\gamma_n}{2}\|K(b_n)-S(a_n)\|^2\\
&\quad -\lambda_n^*[\langle\Theta_n-p,\nabla f(p)-\nabla f(w)\rangle+\langle\Phi_n-q,\nabla g(q)-\nabla g(u)\rangle]\\
&\quad -\beta_{1,i}\beta_{1,k}\phi(\|\nabla f(T_{H_{1,i}}^{f,r_n}d_n)-\nabla f(T_{H_{1,k}}^{f,r_n}d_n)\|)\\
&\quad -\beta_{1,i}\beta_{1,k}\psi(\|\nabla g(T_{H_{2,i}}^{g,r_n}d_n)-\nabla g(T_{H_{2,k}}^{g,r_n}d_n)\|)\\
&\quad -\alpha_{0,n}\alpha_{i,n}[\phi(\|\nabla f(u_n)-\nabla f(w_{i,n})\|)+\psi(\|\nabla g(v_n)-\nabla g(l_{i,n})\|)]\\
&\le (1-\lambda_n^*(1-\epsilon))[D_f(p,x_n)+D_g(q,y_n)]+\chi_n\\
&\quad -\lambda_n^*[\langle\Theta_n-p,\nabla f(p)-\nabla f(w)\rangle+\langle\Phi_n-q,\nabla g(q)-\nabla g(u)\rangle]
\end{aligned}\tag{3.25}$$

$$\begin{aligned}
&\le (1-\lambda_n^*(1-\epsilon))[D_f(p,x_n)+D_g(q,y_n)]+\chi_n\\
&\quad +\lambda_n^*(1-\epsilon)\frac{1}{1-\epsilon}\left\langle\Theta_n-p,\nabla f(w)-\nabla f(p)\right\rangle\\
&\quad +\lambda_n^*(1-\epsilon)\frac{1}{1-\epsilon}\left\langle\Phi_n-q,\nabla g(u)-\nabla g(q)\right\rangle.
\end{aligned}\tag{3.26}$$

By setting $\zeta_n:=\lambda_n^*(1-\epsilon)=\lambda_n(1-\alpha_{0,n})(1-\epsilon)$, $\tau_n:=D_f(p,x_n)+D_g(q,y_n)$,

$$\xi_n \;:=\; \frac{1}{1-\epsilon}[\langle\Theta_n-p,\nabla f(w)-\nabla f(p)\rangle+\langle\Phi_n-q,\nabla g(u)-\nabla g(q)\rangle+\frac{\chi_n}{\lambda_n(1-b)}]$$

and (3.26), we have

$$\tau_{n+1}\le(1-\zeta_n)\tau_n+\zeta_n\xi_n. \tag{3.27}$$

Now, suppose that a subsequence $\{\tau_{n_k}\}$ of a sequence $\{\tau_n\}$ such that

$$\liminf_{k\to\infty}(\tau_{n_k+1}-\tau_{n_k})\ge 0. \tag{3.28}$$

From (3.25), (3.28) and the fact that $\{\lambda_n\}\subset(0,1)$, $\rho\le\gamma_n$, $0<a\le\alpha_{i,n},\beta_{1,n},\beta_{1,n}\le b<1$, $i=0,1,2,\ldots,N$, for all $n\ge 0$ and $\lambda_n\to 0$ as $n\to\infty$, we obtain

$$D_f(a_{n_k},x_{n_k})\to 0, D_g(b_{n_k},y_{n_k})\to 0 \text{ as } k\to\infty, \tag{3.29}$$

$$D_f(T_{H_{1,i}}^{f,r_n}d_n,d_n)\to 0, D_g(T_{H_{2,i}}^{g,r_n}e_n,e_n)\to 0 \text{ as } k\to\infty, \tag{3.30}$$

$$\phi(\|\nabla f(u_{n_k})-\nabla f(w_{i,n_k})\|)\to 0,\psi(\|\nabla g(v_{n_k})-\nabla g(l_{i,n_k})\|)\to 0 \text{ as } k\to\infty \tag{3.31}$$

$$\phi(\|\nabla f(T_{H_{1,i}}^{f,r_{n_k}}d_{n_k})-\nabla f(T_{H_{1,j}}^{f,r_{n_k}}d_{n_k})\|)\to 0 \text{ as } k\to\infty, \tag{3.32}$$

$$\psi(\|\nabla g(T_{H_{2,i}}^{g,r_{n_k}}e_{n_k})-\nabla g(T_{H_{2,j}}^{g,r_{n_k}}e_{n_k})\|)\to 0 \text{ as } k\to\infty \tag{3.33}$$

and

$$\|S(a_{n_k})-K(b_{n_k})\|\to 0 \text{ as } k\to\infty. \tag{3.34}$$

Since $\lambda_n\to 0$ as $n\to\infty$, $\{D_f(h_n,w)\}$ is bounded and by Lemma 2.2 and 2.3, we have

$$\begin{aligned}
D_f(h_{n_k},u_{n_k}) &= D_f(h_{n_k},P_C^f\Theta_{n_k})\le D_f(h_{n_k},\Theta_{n_k})\\
&= D_f(h_{n_k},\nabla f^*[\lambda_{n_k}\nabla f(w)+(1-\lambda_{n_k})\nabla f(h_{n_k})])\\
&\le \lambda_{n_k}D_f(h_{n_k},w)+(1-\lambda_{n_k})D_f(h_{n_k},h_{n_k})=\lambda_{n_k}D_f(h_{n_k},w)\to 0 \text{ as } k\to\infty,
\end{aligned}\tag{3.35}$$

and similarly, we obtain

$$D_g(z_{n_k}, v_{n_k}), D_g(z_{n_k}, \Phi_{n_k}) \to 0 \text{ as } k \to \infty. \tag{3.36}$$

