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A Weighted Monotone Iterative Approach to Periodic Impulsive Fractional Evolution Equations

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ABSTRACT. This work is devoted to the study of periodic boundary value problems associated with impulsive Riemann–Liouville fractional evolution equations in ordered Banach spaces. Employing the framework of lower and upper solutions together with a monotone iterative technique, we derive conditions for the existence of minimal and maximal mild solutions inside a suitable ordered interval. The analysis relies on semigroup methods, fractional calculus tools, and measure of noncompactness arguments. Furthermore, sufficient criteria guaranteeing the convergence of the constructed iterative sequences as well as the uniqueness of solutions are obtained. To support the theoretical developments, an illustrative example is included to show the effectiveness and applicability of the proposed approach.

1. INTRODUCTION

Impulsive fractional differential equations play a significant role in the mathematical modeling of systems involving memory effects and sudden state transitions. Such models arise naturally in various applied contexts, including biological processes, engineering systems, and other phenomena characterized by abrupt transitions (see [5, 22]). In particular, fractional equations with impulses at fixed moments have been investigated in a variety of settings, reflecting their relevance in modeling discontinuous dynamics such as harvesting processes, disease control, and population evolution.

The study of nonlinear impulsive fractional differential equations constitutes an important and developing branch of fractional calculus, supported by both theoretical interest and practical applications. These models provide a more accurate framework for describing complex systems where classical integer-order equations are insufficient (see [6]). Furthermore, the work of Heymans and Podlubny [24] established that initial conditions formulated via Riemann–Liouville fractional operators admit meaningful physical interpretations, particularly in viscoelasticity, thereby enhancing the applicability of such models.

Periodic and antiperiodic boundary value problems for fractional differential equations have also received significant attention. Various analytical techniques, including fixed point methods and approaches based on measures of noncompactness, have been employed to investigate the existence of solutions in these problems (see [13, 14]).

Among the available techniques, the method of lower and upper solutions has proven to be a powerful tool for establishing existence results in both initial and boundary value problems (see [12]). When combined with suitable monotonicity assumptions, this approach leads to the monotone iterative technique, which generates sequences converging

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to extremal solutions within a prescribed order interval (see [10, 11]). This iterative framework is particularly useful due to its constructive nature and its potential applicability in numerical approximations.

Overall, the monotone iterative technique based on lower and upper solutions provides an effective and flexible approach for analyzing nonlinear fractional problems, yielding monotone sequences that converge to minimal and maximal solutions under appropriate conditions (see [7, 10]).

In [10, 38], discussed the monotone iterative method for the for the following initial value problem involving Riemann-Liouville fractional derivative

$$\begin{cases} {}^{L}D^{\alpha}u(t) = f(t,u), \quad t \in (0,T]; \\ t^{1-\alpha}u(t)|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $0 < T < \infty$, and ${}^{L}D^{\alpha}$ is Riemann-Liouville fractional derivative of order $0 < \alpha < 1$. Moreover, the semigroup theoretical to find the existence and uniqueness of mild solution to such problems has been introduced in this paper is concerned the results for the following weighted fractional semilinear evolution equations in Banach spaces

$$\begin{cases} {}^{RL}D^{q}_{0^+}u(t) + Au(t) = f(t,u(t),Gu(t)), \quad t \in I' := (0,b], \\ \left(I^{1-q}_{0^+}u\right)(t) = u_0, \end{cases} \tag{1.2}$$

where ${}^{RL}D^{q}$ is the Riemann-Liouville fractional derivative of order q, $0 < q < 1$, with lower limit zero, $I^{1-q}_{0^+}$ is Riemann-Liouville integral of order $1-q$ and $-A$ is the infinitesimal generator of semigroup in Banach space. The operator G is given by

$$Gu(t) = \int_0^t K(t,s)u(s)ds,$$

is a Volterra integral operator with integral kernel $K \in C(\Delta, \mathbb{R}^+)$, $\Delta = \{(t,s) : 0 \leq s \leq t \leq b\}$.Motivated by the developments discussed above, we consider the existence of solutions for a class of periodic boundary value problems involving semilinear impulsive Riemann–Liouville fractional differential equations of the form

$$\begin{cases} {}^{RL}D^{\alpha}_{t}u(t) + Au(t) = f(t,u(t),Hu(t)), \quad t \in J := (0,\omega],\ t \neq t_k; \\ \Delta I^{1-\alpha}_{t_k}u(t_k) = G_k(u(t_k)), \quad k = 1,2,\dots,m; \\ \widetilde{u}(0) = \widetilde{u}(\omega), \end{cases} \tag{1.3}$$

Here, ${}^{RL}D^{\alpha}_{t}$ denotes the Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$. The operator A is assumed to be a closed and densely defined linear operator such that $-A$ generates a strongly continuous semigroup $\{\mathcal{U}(t)\}_{t\geq 0}$ of bounded linear operators on a Banach space E. Moreover, there exists a constant $M \geq 1$ satisfying $\sup\limits_{t\in[0,\omega]} \|\mathcal{U}(t)\| \leq M$. The nonlinear mappings $f : [0,\omega] \times E \to E$ and $G_k : E \to E$ will be specified later. The constant $\omega > 0$ represents the period, and the points $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = \omega$ denote the moments at which impulses occur. The operator H is defined by

$$Hu(t) = \int_0^t \mathcal{K}(t,s)u(s)\,ds,\ \mathcal{K} \in C(\mathcal{D},\mathbb{R}), \tag{1.4}$$

where $\mathcal{D} = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le \omega\}$. The term $\Delta I_{t_k}^{1-\alpha} u(t_k)$ describes the jump of the function at the impulsive point $t = t_k$, defined by

$$\Delta I_{t_k}^{1-\alpha} u(t_k) = I_{t_k^+}^{1-\alpha} u(t_k^+) - I_{t_k^-}^{1-\alpha} u(t_k^-) = \Gamma(\alpha) \left[\lim_{t \to t_k^+} (t - t_k)^{1-\alpha} u(t) - \lim_{t \to t_k^-} (t - t_k)^{1-\alpha} u(t) \right].$$

Here, $I_{t_k^+}^{1-\alpha} u(t_k^+)$ and $I_{t_k^-}^{1-\alpha} u(t_k^-)$ denote the right-hand and left-hand limits of $I_t^{1-\alpha} u(t)$ at $t = t_k$, respectively. In addition, we use the notation $\widetilde{u}(t) = t^{1-\alpha} u(t)$.

The analysis of periodic impulsive fractional evolution equations is considerably more intricate than that of classical differential systems because of the interaction between fractional dynamics, impulsive discontinuities, and integral operators. The nonlocal nature of the Riemann–Liouville fractional derivative introduces hereditary characteristics into the model, making the construction of suitable solution spaces and the verification of compactness conditions technically demanding. In addition, the impulsive terms generate discontinuities at prescribed moments, which must be incorporated carefully within the fractional setting while maintaining the periodic behavior of solutions.

In this work, we investigate a class of impulsive fractional evolution equations with periodic boundary conditions in an ordered Banach space. The main purpose is to establish the existence and uniqueness of extremal mild solutions by combining lower and upper solution techniques with a monotone iterative approach and semigroup methods. Through this procedure, we generate monotone sequences that converge to the smallest and largest mild solutions of the problem. Compared with methods based exclusively on fixed point arguments, the present approach provides an explicit and constructive approximation process for the solutions.

A further challenge arises from the possible noncompactness of the corresponding solution operator. To deal with this issue, we apply methods related to measures of noncompactness, which permit the derivation of existence results without imposing strong compactness assumptions. In addition, suitable analytical estimates together with an extended form of Gronwall's inequality are employed to prove the convergence and stability of the iterative sequences.

The results obtained here extend several earlier studies in fractional differential equations by simultaneously considering periodic conditions, impulsive phenomena, and integral terms within a single theoretical framework. Hence, the paper contributes to the development of constructive analytical techniques for a broader class of fractional evolution problems.