Then, from (3.17), (3.18), (3.29-3.36), the property of S, K, J_{E_3}, ϕ and ψ and Lemma 2.5, we obtain that

$$\|x_{n_k} - a_{n_k}\| \to 0, \|y_{n_k} - b_{n_k}\| \to 0 \text{ as } k \to \infty \tag{3.37}$$

$$\|T^{f,r_n}_{H_{1,i}} d_n - d_n\| \to 0, \|T^{g,r_n}_{H_{2,i}} e_n - e_n\| \to 0 \text{ as } k \to \infty, \tag{3.38}$$

$$\|\nabla f(T^{f,r_{n_k}}_{H_{1,i}} d_{n_k}) - \nabla f(T^{f,r_{n_k}}_{H_{1,j}} d_{n_k})\| \to 0 \text{ as } k \to \infty, \tag{3.39}$$

$$\|\nabla g(T^{g,r_{n_k}}_{H_{2,i}} e_{n_k}) - \nabla g(T^{g,r_{n_k}}_{H_{2,j}} e_{n_k})\| \to 0 \text{ as } k \to \infty, \tag{3.40}$$

$$\begin{aligned} &\|\nabla f(u_{n_k}) - \nabla f(w_{i,n_k})\| \to 0, \|\nabla g(v_{n_k}) - \nabla g(l_{i,n_k})\| \to 0 \text{ as } k \to \infty, \\ &\qquad \|h_{n_k} - u_{n_k}\| \to 0, \|h_{n_k} - \Theta_{n_k}\| \to 0, \end{aligned} \tag{3.41}$$

$$\|z_{n_k} - v_{n_k}\| \to 0, \|z_{n_k} - \Phi_{n_k}\| \to 0 \text{ as } k \to \infty \tag{3.42}$$

and

$$\|e_{n_k} - b_{n_k}\| \to 0, \|d_{n_k} - a_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{3.43}$$

Then, from (3.5), (3.39), (3.40) and $0 < a \le \beta_{1,i}, \beta_{2,i} \le b < 1, \forall i$, we have

$$\|\nabla f(h_{n_k}) - \nabla f(T^{f,r_{n_k}}_{H_{1,j}} d_{n_k})\| \le b \sum_{i=1}^{N} \|\nabla f(T^{f,r_{n_k}}_{H_{1,i}} d_{n_k}) - \nabla f(T^{f,r_{n_k}}_{H_{1,j}} d_{n_k})\| \to 0, \tag{3.44}$$

$$\|\nabla g(z_{n_k}) - \nabla g(T^{g,r_{n_k}}_{H_{2,j}} e_{n_k})\| \le b \sum_{i=1}^{N} \|\nabla g(T^{g,r_{n_k}}_{H_{2,i}} e_{n_k}) - \nabla g(T^{g,r_{n_k}}_{H_{2,j}} e_{n_k})\| \to 0, \tag{3.45}$$

as $k \to \infty$. And also, from (3.41), (3.42) and the uniformly continuity of ∇f and ∇g, we have

$$\begin{aligned} \|\nabla f(x_{n_{k+1}}) - \nabla f(u_{n_k})\| &\le \alpha_{0,n_k} \|\nabla f(h_{n_k}) - \nabla f(u_{n_k})\| \\ &\quad + \sum_{i=1}^{N} \alpha_{i,n_k} \|\nabla f(w_{i,n_k}) - \nabla f(u_{n_k})\| \\ &\le b\|\nabla f(h_{n_k}) - \nabla f(u_{n_k})\| + b \sum_{i=1}^{N} \|\nabla f(w_{i,n_k}) - \nabla f(u_{n_k})\| \to 0. \end{aligned} \tag{3.46}$$

as $k \to \infty$ and similarly

$$\|\nabla g(y_{n_{k+1}}) - \nabla g(v_{n_k})\| \to 0 \text{ as } k \to \infty. \tag{3.47}$$

From (3.39-3.41), (3.44-3.47) and the uniformly continuity of ∇f^* and ∇g^*, we have

$$\|T^{f,r_{n_k}}_{H_{1,i}} d_{n_k} - T^{f,r_{n_k}}_{H_{1,j}} d_{n_k}\| \to 0, \|T^{g,r_{n_k}}_{H_{2,i}} e_{n_k} - T^{g,r_{n_k}}_{H_{2,j}} e_{n_k}\| \to 0 \text{ as } k \to \infty, \tag{3.48}$$

$$\|u_{n_k} - w_{i,n_k}\| \to 0, \|v_{n_k} - l_{i,n_k}\| \to 0 \text{ as } k \to \infty, \tag{3.49}$$

$$\|h_{n_k} - T^{f,r_{n_k}}_{H_{1,j}} d_{n_k}\| \to 0, \|z_{n_k} - T^{g,r_{n_k}}_{H_{2,j}} e_{n_k}\| \to 0 \text{ as } k \to \infty, \tag{3.50}$$

$$\|x_{n_{k+1}} - u_{n_k}\| \to 0, \|y_{n_{k+1}} - v_{n_k}\| \to 0 \text{ as } k \to \infty \tag{3.51}$$

From (3.37), (3.38), (3.43), (3.46), (3.47), (3.50) and (3.51), we get

$$\begin{aligned} \|x_{n_{k+1}} - x_{n_k}\| &\le \|x_{n_{k+1}} - u_{n_k}\| + \|u_{n_k} - h_{n_k}\| + \|h_{n_k} - T^{f,r_{n_k}}_{H_{1,j}} d_{n_k}\| \\ &\quad + \|T^{f,r_{n_k}}_{H_{1,j}} d_{n_k} - d_{n_k}\| + \|d_{n_k} - a_{n_k}\| + \|a_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty \end{aligned} \tag{3.52}$$

and similarly we obtain

$$\|y_{n_{k+1}} - y_{n_k}\| \to 0 \text{ as } k \to \infty \tag{3.53}$$

Then, by (3.51),(3.52) and (3.53), we obtain

$$\|x_{n_k} - u_{n_k}\| \le \|x_{n_k} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - u_{n_k}\| \to 0 \text{ as } k \to \infty \tag{3.54}$$

and

$$\|y_{n_k} - v_{n_k}\| \le \|y_{n_k} - y_{n_{k+1}}\| + \|y_{n_{k+1}} - v_{n_k}\| \to 0 \text{ as } k \to \infty \tag{3.55}$$