The paper is arranged as follows. Section 2 contains the preliminary material, including fundamental definitions, notation, and auxiliary results needed in the sequel. In Section 3, we establish the main existence and uniqueness results for extremal mild solutions associated with system (2.7). The final section presents a concrete example illustrating the applicability of the theoretical findings.

2. Preliminaries

Let E be an ordered Banach space with the norm $\|\cdot\|_E$. Define a partial order $\le$, whose positive cone $P = \{u \in E, u \ge \theta\}$ (θ are the zero element of E) is normal with normal constant N. Let $C([0,\omega], E)$ be the spaces of E-valued continuous functions on $[0,\omega]$, respectively, endowed with the uniform norm topology

$$\|u\|_\infty = \sup\{\|u(t)\|, t \in [0,\omega]\}.$$

Set $L^1([0,\omega],E)$ denote the Banach space of functions, $u:[0,\omega]\to E$ which are Bochner integrable and normed by

$$\|u\|_{L^1}=\int_0^\omega \|u(t)\|dt.$$

Let $C_{1-\alpha}([0,\omega],E)=\left\{u(\cdot)\in C((0,\omega],E):\lim_{t\to 0^+} t^{1-\alpha}u(t)\ \text{ exists }\right\}$, with the norm is given by

$$\|u\|_\alpha=\sup_{t\in[0,\omega]}\left\{t^{1-\alpha}\|u\|_E\right\},$$

it easy to see $(C_{1-\alpha}([0,\omega],E),\|x\|_\alpha)$ is a Banach space. For investigation of impulsive conditions, consider the piecewise continuous Banach space $AC_{1-\alpha}([0,\omega],E)=\{u(\cdot)\in C((t_k,t_{k+1}],E)$ and $\lim_{t\to t_k^+}(t-t_k)^{1-\alpha}u(t)$ exists, $k=0,1,2,\dots,m\}$, with the norm

$$\|u\|_{AC_{1-\alpha}}=\max\left\{\sup_{t\in[t_k,t_{k+1}]}(t-t_k)^{1-\alpha}\|u\|_E : k=1,2,\dots,m\right\}.$$

Definition 2.1. [31] *Let E be an ordered Banach space with zero element θ. A cone $P\subset E$ is called normal if there exists a real number $N>0$, such that for all $u,v\in E$*

$$\theta\le u\le v\Rightarrow \|u\|_E\le N\|v\|_E.$$

The smallest positive number N satisfying the above condition is called the normal constant of P.

Definition 2.2. [31] *Let E be an ordered Banach space. A cone $P\subset E$ is called regular if every increasing sequence which is bounded from above is convergent i.e if $\{u_n\}$ be a sequence such that*

$$u_1\le u_2\le\cdots\le u_n\le\cdots\le v.$$

For some $v\in E$, then there is $u\in E$ with $\|u_n-u\|\to 0$ as $n\to\infty$. Equivalently , a cone $P\subset E$ is called regular if every decreasing sequence which is bounded from below is convergent. Clearly, a regular cone is a normal cone.

Definition 2.3. [25] *We collect some notations from the fractional analysis, let us recall the following definitions from fractional calculus. We lend some concepts of Riemann-Liouville fractional calculus. Let, $0<\alpha<1$. If $x\in L^1([0,\omega],E)$, then*

$$\left(I^\alpha u\right)(t):=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds,$$

exists a.e. on $[0,\omega]$ and it is called Riemann-Liouville fractional integral of order $\alpha>0$. Here Γ denotes the Gamma function. So, $u\in L^1([0,\omega],E)$ is such that $t\to I^{1-\alpha}u(t)$ is differentiable a.e. on $[0,\omega]$, then

$$\left({}^{RL}D^\alpha u\right)(t):=\frac{d}{dt}\left(I^{1-\alpha}u\right)(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-s)^{-\alpha}u(s)ds,$$

and exists a.e. on $[0,\omega]$ and it is called the Riemann-Liouville fractional derivative of order α. The previous integral is taken in Bochner sense. Let $\Psi_\alpha(t):\mathbb{R}\to\mathbb{R}$, defined by

$$\Psi_\alpha(t)=\begin{cases}\dfrac{t^{1-\alpha}}{\Gamma(\alpha)}, & \text{if } t>0,\\[2mm] 0, & \text{if } t\le 0.\end{cases}$$

Then

$$I^\alpha u(t)=(\Psi_\alpha * u)(t)\ \ \text{and}\ \ {}^{RL}D^\alpha u(t)=\frac{d}{dt}(\Psi_{1-\alpha}*u)(t).$$

Here $\gamma(\cdot) * \beta(\cdot)$ *is convolution, i.e.* $\int_0^t \gamma(t-s)\beta(s)ds$.

The following lemma is a variant of the classical Arzela-Ascoli theorem. For Ω a subset of the space $C_{1-\alpha}([0,\omega], E)$, define Ω_α by $\Omega_\alpha = \{u_\alpha : u \in \Omega\}$ and

$$u(t) = \begin{cases} t^{1-\alpha}u(t), & \text{if } t \in (0,\omega], \\ \lim\limits_{t\to 0^+} t^{1-\alpha}u(t); & \text{if } t = 0. \end{cases}$$

It is clear that $u_\alpha(\cdot) \in C([0,\omega], E)$.

Lemma 2.1. [10] *A set* $\Omega \subset C_{1-\alpha}([0,\omega], E)$ *is relatively compact if and only if* Ω_α *is relatively compact in* $C([0,\omega], E)$.

Lemma 2.2. *Let* $\alpha \in (0,1)$. *If* $u \in AC_{1-\alpha}([0,\omega], E)$ *and* $I_t^{1-\alpha}u(t) \in C([0,\omega], E)$, *then*

$$I_t^{\alpha}\, {}^{RL}D_t^{\alpha}u(t) = \begin{cases} u(t) - I_t^{1-\alpha}u(t)|_{t=0}\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t \in (0,t_1]; \\ u(t) - \sum\limits_{i=1}^{k} \dfrac{\Delta I_t^{1-\alpha}u(t_k)}{\Gamma}(t-t_j)^{\alpha-1} - I_t^{1-\alpha}u(t)|_{t=0}\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}, & t \in (t_k, t_k+1], \end{cases}$$

where $\Delta I_t^{1-\alpha}u(t_k) = I_t^{1-\alpha}u(t_k^+) - I_t^{1-\alpha}u(t_k^-), k = 1, 2, \cdots, m$.

It, is well known that for $u_0 \in E$ and $h \in L^1([0,\omega], E)$, the initial value problem of the linear evolution fractional differential equation order $\alpha \in (0,1)$

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}u(t) + Au(t) = h(t), & t \in (0,\omega], \\ I_{0^+}^{1-\alpha}u(t)|_{t=0} = u_0 \end{cases} \tag{2.5}$$

has a unique mild solution $u \in C_{1-\alpha}([0,\omega], E)$ given by

$$u(t) = t^{\alpha-1}U_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}U_\alpha(t-s)h(s)ds, \quad t \in (0,\omega], \tag{2.6}$$

where

$$U_\alpha(t) = \alpha\int_0^\infty \Theta\zeta_\alpha(\Theta)\mathcal{U}\left(t^\alpha\Theta\right)d\Theta,$$

$$\zeta_\alpha(\Theta) = \frac{1}{\alpha}\Theta^{-1-\frac{1}{\alpha}}\Upsilon_\alpha\left(\Theta^{-\frac{1}{\alpha}}\right),$$

$$\Upsilon_\alpha(\Theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\Theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\ 0 < \Theta < \infty.$$

ζ_α is a probability density function defined on $(0,\infty)$, i.e., $\zeta_\alpha(\Theta) \geq 0$ and $\int_0^\infty \zeta_\alpha(\Theta)d\Theta = 1$.

Lemma 2.3. *The operator* $\{U_\alpha(t),\ t \geq 0\}$ *is a bounded linear operator such that*

(i) $\|U_\alpha(t)y\| \leq \dfrac{M}{\Gamma(\alpha)}\|y\|$, *for any* $y \in E$.

(ii) *the operator* $\{U_\alpha(t),\ t \geq 0\}$ *is strongly continuous i.e. for every* $y \in E$ *and* $0 < t_1 < t_2 \leq \omega$, *we have*

$$\|U_\alpha(t_2)y - U_\alpha(t_1)y\| \to 0, \quad t_2 \to t_1.$$

(iii) *If* $\mathcal{U}(t)$ *is compact, then* $U_\alpha(t)$ *is also compact operator for every* $t > 0$.

To prove our main result, for any $h \in L^1([0,\omega],E)$, we consider the periodic boundary value problem of linear impulsive evolution fractional differential equation order $\alpha \in (0,1)$, (PBVIF)

$$\begin{cases} {}^{RL}D_t^{\alpha}u(t) + Au(t) = h(t) \quad t \in (0,\omega], t \neq t_k; \\ \Delta I_{t_k}^{1-\alpha}u(t_k) = y_k, \quad k = 1,2,\dots,m; \\ \widetilde{u}(0) = \widetilde{u}(\omega), \end{cases} \tag{2.7}$$

where $y_k \in E,\ k = 1,2,\dots,m$, and $u_0 \in E$.

Lemma 2.4. *Let $\{\mathcal{U}(t)\}_{t\geq 0}$ be C_0-semigroup in E generated $-A$, for any $h \in L^1([0,\omega],E)$, $u_0 \in E$ and $y_k \in E,\ k = 1,2,\dots,m$. Then the linear (PBVIF)* (2.7) *has a unique mild solution $u(\cdot) \in AC_{1-\alpha}([0,\omega],E)$ given by*

$$u(t) = t^{\alpha-1}U_\alpha(t)J(h) + \int_0^t (t-s)^{\alpha-1}U_\alpha(t-s)h(s)ds + \sum_{0<t_k<t}(t-t_k)^{\alpha-1}U_\alpha(t-t_k)y_k,\ t \in (0,\omega], \tag{2.8}$$

where

$$J(h) = (I-U_\alpha(\omega))^{-1}\omega^{1-\alpha}\left[\int_0^\omega (\omega-s)^{\alpha-1}U_\alpha(\omega-s)h(s)ds + \sum_{k=1}^{m}(\omega-t_k)^{\alpha-1}U_\alpha(\omega-t_k)y_k\right].$$

Proof. For any $h \in L^1([0,\omega],E)$, we first show that the initial value problem (IVPF) of linear impulsive evolution equation

$$\begin{cases} {}^{RL}D_{0^+}^{\alpha}u(t) + Au(t) = h(t), \quad t \in (0,\omega],\ t \neq t_k \\ \Delta I_{t_k}^{1-\alpha}|_{t=t_k}u(t) = y_k, \quad k = 1,2,\dots,m, \\ I_{0^+}^{1-\alpha}u(t)|_{t=0} = u_0 \end{cases} \tag{2.9}$$

has unique mild solution $u \in AC_{1-\alpha}([0,\omega],E)$ given by

$$u(t) = t^{\alpha-1}U_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}U_\alpha(t-s)h(s)ds + \sum_{0<t_k<t}(t-t_k)^{\alpha-1}U_\alpha(t-t_k)y_k,\ t \in (0,\omega], \tag{2.10}$$

where $u_0, y_k \in E$, $k = 1,2,\dots,m$.

Let $y_0 = \phi$, denote $I_k = [t_{k-1},t_k]$, $k = 1,2,\dots,m+1$, where $t_0 = 0$ and $t_{m+1} = \omega$. If $u \in AC_{1-\alpha}([0,\omega],E)$ is a mild solution of (IVPF) (2.9). Then the restriction of u on I_k satisfies the initial value problem of linear evolution fractional differential equation without impulse

$$\begin{cases} {}^{RL}D_t^{\alpha}u(t) + Au(t) = h(t), \quad t_{k-1} < t \leq t_k, \\ I_t^{1-\alpha}|_{t=t_{k-1}^+}u(t) = I_t^{1-\alpha}|_{t=t_{k-1}}u(t) + y_{k-1}. \end{cases}$$

Hence, on $(t_{k-1},t_k]$, $u(\cdot)$ can be expressed by

$$u(t) = (t-t_{k-1})^{\alpha-1}U_\alpha(t-t_{k-1})(u(t_{k-1}) + y_{k-1}) + \int_{t_{k-1}}^t (t-s)^{\alpha-1}U_\alpha(t-s)h(s)ds.$$

Iterating successive in the above equality with $u(t_k)$, $i = k-1, k-2, \dots\dots, 1$, we see that it satisfies (2.10). Inversely, we can verify directly that the function $u \in AC_{1-\alpha}([0,\omega],E)$ is a mild solution of (IVPIF) (2.9). Hence (IVPIF) (2.9) has a unique mild solution $u \in AC_{1-\alpha}([0,\omega],E)$ given by (2.10). Next, we show that the linear (PBVIF) (2.7) has unique mild solution $u \in AC_{1-\alpha}([0,\omega],E)$ given by (2.8). If a function $u \in AC_{1-\alpha}([0,\omega],E)$ defined by (2.8) is a solution of the linear (PBVIF) (2.7), then $u_0 = u(\omega)$, and

$$(I - U_\alpha(\omega))u_0 = \omega^{1-\alpha}\left[\int_0^\omega (\omega-s)^{\alpha-1}U_\alpha(\omega-s)h(s)ds + \sum_{k=1}^{m}(t-t_k)^{\alpha-1}U_\alpha(\omega-t_k)y_k\right].$$

It follows that $I - U_\alpha(\omega)$ has a bounded inverse operator $(I-U_\alpha)^{-1}$, which is a positive operator when $\{U(t)\}_{t\geq 0}$ is a positive semigroup. Hence we choose

$$\begin{aligned} u_0 &= (I - U_\alpha(\omega))^{-1}\omega^{1-\alpha}\left[\int_0^\omega (\omega-s)^{\alpha-1}U_\alpha(\omega-s)h(s)ds + \sum_{k=1}^{m}(t-t_k)^{\alpha-1}U_\alpha(\omega-t_k)y_k\right] \\ &= J(h). \end{aligned}$$

Then u_0 is the unique initial value of the (IVPIF)(2.7) in E, which satisfies $\widetilde{u}(0) = u_0 = \widetilde{u}(\omega)$. Combining this fact with (2.8), it follows that (2.8) is satisfied. Inversely, we can verify directly that the function $AC_{1-\alpha}([0,\omega],E)$ defined by (2.8) is a solution of the linear (PBVIF)(2.7). Therefore the conclusion of Lemma 2.4. $\square$

Definition 2.4. *A function $v \in AC_{1-\alpha}([0,\omega],E)$ is called a lower solution of* (1.3)*, if it satisfies the following inequality*

$$\begin{cases} {}^{RL}D_t^\alpha v(t) + Av(t) \leq f(t,v(t),Hv(t)), \quad t\in(0,\omega],\ t\neq t_k; \\ \Delta I_{t_k}^{1-\alpha}v(t_k) \leq G_k(v(t_k)), \quad k=1,2,\dots,m; \\ \widetilde{v}(0) \leq \widetilde{v}(\omega). \end{cases}$$

If all the inequalities are reversed, it is called an upper solution of (1.3).

Next, we recall some definitions and properties of measure of noncompactness, for more details, we refer the reader to [4, 8, 26, 35].