Now, since $(\Theta_{n_k}, \Phi_{n_k})$ is bounded and $E_1 \times E_2$ is reflexive, there exists $(\Theta, \Phi) \in C \times D$ and a subsequence $\{(\Theta_{n_{k_j}}, \Phi_{n_{k_j}})\}$ of $\{(\Theta_{n_k}, \Phi_{n_k})\}$ such that $(\Theta_{n_{k_j}}, \Phi_{n_{k_j}}) \rightharpoonup (\Theta, \Phi)$ and

$$\begin{aligned} &\limsup_{k\to\infty} \langle (\Theta_{n_k}, \Phi_{n_k}) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q)) \rangle \\ &= \lim_{j\to\infty} \langle (\Theta_{n_{k_j}}, \Phi_{n_{k_j}}) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q)) \rangle. \end{aligned} \tag{3.56}$$

But $(\Theta_{n_{k_j}}, \Phi_{n_{k_j}}) \rightharpoonup (\Theta, \Phi)$ implies that $\Theta_{n_{k_j}} \rightharpoonup \Theta$ and $\Phi_{n_{k_j}} \rightharpoonup \Phi$. Furthermore, from (3.37), (3.42), (3.43), (3.48), (3.49) and (3.50) we have

$$h_{n_{k_j}} \rightharpoonup \Theta, u_{n_{k_j}} \rightharpoonup \Theta, T^{f,r_{n_k}}_{H_{1,i}} d_{n_{k_j}} \rightharpoonup \Theta, w_{i,n_{k_j}} \rightharpoonup \Theta \text{ as } j \to \infty, \tag{3.57}$$

$$x_{n_{k_j}} \rightharpoonup \Theta, a_{n_{k_j}} \rightharpoonup \Theta, d_{n_{k_j}} \rightharpoonup \Theta \text{ as } j \to \infty, \tag{3.58}$$

$$z_{n_{k_j}} \rightharpoonup \Phi, v_{n_{k_j}} \rightharpoonup \Phi, T^{g,r_{n_k}}_{H_{2,i}} e_{i,n_{k_j}} \rightharpoonup \Phi, l_{i,n_{k_j}} \rightharpoonup \Phi \text{ as } j \to \infty, \tag{3.59}$$

$$y_{n_{k_j}} \rightharpoonup \Phi, b_{n_{k_j}} \rightharpoonup \Phi, e_{n_{k_j}} \rightharpoonup \Phi \text{ as } j \to \infty. \tag{3.60}$$

Now, We show that $(\Theta, \Phi) \in \Omega$. Set $h^*_{n_{k_j}} = T^{f,r_{n_k}}_{H_{1,i}} d_{n_{k_j}}$ and $z^*_{n_{k_j}} = T^{g,r_{n_k}}_{H_{2,i}} e_{n_{k_j}}$. Then,

$$H_{1,i}(h^*_{n_{k_j}}, y) + \frac{1}{r_{n_{k_j}}} \left\langle y - h^*_{n_{k_j}}, \nabla f(h^*_{n_{k_j}}) - \nabla f(d_{n_{k_j}}) \right\rangle \ge 0, \forall y \in C$$

and

$$H_{2,i}(z^*_{n_{k_j}}, z) + \frac{1}{r_{n_{k_j}}} \left\langle z - z^*_{n_{k_j}}, \nabla g(z^*_{n_{k_j}}) - \nabla g(e_{n_{k_j}}) \right\rangle \ge 0, \forall z \in D.$$

Thus, by Condition (A2), we have

$$\begin{aligned} H_{1,i}(y, h^*_{n_{k_j}}) &\le -H_{1,i}(h^*_{n_{k_j}}, y) + \frac{1}{r_{n_{k_j}}} \left\langle y - h^*_{n_{k_j}}, \nabla f(h^*_{n_{k_j}}) - \nabla f(d_{n_{k_j}}) \right\rangle \\ &\le \|y - h^*_{n_{k_j}}\| \frac{\|\nabla f(h^*_{n_{k_j}}) - \nabla f(d_{n_{k_j}})\|}{r_{n_{k_j}}} \\ &\le Q_1 \frac{\|\nabla f(h^*_{n_{k_j}}) - \nabla f(d_{n_{k_j}})\|}{r_{n_{k_j}}}, \end{aligned} \tag{3.61}$$

where $Q_1 = \max\limits_{1\le i\le N} \sup\limits_{k\ge 0} \|y - h^*_{n_{k_j}}\|$ and similarly

$$H_{2,i}(z, z^*_{n_{k_j}}) \le Q_2 \frac{\|\nabla g(z^*_{n_{k_j}}) - \nabla g(e_{n_{k_j}})\|}{r_{n_{k_j}}}, \tag{3.62}$$

where $Q_2 = \max\limits_{1\le i\le N} \sup\limits_{k\ge 0} \|z - z^*_{n_{k_j}}\|$. From the facts that $h^*_{n_{k_j}} \rightharpoonup \Theta$ and $z^*_{n_{k_j}} \rightharpoonup \Phi$, **Condition A** (A4), $r_n \ge \vartheta$, for all $n \ge 0$ and taking limits on both sides of the inequality (3.61) and