Definition 2.5. *Let E be a Banach space and $(\mathcal{A},\geq)$ a partially ordered set. A map $\beta_E : \mathcal{P}(E) \longrightarrow \mathcal{A}$ is called a measure of noncompactness (MNC) on E, if for every subset $\Omega \in \mathcal{P}(E)$, we have*

$$\beta_E(\overline{co}\Omega) = \beta_E(\Omega).$$

Notice that if D is dense in Ω, then $\overline{co}\Omega = \overline{co}D$ and hence $\beta_E(\Omega) = \beta_E(D)$.

Definition 2.6. [8, 3] *A measure of noncompactness β_E is called*

- (a) *Monotone, if $\Omega_0, \Omega_1 \in \mathcal{P}(E)$, $\Omega_0 \subset \Omega_1$ implies $\beta_E(\Omega_0) \leq \beta_E(\Omega_1)$.*
- (b) *Nonsingular, if $\beta_E(\{a\}\cup\Omega) = \beta_E(\Omega)$ for every $a \in E$ and $\Omega \in \mathcal{P}(E)$.*
- (c) *Invariant with respect to the union with compact sets, if $\beta_E(K\cup\Omega) = \beta_E(\Omega)$ for every relatively compact set $K \subset E$ and $\Omega \in \mathcal{P}(E)$.*
- (d) *Real, if $\mathcal{A} = \overline{\mathbb{R}_+} = [0,\infty]$ and $\beta_E(\Omega) < \infty$ for every bounded Ω.*
- (e) *Regular, if th condition $\beta_E(\Omega) = 0$ is equivalent to the relative compactness of Ω.*
- (f) *If $\{W_n\}_n^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset and*

$$\lim_{n\to+\infty}\beta_E(W_n) = 0,$$

  then $\cap_{n=1}^{+\infty}W_n$ is nonempty and compact,
- (g) *Algebraically semiadditive, if $\beta_E(\Omega_0+\Omega_1) \leq \beta_E(\Omega_0)+\beta_E(\Omega_1)$ for every $\Omega_0,\Omega_1 \in \mathcal{P}(E)$.*

As example of an MNC, are may consider the Hausdorff measure

$$\chi_E(\Omega) = \inf\{\epsilon > 0, \ \textit{for which } \Omega \textit{ has a finite } \epsilon\textit{-net in } E\}.$$

For any $W \subset AC(J, E)$, we define

$$\int_0^t W(s)ds = \left\{ \int_0^t u(s)ds : u \in W, \ \text{ for } t \in J = [0, \omega] \right\},$$

where $W(s) = \{u(s) \in E : u \in w\}$.

Lemma 2.5. [26] *If $\{b_n\}_n^\infty \subset L^1([0, \omega], E)$ and there exists an $\eta \in L^1([0, \omega], E)$ such that $\|b_n(t)\| \leq \eta(t)$, a.e. $t \in [0, \omega]$, then $\beta\left(\{b_n(t)\}_{n=1}^\infty\right)$ is integrable and*

$$\beta\left(\left\{ \int_0^t b_n(s)ds \right\}_{n=1}^\infty\right) \leq 2\int_0^t \beta\left(\{b_n(s)\}_{n=1}^\infty\right) ds.$$

Lemma 2.6. [26] *Let E be a Banach space , $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that $\beta_E(D) \leq 2\beta_E(D_0)$.*

Generalized Gronwall inequality for fractional differential equation

Lemma 2.7. [37] *Suppose $a \geq 0$, $\gamma > 0$, $c(t)$ and $z(t)$ be the nonnegative locally integrable functions on $0 \leq t < \omega < +\infty$ with*

$$z(t) \leq c(t) + a\int_0^t (t-s)^{\gamma-1} z(s)ds,$$

then

$$z(t) \leq c(t) + \int_0^t \left[\sum_{n=1}^\infty \frac{(a\Gamma(\gamma))^n}{\Gamma(n\gamma)}(t-s)^{n\gamma-1}c(s)\right] ds, \quad 0 \leq t < \omega.$$

3. Main Results

In this section we will prove our results, we require the following assumptions

(F_1) the function $f(t, \cdot) : E \to E$ is continuous for a.e. $t \in [0, \omega]$ and for all $v \in E$, the function $f(\cdot, v) : [0, \omega] \to E$ is strongly measurable.

(F_2) there exist a constant $\lambda > 0$ such that

$$f(t, u_2, \nu_2) - f(t, u_1, \nu_1) \geq -\lambda(u_2 - u_1),$$

for any $t \in [0, \omega]$ and $v_0(t) \leq u_1 \leq u_2 \leq w_0(t)$, $Hv_0(t) \leq \nu_1 \leq \nu_2 \leq Hw_0(t)$.

(F_3) $G_k(u)$ is increasing, continuous and compact on order interval $[v_0(t), w_0(t)]$, for $t \in [0, \omega]$, $k = 1, 2, \ldots, m$.

(F_4) there exists a constant $L > 0$ such that for all $t \in [0, \omega]$,

$$\beta_E(\{f(t, u_n, \nu_n)\}) \leq L\left(\beta_E(\{u_n\}) + \beta_E(\{\nu_n\})\right)$$

and increasing or decreasing sequence $\{u_n\} \subset [v_0(t), w_0(t)]$ and $\{\nu_n\} \subset [Hv_0, Hw_0]$

3.1. The case that $U(t)$ is compact.

Theorem 3.1. *Let E be an ordered Banach space, whose positive cone P is normal constant N. Assume that $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a positive compact C_0-semigroup $\{\mathcal{U}(t)\}_{t\geq 0}$ in E. If the (PBVIF)* (1.3) *has a lower $v_0 \in AC_{1-\alpha}([0, \omega], E)$ and an upper solution $w_0 \in AC_{1-\alpha}([0, \omega], E)$ with $v_0 \leq w_0$ and assumptions (F_1)-(F_3) holds, and $M^* < \Gamma(\alpha)$. Then the (PBVIF)* (1.3) *has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0.*

Proof. Let $\mathcal{E} = [v_0, w_0] = \{u \in AC_{1-\alpha}([0,\omega], E) : v_0 \le u \le w_0\}$, we consider the periodic boundary value problem of impulsive evolution fractional differential equation in E.

$$\begin{cases} {}^{RL}D_t^\alpha u(t) + (A+\lambda)u(t) = F(t, u(t), Hu(t)) \quad t \in (0,\omega],\ t \neq t_k; \\ \Delta I_{t_k}^{1-\alpha}|_{t=t_k} u(t) = G_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ \widetilde{u}(0) = \widetilde{u}(\omega). \end{cases} \tag{3.11}$$

where $F(t, u(t), Hu(t)) = f(t, u(t), Hu(t)) + \lambda u(t)$. Also $\lambda > 0$, $-(A+\lambda)$ generates a C_0-semigroup $S(t) = e^{-\lambda t}\mathcal{U}(t)(t \ge 0)$ on E and $S(t)$ is positive and continuous in the uniform operator topology for $t > 0$. That is there exists $M^* \ge 1$ such that $\sup\limits_{t \in [0,+\infty)} \|S(t)\| \le M^*$.

We have

$$\|S_\alpha(t)u\| \le \frac{M^*}{\Gamma(\alpha)}\|u\|, \quad t \ge 0.$$

Define a map $\Upsilon : \mathcal{E} \to AC_{1-\alpha}([0,\omega], E)$, for all $t \in (0,\omega]$ by

$$\begin{aligned} \Upsilon u(t) = {}& t^{\alpha-1}S_\alpha(t)J(F) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)F(s, u(s), Hu(s))ds \\ & + \sum_{0<t_k<t} (t-t_k)^{\alpha-1}S_\alpha(t-t_k)G_k(u(t_k)), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} J(F) = {}& (I - S_\alpha(\omega))^{-1}\omega^{1-\alpha}\Bigg[\int_0^\omega (\omega-s)^{\alpha-1}S_\alpha(\omega-s)F(s, u(s), Hu(s))ds \\ & + \sum_{k=1}^m (\omega-t_k)^{\alpha-1}S_\alpha(\omega-t_k)G_k(u(t_k))\Bigg]. \end{aligned}$$

It is clear that Υ is well defined. The rest of the proof is divided into four steps
Step 1: The map Υ is continuous in E.
Let $\{u_n\} \in E$ be a sequence such that $\{u_n\} \to u \in E$ as $n \to \infty$. Using assumptions (F_1) and (F_3), for almost every $t \in [0,\omega]$, $t \neq t_k$, by Lebesgue dominated convergence theorem, we get

$$\begin{aligned} & t^{1-\alpha}\|\Upsilon u_n(t) - \Upsilon u(t)\| \le \\ & \frac{(M^*)^2\omega^{1-\alpha}}{\Gamma(\alpha)(\Gamma(\alpha)-M^*)}\Bigg[\int_0^\omega (\omega-s)^{\alpha-1}\|F(s, u_n(s), Hu_n(s)) - F(s, u(s), Hu(s))\|ds \\ & + \sum_{k=1}^m (\omega-t_k)^{\alpha-1}\|G_k(t_k, u_n(t_k)) - G_k(u(t_k))\|\Bigg] \\ & + \frac{M^* t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|F(s, u_n(s), Hu_n(s)) - F(s, u(s), Hu(s))\|ds, \\ & + \frac{M^*}{\Gamma(\alpha)}\sum_{0<t_k<t}(t-t_k)^{\alpha-1}\|G_k(t_k, u_n(t_k)) - G_k(u(t_k))\| \\ & \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence the map Υ is continuous in E.
Step 2: Υ is increasing monotonic operator.
We show that $v_0 \le \Upsilon v_0$, $\Upsilon w_0 \le w_0$. For this $h(t) = {}^{RL}D_t^\alpha v_0(t) + Av_0(t) + \lambda v_0(t)$, by (3.11), $h \in AC_{1-\alpha}([0,\omega], E)$ and $h(t) \le f(t, v_0(t), Hv_0(t)) + \lambda v_0(t)$, $t \in [0,\omega]$, $t \neq t_k$. By Lemma

2.4, we have

$$v_0(t) = t^{\alpha-1}S_\alpha(t)\widetilde{v}(0) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)h(s)ds$$
$$+ \sum_{0<t_k<t}(t-t_k)^{\alpha-1}S_\alpha(t-t_k)G_k(v_0(t_k)),\ t\in(0,\omega]$$
$$\le t^{\alpha-1}S_\alpha(t)\widetilde{v}(0) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)(f(s,v_0(s),Hv_0(s))+\lambda v_0(s))ds$$
$$+ \sum_{0<t_k<t}(t-t_k)^{\alpha-1}S_\alpha(t-t_k)G_k(v_0(t_k)),$$

for $t\in[0,\omega]$, especially we have

$$\widetilde{v}(\omega) \le S_\alpha(\omega)\widetilde{v}(0) + \omega^{1-\alpha}\left[\int_0^\omega (\omega-s)^{\alpha-1}S_\alpha(\omega-s)(f(s,v_0(s),Hv_0(s))+\lambda v_0(s))ds\right.$$
$$\left.+\sum_{k=1}^m(\omega-t_k)^{\alpha-1}S_\alpha(\omega-t_k)G_k(v_0(t_k))\right].$$

Combining this inequality with $\widetilde{v}(0)=\widetilde{v}(\omega)$, it follows that

$$\widetilde{v}(0) \le (I-S_\alpha(\omega))^{-1}\omega^{1-\alpha}\left[\int_0^\omega (\omega-s)^{\alpha-1}S_\alpha(\omega-s)(f(s,v_0(s),Hv_0(s))+\lambda v_0(s))ds\right.$$
$$\left.+\sum_{k=1}^m(\omega-t_k)^{\alpha-1}S_\alpha(\omega-t_k)G_k(v_0(t_k))\right] = J(v_0).$$

On the other hand, from (3.12), we have

$$\Upsilon v_0(t) = t^{\alpha-1}S_\alpha(t)J(v_0) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)(f(s,v_0(s),Hv_0(s))+\lambda v_0(s))ds$$
$$+ \sum_{0<t_k<t}(t-t_k)^{\alpha-1}S_\alpha(t-t_k)G_k(v_0(t_k)),\ t\in(0,\omega].$$

Therefore, $\Upsilon v_0(t) - v_0(t) \ge S_\alpha(t)(t^{\alpha-1}J(v_0)-\widetilde{v}(0)) \ge 0$ for all $t\in[0,\omega]$. It implied that $v_0 \le \Upsilon v_0$. Similarly it can be show that $\Upsilon w_0 \le w_0$.

Next, we show that $\Upsilon : [v_0,w_0]\to[v_0,w_0]$ is completely continuous operator. Let

$$\Theta_1 u(t) = \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)(f(s,u(s),Hu(s))+\lambda u(s))ds,$$
$$\Theta_2 u(t) = \sum_{0<t_k<t}(t-t_k)^{\alpha-1}S_\alpha(t-t_k)G_k(u(t_k)),\quad u\in[v_0,w_0].$$

Step 3: Υ is equicontinuous on $[0,\omega]$.

For any $u\in E$ and $t_1,t_2\in[0,\omega]$ such that $0<t_1<t_2\le\omega$, we have

$$\left\|t_2^{1-\alpha}\Upsilon_1 u(t_2) - t_1^{1-\alpha}\Upsilon_1 u(t_1)\right\| \le \left\|t_2^{1-\alpha}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}S_\alpha(t_2-s)F(s,u(s),Hu(s))ds\right\|$$
$$+\left\|\int_0^{t_1}\left[t_2^{1-\alpha}(t_2-s)^{\alpha-1} - t_1^{1-\alpha}(t_1-s)^{\alpha-1}\right]S_\alpha(t_2-s)F(s,u(s,Hu(s)))ds\right\|$$
$$+\left\|t_1^{1-\alpha}\int_0^{t_1}(t_1-s)^{\alpha-1}[S_\alpha(t_2-s)-S_\alpha(t_1-s)]F(s,u(s),Hu(s))ds\right\|.$$

By the normality of the cone P, there exists $C_1>0$ such that $\|F(t,u(t),Hu(t))\|\le C_1$, $u\in[v_0,w_0]$, we get

$$\begin{aligned}
&\left\|t_2^{1-\alpha}\Upsilon_1 u(t_2)-t_1^{1-\alpha}\Upsilon_1 u(t_1)\right\| \leq \frac{M^*C_1}{\Gamma(\alpha)}\int_{t_1}^{t_2} t_2^{1-\alpha}(t_2-s)^{\alpha-1}ds\\
&+\frac{M^*C_1}{\Gamma(\alpha)}\int_0^{t_1}\left|t_2^{1-\alpha}\left[t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]+(t_1-s)^{\alpha-1}\left[t_2^{1-\alpha}-t_1^{1-\alpha}\right]\right|ds\\
&C_1\int_0^{t_1} t_1^{1-\alpha}(t_1-s)^{\alpha-1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|ds\\
&=\sum_{i=1}^{4}U_i.
\end{aligned}$$

Using Lemma 2.3(ii), $U_1\to 0$ as $t_2\to t_1$. Moreover, it is easy to see that $U_2, U_3\to 0$ as $t_2\to t_1$. For any $\epsilon\in(0,t)$, we have

$$\begin{aligned}
U_4 \leq & C_1\int_0^{t-\epsilon} t_1^{1-\alpha}(t_1-s)^{\alpha-1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|ds\\
&+C_1\int_{t-\epsilon}^{t} t_1^{1-\alpha}(t_1-s)^{\alpha-1}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|ds\\
\leq & \int_0^{t-\epsilon} t_1^{1-\alpha}(t_1-s)^{\alpha-1}\sup_{s\in[0,t-\epsilon]}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|ds\\
&+\frac{2M^*C_1}{\Gamma(\alpha)}\int_{t-\epsilon}^{t} t_1^{1-\alpha}(t_1-s)^{\alpha-1}ds\\
\leq & \int_0^{t-\epsilon} t_1^{1-\alpha}(t_1-s)^{\alpha-1}\sup_{s\in[0,t-\epsilon]}\|S_\alpha(t_2-s)-S_\alpha(t_1-s)\|ds\\
&+\frac{2M^*C_1t_1^{1-\alpha}}{\Gamma(\alpha+1)}\epsilon^\alpha\\
\to & \, 0 \text{ as } t_2\to t_1 \text{ and } \epsilon\to\ 0.
\end{aligned}$$