(3.62) as $k \to \infty$, we obtain that

$$H_{1,i}(y,\Theta) \leq 0, \forall y \in C, \forall i \tag{3.63}$$

$$H_{2,i}(z,\Phi) \leq 0, \forall z \in D, \forall i. \tag{3.64}$$

Set $\Theta_\lambda = \lambda y + (1-\lambda)\Theta$ and $\Phi_\lambda = \lambda z + (1-\lambda)\Phi$, $\lambda \in (0,1]$ and $y \in C$. Consequently, we get $\Theta_\lambda \in C$. From (3.63), (3.64) and **Condition A** (A1), we obtain

$$\begin{aligned} 0 &= H_{1,i}(\Theta_\lambda,\Theta_\lambda) \leq \lambda H_{1,i}(\Theta_\lambda, y) + (1-\lambda)H_{1,i}(\Theta_\lambda,\Theta) \\ &\leq H_{1,i}(\Theta + \lambda(\Theta - y), y) \end{aligned} \tag{3.65}$$

and similarly

$$0 \leq H_{2,i}(\Phi + \lambda(\Phi - z), z) \tag{3.66}$$

If $\lambda \downarrow 0$, using **Condition A** (A3) we have $H_{1,i}(\Theta, y) \geq 0, \forall y \in C$ and $H_{2,i}(\Phi, z) \geq 0, \forall z \in D$. Hence, $\Theta \in GMEP(H_{1,i})$ and $\Phi \in GMEP(H_{2,i})$, for each $i = 1, 2, \ldots\ldots, N$. Next, we show that $\Theta \in F(T_{1,i})$ and $\Phi \in F(T_{2,i})$, for each $i = 1, 2, \ldots\ldots, N$. Since $u_{n_{k_j}} \rightharpoonup \Theta$, $w_{i,n_{k_j}} \rightharpoonup \Theta$, $v_{n_{k_j}} \rightharpoonup \Phi$, $l_{i,n_{k_j}} \rightharpoonup \Phi$ as $j \to \infty$, we have

$$d(u_{n_{k_j}}, T_{1,i}(u_{n_{k_j}})) \leq \|u_{n_{k_j}} - w_{i,n_{k_j}}\| \rightharpoonup 0 \text{ as } j \to \infty,$$

$$d(v_{n_{k_j}}, T_{2,i}(v_{n_{k_j}})) \leq \|v_{n_{k_j}} - l_{i,n_{k_j}}\| \rightharpoonup 0 \text{ as } j \to \infty$$

and by the demiclosedness of $I - T_{t,i}$, $t = 1, 2$ at zero, we get $\Theta \in F(T_{1,i})$ and $\Phi \in F(T_{2,i})$, for each $i = 1, 2, \ldots\ldots, N$. Then, $\Theta \in \bigcap_{i=1}^{N}[GMEP(H_{1,i}) \cap F(T_{1,i})]$ and $\Phi \in \bigcap_{i=1}^{N}[GMEP(H_{2,i}) \cap F(T_{2,i})]$. Finally, we show that $S(\Theta) = K(\Phi)$. Now, from (3.34), (3.58), (3.60) and the fact that S and K are bounded linear mappings we have $S\Theta = K\Phi$ and hence $(\Theta, \Phi) \in \Omega$. Then by (3.56)and Lemma 2.3, we obtain

$$\begin{aligned} &\limsup_{k\to\infty} \langle (\Theta_{n_k}, \Phi_{n_k}) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q)) \rangle \\ &= \lim_{j\to\infty} \langle (\Theta_{n_{k_j}}, \Phi_{n_{k_j}}) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q)) \rangle \\ &= \lim_{j\to\infty} \langle \Theta_{n_{k_j}} - p, \nabla f(x) - \nabla f(p) \rangle + \lim_{j\to\infty} \langle \Phi_{n_{k_j}} - q, \nabla g(w) - \nabla g(q) \rangle \\ &= \langle (\Theta, \Phi) - (p,q), (\nabla f(x), \nabla g(w)) - (\nabla f(p), \nabla g(q)) \rangle \leq 0. \end{aligned} \tag{3.67}$$

and so

$$\limsup_{k\to\infty} \xi_{n_k} \leq 0. \tag{3.68}$$

Thus, from (3.27), (3.28), (3.68), Lemma 2.8 and using fact that $\sum_{n=1}^{\infty} \zeta_n = \infty$, we conclude that $\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} [D_f(p, x_n) + D_g(q, y_n)] = 0$ and then $\lim_{n\to\infty} D_f(p, x_n) = 0$ and $\lim_{n\to\infty} D_g(q, y_n) = 0$. Thus, by Lemma 2.5 we obtain $\lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} y_n = q$. Therefore, the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to $(p,q) = P^h_\Omega(w,u)$ and this completes the proof. □

If, in Theorem 3.1, we assume that T_i and G_i, $i = 1, 2, \ldots, N$, are single valued quasi-Bregman nonexpansive mappings such that $I - T_i$ and $I - G_i$ are demiclosed at zero, respectively, then we obtain the following theorem.

Theorem 3.2. *Suppose that assumption (A1), (A2) and (A4)- (A10) are satisfied. If* $T_i : C \to C$ *and* $G_i : D \to D$, $i = 1, 2, \dots, N$, *are single valued quasi-Bregman nonexpansive mappings such that* $I - T_i$ *and* $I - G_i$ *are demiclosed at zero, respectively, then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm 3.1 converges strongly to* (p, q) *in* Ω, *where* $(p, q) = P^{\psi}_{\Omega}(w, u)$, *where* $\psi : E_1 \times E_2 \to \mathbb{R}$ *is given by* $\psi(x, y) = f(x) + g(y)$.

Let E_i, $i = 1, 2, 3$ be strictly convex and smooth Banach spaces with their respective, duals E_i^*, $i = 1, 2, 3$. If $g(x) = f(x) = \frac{1}{2}\|x\|^2$, then $\nabla f = \nabla g = J$, $\nabla f^* = \nabla g^* = J^{-1}$, $P^f_\Omega = \Pi_\Omega = P^g_\Omega$ and the quasi-Bregman nonexpansive mappings T_i and G_i reduce to quasi-Ψ-nonexpansive mappings. Thus, we obtain the following theorem.

Theorem 3.3. *Let* E_i, $i = 1, 2, 3$ *be strictly convex and smooth reflexive real Banach spaces with their respective, duals* E_i^*, $i = 1, 2, 3$ *and let* $T_j : C \to CB(C)$ *and* $G_j : D \to CB(D)$, $i = 1, 2, \dots, N$ *be quasi-Ψ-nonexpansive mappings with* $T_i(p) = \{p\}, \forall p \in F(T_i)$ *and* $G_i(p) = \{p\}, \forall p \in F(G_i)$ *such that* $I - T_i$ *and* $I - G_i$ *are demiclosed at zero, respectively. If the assumptions (A4)- (A6) and (A8)-(A10) are satisfied and* $\Omega = \{(p, q) \in [\bigcap_{j=1}^{N}(GMEP(H_{1,j}) \cap F(T_j))] \times [\bigcap_{j=1}^{N}(GMEP(H_{2,j}) \cap F(G_j))] : S(p) = K(q)\} \neq \emptyset$, *then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm 3.1 with* $\nabla f = J = \nabla g$, $\nabla f^* = J^{-1} = \nabla g^*$ *and* $P^f_\Omega = \Pi_\Omega = P^g_\Omega$ *converges strongly to* (p, q) *in* Ω, *where* $(p, q) = \Pi_\Omega(w, u)$.

4. APPLICATION

4.1. The Split Equality of Monotone Inclusion and Generalized Mixed Equilibrium Problems.

Definition 4.1. *The Split Equality of Monotone Inclusion and Generalized Mixed Equilibrium Problems is defined as finding a point* $(p, q) \in E_1 \times E_2$ *such that* $S(p) = K(q)$ *and*

$$(p,q) \in [\bigcap_{j=1}^{N}(GMEP(H_{1,j}) \cap (\mathfrak{A}_j + \mathcal{A}_j)^{-1}(0))] \times [\bigcap_{j=1}^{N}(GMEP(H_{2,j}) \cap (\mathfrak{B}_j + \mathcal{B}_j)^{-1}(0))], \tag{4.1}$$

where $\mathcal{A}_j : E_1 \to 2^{E_1^*}$ *and* $\mathcal{B}_j : E_2 \to 2^{E_2^*}$ *are maximal monotone mappings,* $\mathfrak{A}_j : E_1 \to E_1^*$ *and* $\mathfrak{B}_j : E_2 \to E_2^*$ *are monotone mappings and* $S : E_1 \to E_3$ *and* $K : E_2 \to E_3$ *are bounded linear mappings with adjoints* $S^* : E_3^* \to E_1^*$ *and* $K^* : E_3^* \to E_2^*$, *respectively. Denote* $\Gamma = \{(p, q) \in [\bigcap_{j=1}^{N}(GMEP(H_{1,j}) \cap (\mathfrak{A}_j + \mathcal{A}_j)^{-1}(0))] \times [\bigcap_{j=1}^{N}(GMEP(H_{2,j}) \cap (\mathfrak{B}_j + \mathcal{B}_j)^{-1}(0))] : S(p) = K(q)\}$.

Let $\mathcal{A} : E \to 2^{E^*}$ and $\mathfrak{A} : E \to E^*$ be mappings. Then the resolvent associated with $\mathcal{A}$ and λ for any $\lambda > 0$ is the mapping $J^{\mathcal{A}}_\lambda : E \to E$ defined by $J^{\mathcal{A}}_\lambda(x) = (\nabla f + \lambda\mathcal{A})^{-1}\nabla f(x), \forall x \in E$, where $f : E \to (-\infty, +\infty]$ is Gâteaux differentiable convex function. We note that if $\mathcal{A} : E \to 2^{E^*}$ and $\mathfrak{A} : E \to E^*$ are maximal monotone and monotone mappings, respectively, then a mapping $T^f_{\mathfrak{A}+\mathcal{A}} := J^{\mathcal{A}}_\lambda(\nabla f - \lambda\mathfrak{A})$ is single valued quasi-Bregman nonexpansive mapping and $F(T^f_{\mathfrak{A}+\mathcal{A}}) = (\mathfrak{A} + \mathcal{A})^{-1}(0)$ (See, for example, [21]). The following theorem approximates the solution of split equality of monotone inclusion and generalized mixed equilibrium problem given in (4.1).

Theorem 4.1. *Assume that conditions (A1),(A2), (A4)-(A6) and (A8)-(A10) are satisfied and* $\Gamma \neq \emptyset$. *If* $T_j = T^f_{\mathfrak{A}_j+\mathcal{A}_j}$ *and* $G_j = T^g_{\mathfrak{B}_j+\mathcal{B}_j}$, *then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm 3.1 converges strongly to* (p, q) *in* Λ, *where* $(p, q) = P^h_\Gamma(w, u)$.

4.2. **The Split Equality of Multi-Objective Constrained Optimization Problem.** Let $p_{i,j} : C \to \mathbb{R}$ be convex smooth functions and $q_{i,j} : D \to \mathbb{R}$ be convex, lower semicontinuous functions, $i = 1, 2$, $j = 1, 2, \dots, N$. We consider the following minimization problem: Find $(p, q) \in C \times D$ such that

$$\begin{aligned} (p,q) \text{ solves } \min_{(x,y)\in C\times D} \quad & (p_{1,j}+q_{1,j})(x) + (p_{2,j}+q_{2,j})(y), j = 1,2,\dots,N \\ \text{s.t.} \quad & (x,y) \in \bigcap_{j=1}^{N} GMEP(H_{1,j}) \times GMEP(H_{2,j}); \\ & Sx = Ky, \end{aligned} \tag{4.2}$$

where $S : E_1 \to E_3$ and $K : E_2 \to E_3$ are bounded linear mappings. Denote $\Delta = \{(z, v) \in C \times D : (z, v) \text{ solve } 4.2\}$. By Fermat's rule, the above minimization problem is equivalent to the problem of finding $(p, q) \in C \times D$ such that $S(p) = K(q)$ and

(4.3)

$$(p,q) \in [\bigcap_{j=1}^{N}(GMEP(H_{1,j})\cap(\nabla p_{1,j}+\partial q_{1,j})^{-1}(0))]\times[\bigcap_{j=1}^{N}(GMEP(H_{2,j})\cap(\nabla p_{2,j}+\partial q_{2,j})^{-1}(0))],$$

where $\nabla p_{i,j}$ are gradient of $f_{i,j}$ and $\partial q_{i,j}$ are subdifferential of $q_{i,j}$, $i = 1, 2$, $j = 1, 2, \dots, N$. Note that $\nabla p_{i,j}$ and $\partial q_{i,j}$ are monotone and maximal monotone mappings, respectively.