Hence $\Upsilon_1([v_0,w_0])$ is equicontinuous operator. The same idea can be used to prove equicontinuous of Υ_2. Hence $\Upsilon([v_0,w_0])$ is equicontinuous in $AC_{1-\alpha}([0,\omega],E)$.
Step 4: The set $Y(t)=\{\Upsilon u(t): u\in[v_0,w_0]\}$, $t\in[0,\omega]$, is precompact in $[v_0,w_0]$.
Let

$$\Upsilon u(t)=\Upsilon u(t)=t^{\alpha-1}S_\alpha(t)J(F)+\Upsilon_1 u(t)+\Upsilon_2 u(t)$$

and $\widetilde{\Upsilon}u(0)=J(F)=\widetilde{u}(\omega)$ is precompact in E. For any $t\in[0,\omega]$, choose $\epsilon\in(0,t)$ and $\nu>0$ such that

$$\begin{aligned}
&\Upsilon_1 u^{\epsilon,\nu}(t)=\alpha\int_0^{t-\epsilon}\int_\nu^\infty(t-s)^{\alpha-1}\Theta\zeta_\alpha(\Theta)\mathcal{U}((t-s)^\alpha\Theta)F(s,u(s),Hu(s))d\Theta ds\\
&\leq \alpha\mathcal{U}(\epsilon^\alpha\nu)\int_0^{t-\epsilon}\int_\nu^\infty(t-s)^{\alpha-1}\Theta\zeta_\alpha(\Theta)\mathcal{U}((t-s)^\alpha\Theta-\epsilon^\alpha\nu)F(s,u(s),Hu(s))d\Theta ds
\end{aligned}$$

Note that $\Theta\geq\nu$ and $t-\epsilon\geq s$ so $(t-s)^\alpha\Theta-\epsilon^\alpha\nu\geq 0$. Therefore from Lemma 2.3(iii), the operators $\mathcal{U}(\epsilon^\alpha\nu)$ and $\mathcal{U}(t^\alpha\Theta-\epsilon^\alpha\nu)$ are compact. Hence $\Upsilon_1 u^{\epsilon,\nu}(t)$ is precompact in E.

Now, we have

$$
\begin{aligned}
& t^{1-\alpha}\|\Upsilon_1 u(t)-\Upsilon_1 u^{\epsilon,\nu}(t)\| \\
& = \left\|\alpha t^{1-\alpha}\int_0^t\int_\nu^\infty (t-s)^{\alpha-1}\Theta\zeta_\alpha(\Theta)\mathcal{U}((t-s)^\alpha\Theta)F(s,u(s),Hu(s))d\Theta ds\right\| \\
& + \left\|\alpha t^{1-\alpha}\int_{t-\epsilon}^t\int_\nu^\infty (t-s)^{\alpha-1}\Theta\zeta_\alpha(\Theta)\mathcal{U}((t-s)^\alpha\Theta)F(s,u(s),Hu(s))d\Theta ds\right\| \\
& \leq \alpha M^* C_1\omega^{1-\alpha}\int_0^t (t-s)^{\alpha-1}ds\int_0^\nu \Theta\zeta_\alpha(\Theta)d\Theta \\
& +\alpha M^* C_1\omega^{1-\alpha}\int_{t-\epsilon}^t (t-s)^{\alpha-1}ds\int_0^\infty \Theta\zeta_\alpha(\Theta)d\Theta \\
& \to 0 \text{ as } \epsilon\to 0 \text{ and } \nu\to\ 0.
\end{aligned}
$$

i.e. precompact sets $\Upsilon_1 u^{\epsilon,\nu}(t)$ are arbitrarily closed to the sets $\{\Upsilon_1 u(t) : u \in [v_0, w_0]\}$. Hence the set $\{\Upsilon_1 u(t) : u \in [v_0, w_0]\}$ is precompact in $[v_0, w_0]$. Moreover, for $0 < t_k < t$, $k = 1, \ldots, m$, we get $\{\Upsilon_2 u(t) : u \in [v_0, w_0]\}$ is precompact in $[v_0, w_0]$. Hence, $\Upsilon([v_0, w_0])$ is precompact in $AC_{1-\alpha}([0, \omega], E)$, by the Arzela-Ascoli theorem. So $\Upsilon : [v_0, w_0] \to [v_0, w_0]$ is completely continuous. Hence Υ has minimal and maximal fixed points $\underline{u}$ and $\overline{u}$ in $[v_0, w_0]$ and therefore, they are the minimal and maximal mild solutions of the (PBVIF) (1.3) in $[v_0, w_0]$, respectively.

□

3.2. **The case that $U(t)$ is not compact.**

Theorem 3.2. *Let E be an ordered Banach space, whose positive cone P is normal constant N. Assume that $A : D(A) \subset E \to E$ be a closed linear operator and $-A$ generates a positive compact C_0-semigroup $\{U(t)\}_{t\geq 0}$ in E. If the (PBVIF)* (1.3) *has a lower $v_0 \in AC_{1-\alpha}([0, \omega], E)$ and an upper solution $w_0 \in AC_{1-\alpha}([0, \omega], E)$ with $v_0 \leq w_0$ and assumptions (F_1)-(F_4) holds, $M^* < \Gamma(\alpha)$ and there exists a constants $M^*, L, \mathcal{K}_0, \lambda$ such that for all $t \in [0, \omega]$,*

$$\frac{2M^*\omega^{1-\alpha}(L + 2\omega L\mathcal{K}_0 + \lambda)}{\Gamma(\alpha)} < 1.$$

Then the (PBVIF) (1.3) *has minimal and maximal mild solutions $\underline{u}$ and $\overline{u}$ between v_0 and w_0.*

Proof. From Theorem 3.1, we know that $\Upsilon : [v_0, w_0] \to [v_0, w_0]$ is a continuously increasing operator. Now we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = \Upsilon v_{n-1}, \quad w_n = \Upsilon w_{n-1}, \quad n = 1, 2, \ldots. \tag{3.13}$$

Then from the monotonicity of Υ, it follows that

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_2 \leq w_1 \leq w_0. \tag{3.14}$$

We prove that $\{v_n\}$ and $\{w_n\}$ are convergence in $[0, \omega]$.

For convenience, we denote $B = \{v_n : n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1} : n \in \mathbb{N}\}$. Then $B = \Upsilon(B_0)$. Let $I_1' = (0, t_1]$, $I_k' = (t_k, t_k]$, $k = 2, 3, \cdots, m + 1$. From $B_0 = B \cup \{v_0\}$ it follows that $\beta_E(B_0(t)) = \beta_E(B(t))$ for $t \in (0, \omega]$. Let $\varphi(t) := \beta_E(B(t))$, $t \in (0, \omega]$, going from I_1' to I_{m+1}' interval by interval we show that $\varphi(t) \equiv 0$ in $[0, \omega]$.