Corollary 4.1. *Let $p_{i,j} : E_i \to \mathbb{R}$ be convex smooth functions and $q_{i,j} : E_i \to \mathbb{R}$ be convex, lower semicontinuous functions, $i = 1, 2$, $j = 1, 2, \dots, N$. Assume that conditions (A1),(A2), (A4)-(A6) and (A8)-(A10) are satisfied. If $\Delta \neq \emptyset$ and $T_j = T^{f}_{\nabla p_{1,j}+\partial q_{1,j}}$ and $G_j = T^{g}_{\nabla p_{2,j}+\partial q_{2,j}}$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 3.1 converges strongly to (p, q) in Δ, where $(p, q) = P^h_\Delta(w, u)$, where $h = f_1 + f_2$.*

4.3. **The Multiple-Sets Split Equality Feasibility Problems.** Given nonempty, closed and convex sets C_j and Q_j, $j = 1, 2, \dots, N$, in E_1 and E_2, respectively. The multiple-sets split equality feasibility problems is defined as to find p and q for which

$$p \in \bigcap_{j=1}^{N} C_j \text{ and } q \in \bigcap_{j=1}^{N} Q_j \text{ such that } Sp = Kq, \tag{4.4}$$

where $S : E_1 \to E_3$ and $K : E_2 \to E_3$ are bounded linear mappings. Denote $\Upsilon = \{(u, v) \in E_1 \times E_2 : (z, v) \text{ solve } 4.4\}$. Recall that the indicator functions i_{C_j} and i_{Q_j} of C_j and Q_j given by

$$i_{C_j}(x) = \begin{cases} 0 & \text{if } x \in C_j \\ \infty & \text{if } x \notin C_j \end{cases} \quad \text{and } i_{Q_j}(x) = \begin{cases} 0 & \text{if } x \in Q_j \\ \infty & \text{if } x \notin Q_j \end{cases},$$

respectively. It is known that i_{C_j} and i_{Q_j} are proper convex, lower semicontinuous and convex functions with its subdifferential ∂i_{C_j} and ∂i_{Q_j} are maximal monotones.

In fact, we set $T_j(x) := J_\lambda^{\partial i_{C_j}}(x)$ and $G_j(y) := J_\lambda^{\partial i_{Q_j}}(y)$ are quasi-Bregman nonexpansive mapping, $F(T_j) = C_j$ and $F(G_j) = Q_j$ and $H_{i,j} = 0$, $i = 1, 2$, $j = 1, 2, \dots, N$. So we obtain the following result.

Corollary 4.2. *Assume that conditions (A1),(A2), (A6) and (A8)-(A10) are satisfied. If $\Upsilon \neq \emptyset$, $T_j := J_\lambda^{\partial i_{C_j}}$, $G_j := J_\lambda^{\partial i_{Q_j}}$ and $H_{i,j} = 0$, $i = 1, 2$, $j = 1, 2, \dots, N$, then the sequence $\{(x_n, w_n)\}$ generated by Algorithm 3.1 converges strongly to (p, q) in Υ, where $(p, q) = P^h_\Upsilon(w, u)$, where $h = f_1 + f_2$.*

5. NUMERICAL EXAMPLES

Let $C = [0,1]$ and $D = [-1,0]$ be nonempty, closed and convex subsets of $E_i = \mathbb{R}$ with Euclidean norm, where $i = 1,2$, respectively and $E = \mathbb{R}$. Let $f : E_1 \to \mathbb{R}$ and $g : E_2 \to \mathbb{R}$ be defined by $f(x) = \frac{1}{2}\|x\|^2, \forall x \in E_1$ and $g(y) = \frac{1}{2}\|y\|^2, \forall y \in E_2$, respectively. Then f and g are strongly coercive, lower semi-continuous, strongly convex, bounded and uniformly Fréchet differentiable Legendre functions on bounded subsets with strongly convex conjugates f^* and g^*, respectively. Define $T_k : C \to CB(C)$, $k = 1,2,\ldots,N$ by $T_k(x) = \left[0, \frac{1}{2k}x\right]$, for all $x \in C$ and $G_k : D \to CB(D), k = 1,2,\ldots,N$ by $G_k(y) = \left[\frac{1}{3k}y, 0\right]$, for all $y \in D$. In this particular case, $F(T_k) = \{0\}$ and $F(G_k) = \{0\}$, for $k = 1,2,...,N$. Then for each $x \in C$, we have $d(0, T_k x) = 0$. Therefore,

$$\sup_{a \in T_k 0} d(a, T_k x) \leq \frac{1}{4k^2}\|0 - x\|^2 \leq D_f(0,x).$$

It follows from a similar argument that

$$\sup_{b \in T_k x} d(b, T_k p) \leq \frac{1}{4k^2}\|0 - x\|^2 \leq D_f(0,x).$$

Therefore, for every $p \in Fix(T_k)$ and for every $x \in C$ we have

$$H(T_k 0, T_k x) \leq D_f(0,x),$$

so that T_k is a quasi-Bregman nonexpansive mapping. Let $\{x_n\}$ be a sequence in C such that $x_n \to x$ and $d(x_n, T_k x_n) \to 0$, where $d(x_n, T_k x_n) = \inf_{y \in T_k x_n} D_f(x_n, y) = \inf_{y \in T_k x_n} \|x_n - y\|^2$. Thus, we that

$$\begin{aligned} d(x_n, T_k x_n) &= \inf_{y \in T_k x_n} D_f(x_n, y) = \inf_{0 \leq y \leq \frac{1}{2k} x_n} \|x_n - y\|^2 \\ &= \|x_n - \frac{1}{2k} x_n\|^2 = \frac{(2k-1)^2}{4k^2} x_n^2. \end{aligned}$$

Since $d(x_n, T_k x_n) \to 0$, we have $x_n \to 0$. So that $x = 0$; and hence $x \in T_k x$. Therefore, $I - T_k$ is demiclosed at zero. Similar arguments lead us to the conclusion that for each $k = 1,2,\cdots,N$, G_k is a quasi-Bregman nonexpansive and $I - G_k$ is demiclosed at zero.

We define $F_{1,i} : C \times C \to \mathbb{R}$ and $F_{2,i} : D \times D \to \mathbb{R}$ by $F_{1,i}(x,y) = \frac{1}{2i+1} x\,(y - x)$, $\forall x, y \in C$ and $F_{2,i}(u,v) = \frac{1}{3i+2} u\,(v - u)$, $\forall u, v \in D$, respectively, $B_{1,i} : C \to E_1^*$ and $B_{2,i} : D \to E_2^*$ by $B_{1,i}(x) = \frac{1}{4i+3}x, \forall x \in C$ and $B_{2,i}(u) = \frac{2}{5i+1}u, \forall u \in D$ respectively, and $\varphi_{1,i} : C \to \mathbb{R}$ and $\varphi_{2,i} : D \to \mathbb{R}$ by $\varphi_{1,i}(x) = 2i + 3, \forall x \in C$ and $\varphi_{2,i}(u) = 5i - 1, \forall u \in D$, respectively, $i = 1,2,\ldots,N$. Then $B_{1,i}$, $B_{2,i}$, for $i = 1,\ldots,N$ are monotone mappings, and $F_{1,i}$, $F_{2,i}$, for $i = 1,\ldots,N$ are bi-function satisfying Condition A. Thus, a common solution set of the generalized mixed equilibrium problems is $\bigcap_{i=1}^N GMEP(F_{1,i}, \varphi_{1,i}, B_{1,i}) = \{0\} = \bigcap_{i=1}^N GMEP(F_{2,i}, \varphi_{2,i}, B_{2,i})$. Thus, we get $\Omega = \{(p,q) \in [\bigcap_{j=1}^N (GMEP(H_{1,j}) \cap F(T_j))] \times [\bigcap_{j=1}^N (GMEP(H_{2,j}) \cap F(G_j))] : S(p) = K(q)\} = \{(0,0)\}$, where $S(x) = -x, \forall x \in C$ and $K(u) = u, \forall u \in D$ are bounded linear mappings. With these mappings, Algorithm 3.1 reduces to Algorithm 5.1.

Algorithm 5.1

Initialization: Choose $(w,u), (x_0,y_0), (x_1,y_1) \in C \times D$, $0 < \sigma, \rho$. Define the algorithm as follows:

Step 0: Compute

$$\overline{\sigma}_n = \begin{cases} \min\left\{\frac{\chi_n}{|x_n - x_{n-1}| + |y_n - y_{n-1}|}, \sigma\right\} \text{ if } x_n \neq x_{n-1} \& y_n \neq y_{n-1} \\ \sigma \text{ otherwise }, \end{cases} \tag{5.1}$$

Step 1: Choose:

$$\sigma_n = \overline{\sigma}_n - \frac{n}{n+1}|\sigma - \overline{\sigma}_n|$$

Step 2: Compute

$$\begin{aligned} a_n &= x_n + \sigma_n(x_n - x_{n-1}), \\ b_n &= y_n + \sigma_n(y_n - y_{n-1}). \end{aligned} \tag{5.2}$$

Step 3: Compute

$$\rho_n = \min\left\{\rho + 1, \frac{\mu|a_n + b_n|^2}{2|a_n - b_n|^2 + 2|a_n + b_n|^2}\right\}, \tag{5.3}$$

Step 4: Choose:

$$\gamma_n = \rho_n - \frac{n}{n+1}|\rho - \rho_n|$$

Step 5: Compute

$$\begin{aligned} d_n &= a_n - \gamma_n(a_n + b_n), \\ e_n &= b_n - \gamma_n(b_n + a_n), \end{aligned} \tag{5.4}$$

Step 6: Compute

$$\begin{aligned} h_n &= \sum_{i=1}^{N} \beta_{1,i} \frac{8i^2 + 10i + 3}{8i^2 + (6r_n + 10)i + 3(r_n + 1)} d_n, \\ z_n &= \sum_{i=1}^{N} \beta_{2,i} \frac{15i^2 + 13i + 2}{15i^2 + (11r_n + 13)i + (5r_n + 2)} e_n, \end{aligned} \tag{5.5}$$

Step 7: Compute

$$u_n = \begin{cases} \min\{-\lambda_n w - (1-\lambda_n)h_n, -1 + \lambda_n w + (1-\lambda_n)h_n\} \text{ if } \lambda_n w + (1-\lambda_n)h_n \notin [0,1] \\ \lambda_n w + (1-\lambda_n)h_n \text{ otherwise} \end{cases}$$

$$v_n = \begin{cases} \min\{-1 - \lambda_n u - (1-\lambda_n)z_n, \lambda_n u + (1-\lambda_n)z_n\} \text{ if } \lambda_n u + (1-\lambda_n)z_n \notin [-1,0] \\ \lambda_n u + (1-\lambda_n)z_n \text{ otherwise} \end{cases}$$

Step 8: Choose:

$$m_{i,n} = \frac{1}{2i(n^2+1)}|u_n| \tag{5.6}$$

$$l_{i,n} = -\frac{1}{3i(n^2+1)}|v_n| \tag{5.7}$$

Step 9: Compute

$$\begin{aligned} x_{n+1} &= \alpha_{0,n} h_n + \sum_{i=1}^{N} \alpha_{i,n} m_{i,n}, \\ y_{n+1} &= \alpha_{0,n} z_n + \sum_{i=1}^{N} \alpha_{i,n} l_{i,n}. \end{aligned} \tag{5.8}$$

Set $n := n + 1$ and go to **Step 0**.