For $t \in [0,\omega]$ there exists a I'_k such that $t \in I'_k$. By using Lemma 2.5 and 2.6, by (1.4), we have

$$\begin{aligned}\beta_E(H(B_0)(t)) &= \beta_E\left(\left\{\int_0^t \mathcal{K}(t,s)\nu_{n-1}(s)ds : n \in \mathbb{N}\right\}\right)\\ &\le \sum_{j=1}^{k-1}\beta_E\left(\left\{\int_{t_{j-1}}^{t_j} \mathcal{K}(t,s)\nu_{n-1}(s)ds : n \in \mathbb{N}\right\}\right)\\ &\quad + \beta_E\left(\left\{\int_{t_{j-1}}^{t} \mathcal{K}(t,s)\nu_{n-1}(s)ds : n \in \mathbb{N}\right\}\right)\\ &\le 2\mathcal{K}_0\sum_{j=1}^{k-1}\int_{t_{j-1}}^{t_j}\beta_E(B_0(s))ds + 2\mathcal{K}_0\int_{t_{j-1}}^{t}\beta_E(B_0(s))ds\\ &= 2\mathcal{K}_0\sum_{j=1}^{k-1}\int_{t_{j-1}}^{t_j}\varphi(s)ds + 2\mathcal{K}_0\int_{t_{j-1}}^{t}\varphi(s)ds\\ &= 2\mathcal{K}_0\int_0^t\varphi(s)ds.\end{aligned}$$

Therefore

$$\int_0^t \beta_E(H(B_0)(s))ds \le 2\mathcal{K}_0\int_0^t\varphi(s)ds. \tag{3.15}$$

$$\begin{aligned}&\varphi(t) = \beta_E(B(t)) = \beta_E(\Upsilon(B_0(t)))\\ &= \beta_E\left(\left\{t^{\alpha-1}S_\alpha(t)J(F) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)(f(s,v_{n-1}(s),Hv_{n-1}(s)) + \lambda v_{n-1}(s))ds\right\}\right).\end{aligned}$$

So

$$\begin{aligned}\widetilde{\varphi}(t) &= \beta_E\left(\left\{S_\alpha(t)J(F) + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)(f(s,v_{n-1}(s),Hv_{n-1}(s)) + \lambda v_{n-1}(s))ds\right\}\right)\\ &\le \frac{2M^*\omega^{1-\alpha}(L+2\omega L\mathcal{K}_0+\lambda)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\widetilde{\varphi}(s)ds.\end{aligned}$$

Hence by the Gronwall's inequality, $\widetilde{\varphi}(t) \equiv 0$ on $[0,t_1]$. In particular, $\beta_E(B(t_1)) = \beta_E(B_0(t_1)) = \widetilde{\varphi}(t_1) = 0$, this means that $B(t_1)$ and $B_0(t_1)$ are precompact in E. Thus $G_1(B_0(t_1))$ is precompact in E, and $\beta_E(G_1(B_0(t_1))) = 0$.

Now, for $t \in I'_2$, by (3.12) and the above argument for $t \in I'_1$, we have

$$\begin{aligned}&\varphi(t) = \beta_E(B(t)) = \beta_E(\Upsilon(B_0(t)))\\ &= \beta_E\left(\left\{t^{\alpha-1}S_\alpha(t)J(F) + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)(f(s,v_{n-1}(s)) + \lambda v_{n-1}(s))ds\right\}\right).\end{aligned}$$

So

$$\begin{aligned}\widetilde{\varphi}(t) &= \beta_E\left(\left\{S_\alpha(t)J(F) + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)(f(s,v_{n-1}(s)) + \lambda v_{n-1}(s))ds\right\}\right)\\ &\le \frac{2M^*\omega^{1-\alpha}(L+2\omega L\mathcal{K}_0+\lambda)}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}s^{\alpha-1}\widetilde{\varphi}(s)ds.\end{aligned}$$

Again by the Gronwall's inequality, $\widetilde{\varphi}(t) \equiv 0$ in I'_2, from which we obtain that $\beta_E(B_0(t_2)) = 0$ and $\beta_E(G_2(B_0(t_2))) = 0$. Continuing such a process interval by interval up to I'_{m+1}, we can prove that $\varphi(t) \equiv 0$ in every I'_k, $k = 1, 2, \cdots, m+1$. For any I_k if we modify the value of v_n at $t = t_{k-1}$ via $v_n(t_{k-1}) = v_n(t_{k-1}^+)$, $n \in \mathbb{N}$, then $\{v_n\} \subset AC_{1-\alpha}(I_k, E)$ and it is equicontinuous. Since $\beta_E(\{v_n(t)\} \equiv 0$, $\{v_n(t)\}$ is precompact in E for every $t \in I_k$. By the Arzela-Ascoli theorem, $\{v_n(t)\}$ is precompact in $AC_{1-\alpha}(I_k, E)$. Hence, $\{v_n\}$ has a convergent subsequence in $AC_{1-\alpha}(I_k, E)$. Combining this with the monotonicity (3.14) we easily prove that $\{v_n\}$ itself is convergent in $AC_{1-\alpha}(I_k, E)$. In particular, $\{v_n(t)\}$ is uniformly convergent in I'_k. Consequently, $\{v_n(t)\}$ is uniformly convergent over the whole of $[0, \omega]$. Using a similar argument to that for $\{v_n(t)\}$, we can prove that $w_n(t)\}$ is also uniformly convergent in $[0, \omega]$. Hence, $\{v_n(t)\}$ and $\{w_n(t)\}$ are convergent in $AC_{1-\alpha}([0, \omega], E)$. Set

$$\underline{u} = \lim_{n\to\infty} v_n, \quad \overline{u} = \lim_{n\to\infty} w_n, \quad \text{in} \quad AC_{1-\alpha}([0, \omega], E). \tag{3.16}$$

Letting $n \to \infty$ in (3.13) and (3.14), we see that $v_0 \leq \underline{u} \leq \overline{u} \leq w_0$ and

$$\underline{u} = \Upsilon \underline{u}, \quad \overline{u} = \Upsilon \overline{u}. \tag{3.17}$$

By the monotonicity of Υ, it is easy to see that $\underline{u}$ and $\overline{u}$ are the minimal and maximal fixed points of Υ in $[v_0, w_0]$, and therefore, they are the minimal and maximal solutions of (BVPIF)(1.3) in $[v_0, w_0]$, respectively.
This completes the proof of Theorem 3.2. □

Now we discuss the uniqueness of the solution to (BVPIF)(1.3) in $[v_0, w_0]$. if we replace the assumption (F_4) by the following assumption:

(F_5) there exist positive constant C, such that

$$f(t, u_2) - f(t, u_1) \leq C(u_2 - u_1) + \overline{C}(\nu_2 - \nu_1),$$

$$\forall t \in [0, \omega], \quad \text{and} \quad v_0(t) \leq u_1 \leq u_2 \leq w_0(t), \text{ and } Hv_0(t) \leq \nu_1 \leq \nu_2 \leq Hw_0.$$

we have the following unique existence result

Theorem 3.3. *Let E be an ordered Banach space, whose positive cone P is normal, $f \in C([0, \omega]; \times E, E)$ and $I_k \in C(E, E)$, $k = 1, 2, \cdots, m$. If the (BVPIF)(1.3) has a lower solution $v_0 \in AC_{1-\alpha}([0, \omega], E)$ and an upper solution $w_0 \in AC_{1-\alpha}([0, \omega], E)$ with $v_0 \leq w_0$, such that conditions $(F_1) - (F_2)$ and (F_4) holds,and there exists a constants $M^*, C, \mathcal{K}_0, \overline{C}, \lambda$ such that for all $t \in [0, \omega]$,*

$$\frac{NM^* \omega^{1-\alpha} \left[(\lambda + C) + \omega \overline{C} \mathcal{K}_0 \right]}{\Gamma(\alpha)} < 1.$$

Then the (BVPIF)(1.3) has a unique solution between v_0 and w_0 which can be obtained by a monotone iterative procedure starting from v_0 or w_0.