We have the following values for 200 iterations corresponding to the initialization parameters chosen below:
$\mu = 1$, $u = -0.8$, $w = 0.5$, $x_0 = -0.5$, $y_0 = 0.5$, $x_1 = 0.5$, $y_1 = -0.75$, $\sigma = 0.25$, $\rho = 1$, $N = 10$, $\lambda_n = \frac{1}{n}$, $\chi_n = \frac{1}{2^n}$, $n = 1, 2, 3, ...$ $\beta_{1,i} = \frac{1}{N} = \beta_{2,i}$ for each $i = 1, \cdots N$; $\alpha_{i,n} = \frac{1}{N+1}$ for $i = 0, 1, \cdots N$ and $n = 1, 2, \cdots$, $r_n = 1$ for $n = 1, 2, ...$

TABLE 1. Table of Numerical Results For Example

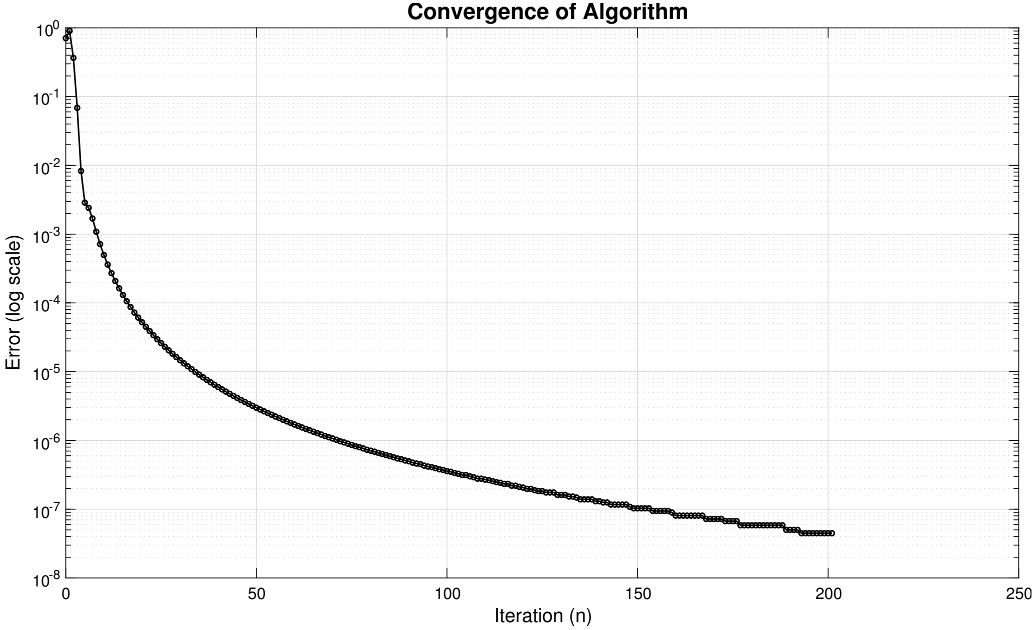
N	(x_n,y_n)	Error(E_n)
0	(- 0.5, 0.5)	0.707106781
1	(0.5, - 0.75)	0.901387819
2	(0.23818885, - 0.27781387)	0.365943267
10	(0.00031265, - 0.00018042)	0.000360973
20	(0.00003896, - 0.00002253)	4.50054×10^{-05}
30	(0.00001144, - 0.00000663)	1.32223×10^{-05}
40	(0.00000477, - 0.00000278)	5.52099×10^{-06}
50	(0.00000243, - 0.00000142)	2.81448×10^{-06}
60	(0.0000014, - 0.00000082)	1.62247×10^{-06}
70	(0.00000088, - 0.00000052)	1.02215×10^{-06}
80	(0.00000059, - 0.00000035)	6.86003×10^{-07}
90	(0.00000041, - 0.00000024)	4.75079×10^{-07}
100	(0.0000003, - 0.00000018)	3.49857×10^{-07}
110	(0.00000023, - 0.00000013)	2.64197×10^{-07}
120	(0.00000017, - 0.0000001)	1.97231×10^{-07}
130	(0.00000014, - 0.00000008)	1.61245×10^{-07}
140	(0.00000011, - 0.00000006)	1.253×10^{-07}
150	(0.00000009, - 0.00000005)	1.02956×10^{-07}
160	(0.00000007, - 0.00000004)	$.06226 \times 10^{-08}$
170	(0.00000006, - 0.00000004)	7.2111×10^{-08}
180	(0.00000005, - 0.00000003)	5.83095×10^{-08}
190	(0.00000004, - 0.00000003)	0.00000005
200	(0.00000004, - 0.00000002)	4.47214×10^{-08}

Remark 5.1. *From Table 1 and Figure 1 above, we observe that the inertial method for converges to the solution of the split equality of the generalized mixed equilibrium and fixed point of multi-valued quasi-Bregman nonexpansive mapping problem.*

6. CONCLUSIONS

In this paper, we introduced and studied an inertial method for approximating the solution of the split equality of the generalized mixed equilibrium and fixed point of multi-valued quasi-Bregman nonexpansive mapping problems. We proved a strong convergence theorem for the developed algorithm in reflexive real Banach spaces. In addition, some applications of the proposed method to the split equality of monotone inclusion and generalized mixed equilibrium problems, the split equality of multi-Objective constrained optimization problem, the multiple-sets split equality feasibility problems and

FIGURE 1. Convergence of Algorithms 5.1. Number of iteration VS Error



the multiple-sets split equality feasibility problems are provided. Moreover, A numerical example is also provided to illustrate the behavior of the proposed algorithm. The results obtained in this paper extend, unify and complement many of the results in the literature. For instance, the results in this paper enhances and generalizes the works of Zegeye and Shahzad [30], Shahzad and Zegeye [27], Alghamdia et al. [1], Nnakwe [20], in the sense that the results in this paper are valid for the class of multi-valued quasi-Bregman nonexpansive mappings more general than quasi-ϕ-nonexpansive mappings in real reflexive Banach spaces more general than 2-uniformly convex and uniformly smooth real Banach spaces.

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