Proof. We firstly prove that (F_1) and (F_5) imply (F_4). For $t \in [0, \omega]$, let $\{u_n\} \subset [v_0, w_0]$, $\{\nu\} \subset [Hv_0(t), Hw_0(t)]$ be increasing sequence. For $m, n \in \mathbb{N}$ with $m > n$, by (F_1) and (F_5),

$$\begin{aligned} \theta &\leq (f(t, u_m, \nu_m) - f(t, u_n, \nu_n)) + \lambda(u_m - u_n) \\ &\leq (C + \lambda)(u_m - u_n) + \overline{C}(\nu_m - \nu_n). \end{aligned}$$

By this and the normality of cone P, we have

$$\begin{aligned} &\|f(t, u_m, \nu_m) - f(t, u_n, \nu_n)\| \\ &\leq N \left[\|(C + \lambda)(u_m - u_n) + \overline{C}\|\nu_m - \nu_n\| \right] + \lambda\|u_m - u_n\| \\ &\leq (\lambda + N\lambda + NC)\|u_m - u_n\| + N\overline{C}\|\nu_m - \nu_n\|. \end{aligned}$$

From this inequality and the definition of the measure of noncompactness, it follows that

$$\begin{aligned} \beta_E(\{f(t, u_n, \nu_n))\}) &\leq (\lambda + N\lambda + NC)\beta_E(\{u_n\}) + N\overline{C}\beta_E(\{\nu_n\}) \\ &\leq L(\beta_E(\{u_n\}), \end{aligned} \tag{3.18}$$

where $L = \max\{(\lambda + N\lambda + NC), N\overline{C}\}$. If $\{u_n\}$, $\{\nu_n\}$ are two decreasing sequences, the above inequality is also valid.
Hence (F_4) holds. Therefore, by Theorem 3.2 the (BVPIF)(1.3) has minimal solution $\underline{u}$ and maximal solution $\overline{u}$ in $[v_0, w_0]$. By the proof of Theorem 3.2,(3.13) and (3.14) equations are valid. Going from I_1' to I_{m+1}' interval by interval we show that $\underline{u}(t) \equiv \overline{u}(t)$ in every I_k'. For $t \in I_1'$, by (3.12) and assumption (F_5), we have

$$\theta \leq \overline{u}(t) - \underline{u}(t) = \Upsilon\overline{u}(t) - \Upsilon\underline{u}(t)$$
$$= \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) \left[f(s, \overline{u}(s), H\overline{u}(s)) - f(s, \underline{u}(s), H\underline{u}(s)) + \lambda(\overline{u}(s)) - \underline{u}(s))\right] ds$$

From this and the normality of cone P it follows that

$$\|\overline{u}(t) - \underline{u}(t)\|_\alpha \leq \frac{NM^*\omega^{1-\alpha}\left[(\lambda + C) + \omega\overline{C}\mathcal{K}_0\right]}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \|\overline{u}(s) - \underline{u}(s)\|_\alpha ds.$$

By this the Gronwall's inequality, we obtain that $\underline{u}(t) \equiv \overline{u}(t)$ in I_1'. For $t \in I_2'$, since $G_1(\overline{u}(t_1) = G_1(\underline{u}(t_1))$ using (3.12) and completely the same argument as above for $t \in I_1'$, we can prove that

$$\|\overline{u}(t) - \underline{u}(t)\|_\alpha \leq \frac{NM^*\omega^{1-\alpha}\left[(\lambda + C) + \omega\overline{C}\mathcal{K}_0\right]}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \|\overline{u}(s) - \underline{u}(s)\|_\alpha ds$$
$$= \frac{NM^*\omega^{1-\alpha}\left[(\lambda + C) + \omega\overline{C}\mathcal{K}_0\right]}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} s^{\alpha-1} \|\overline{u}(s) - \underline{u}(s)\|_\alpha$$

Again, by the Gronwall's inequality, we obtain that $\underline{u}(t) \equiv \overline{u}(t)$ in I_2'.
Continuing such a process interval by interval up to I_{m+1}', we see that $\underline{u}(t) \equiv \overline{u}(t)$ over the whole of $[0, \omega]$. Hence, $u^* := \underline{u} = \overline{u}$ is the unique solution of (BVPIF)(1.3) in $[v_0, w_0]$, which can be obtained by the monotone iterative procedure(3.14) starting from v_0 or w_0. □

4. An example

In this section, we give an example to demonstrate how to utilize our results. We consider the (BVPIF) of impulsive fractional partial differential equation

(4.19)
$$\begin{cases} {}^{RL}D_{0+}^\alpha u(t,x) = \Sigma_{|\alpha| \leq 2m} a_\alpha D_x u(t,x) f(t, x, u(t,x), Hu(t,x)), & (t,x) \in (0, \omega] \times \Omega,\ t \neq t_k, \\ \Delta I_t^{1\alpha} u|_{t=t_k} = G_k(u(x, t_k)), \quad x \in \Omega,\ k = 1, 2, \ldots, m,\ u|_{\partial\Omega} = 0, \\ u(x, 0) = u(x, \omega), \quad x \in \Omega, \end{cases}$$

where $I = [0, \omega]$, $0 < t_1 < t_2 < \ldots < t_m < \omega$, integer $N \geq 1$, let $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$, and ${}^{RL}D_{0+}^\alpha$ is the Riemann-Liouville fractional derivative, $0 < \alpha < 1$, $f : I \times E \to E$ is continuous, $G_k : E \to E$ are also continuous, $k = 1, 2, \ldots, m$, and

$$D_x^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is an n-dimensional multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, coefficient function $a_\alpha(x) \in C^{2m}(\overline{\Omega})$.

Let $E = L^p(\Omega)$ with $1 < p < \infty$, $P = \{u \in L^p(\Omega) : u(x) \geq 0,\ a.e.\ x \in \Omega\}$ and define the operator $A : D(A) \subset E \to E$ is follows

$$D(A) = \left\{u \in W^{2m,p} \cap W_0^{m,p}(\Omega) : u|_{\partial u} = 0\right\}, \quad Au = \sum_{|\alpha|2m} a_\alpha D_x^\alpha u.$$

Then E is a Banach space, P is a normal cone of E and $-A$ generates a positive C_0-semigroup $U(t)(t \geq 0)$ in E, let $f(t, u(t), Hu(t)) = f(t, x, u(t, x), Hu(t, x))$ then problem (4.19) can be written as abstract (1.3). For solving the problem (4.19) the following assumptions are needed.

(H_1) Let $f(t, x, 0) \geq 0$, $I_k(0) \geq 0$, $u(\omega, x) \geq 0$, $x \in \Omega$, and there exists a function $w = w(t, x) \in AC_{1-\alpha}(I, E)$ such that

$$\begin{cases} {}^{RL}D_{0^+}^\alpha w \geq \Sigma_{|\alpha|\leq 2m} a_\alpha D_x u(t, x) f(t, x, w, Hw), \quad (t, x) \in (0, \omega] \times \Omega,\ t \neq t_k, \\ \Delta w|_{t=t_k} \geq G_k(w(x, t_k)), \quad x \in \Omega,\ k = 1, 2, \ldots, m,\ u|_{\partial\Omega} = 0, \\ w(x, 0) \geq w(x, \omega), \quad x \in \Omega, \end{cases}$$

(H_2) There exists a consant $\lambda > 0$ such that

$$f(t, x, x_2, y_2) - f(t, x, x_2, y_1) \geq -\lambda(x_2 - x_1),$$

for any $t \in I$ and $0 \leq x_1 \leq x_2 \leq w(t, x)$ and $0 \leq y_1 \leq y_2 \leq Hw(t, x)$,

(H_3) For any $u_1, u_2 \in [0, w(t, x)]$, with $u_1 \leq u_2$, we have

$$G_k(u_1(t_k, x)) \leq G_k(u_2(t_k, x)), \quad x \in \Omega,\ k = 1, 2, \ldots, m.$$

Assumption (H_1) implies that $v_0 \equiv 0$ and $w_0 \equiv w(t, x)$ are lower and upper solutions of the (BVPIF) (4.19) has minimal and maximal ,ild solutions between 0 and $w(t, x)$ which can be obtained by a monotone iterative procedure starting from 0 and $w(t, x)$ respectively.

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