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A study on pseudo BG-algebras

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ABSTRACT. The concept of BG-algebras was introduced in 2008 and the concept of pseudo BG-algebras, as an extension of BG-algebras, was introduced in 2022. In this article, we not only make a new review of known results on pseudo BG-algebras, but also state and prove some new results both on the properties of pseudo BG-algebras and on the properties of some of their substructures.

1. INTRODUCTION

The study of BCK-algebras was initiated by Y. Imai and K. Iséki ([4]) in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. In order to extend BCK-algebras in a noncommutative form, Georgescu and Iorgulescu ([2]) introduced the notion of pseudo BCK-algebras and studied their properties. The pseudo BCK-algebras is a generalization of BCK-algebras. Then some other pseudo algebras were introduced and discussed, such as, for example, pseudo BCI-algebra ([3], 2008) by W. A. Dudek and Y. B. Jun, pseudo BCH-algebra ([15], 2015) by A. Walendziak, pseudo BH-algebras ([7], 2015) by Y. B. Jun and S. S. Ahn and pseudo UP-algebra ([11], 2020) by this author.

This article is, in a literal sense, a continuation of our previous research on BG-algebras [12, 13, 14] and refers to the research on pseudo BG-algebras wich started with [1, 10]. In [1], the authors introduced the concept of pseudo BG-algebras and, in addition, discussed substructures of sub-algebras and ideals in pseudo BG-algebras. The article [10] discuss the concept of derivations in pseudo BG-algebras.

First, here, in this paper, we prove that every pseudo BG-algebra is a pseudo BH-algebra and give an example to show that the converse need not be true. Then, after showing some of the important properties of this class of logical algebras in a somewhat different way than in the original text [1], we prove that the direct product of any family of pseudo BG-algebras is a pseudo BG-algebra again. Then we deal with the relation between sub-algebras and ideals in pseudo BG-algebras, showing that every sub-algebra is an ideal, on the one hand, while a closed ideal is a sub-algebra, on the other hand. Also, discussing strong and weak ideals in pseudo BG-algebras, we make several propositions about these two substructures.

2. PRELIMINARIES

The terminology and notation in this paper are mainly taken from well known paper [5] and from [1, 12, 13]. We will start with the determination of BG-algebras:

Definition 2.1 ([8], pp. 498)**.** *A BG-algebra is an algebra* $\mathfrak{A} =: (A, *, 0)$ *of type* $(2, 0)$ *which satisfies the following axioms:*

(Re) $(\forall x \in A)(x * x = 0)$.

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(M) $(\forall x \in A)(x * 0 = x)$.

(BG) $(\forall x, y \in A)((x * y) * (0 * y) = x)$.

The class of BG-algebras is squeezed between the classes of B-algebras and BH-algebras, as shown in [8], (pp. 500). A B-algebra is a non-empty set A ([9], pp. 22) with a constant 0 and a binary operation $'*'$ on A satisfying the the conditions (Re), (M) and the following axiom:

(B) $(\forall x, y, z \in A)((x * y) * z = x * (z * (0 * y)))$.

By a BH-algebra ([6], Definition 2.1), we mean an algebra $(A, *, 0)$ of type $(2, 0)$ having the conditions (Re), (M) and the following axiom

(An) $(\forall x, y \in A)((x * y = 0 \wedge y * x = 0) \implies x = y)$.

In this sense, every BG-algebra is a BH-algebra ([8], Proposition 2.8). The converse of previously stated statement need not be true in general, as shown in [8], Example 2.9.

The following determination describes the concept of pseudo BG-algebras:

Definition 2.2 ([1], Definition 3.1)**.** *A pseudo BG-algebra is an algebra* $\mathfrak{A} = (A, \rightarrow, \rightsquigarrow, 0)$ *of type* $(2, 2, 0)$ *which satisfies the following axioms:*

(pRe) $(\forall x \in A)(x \rightarrow x = 0 = x \rightsquigarrow x)$,

(pM) $(\forall x \in A)(x \rightarrow 0 = x = x \rightsquigarrow 0)$,

(pBG_1) $(\forall x, y \in A)((x \rightarrow y) \rightsquigarrow (0 \rightarrow y) = x)$,

(pBG_2) $(\forall x, y \in A)((x \rightsquigarrow y) \rightarrow (0 \rightsquigarrow y) = x)$.

As can be seen, if $\rightsquigarrow = \rightarrow$, then the pseudo BG-algebra $(A, \rightarrow, \rightsquigarrow, 0)$ is a BG-algebra $(A, \rightarrow, 0)$. So, the concept of pseudo BG-algebra is a generalization of the concept of BG-algebra. However, in the general case, for a given pseudo BG-algebra $(A, \rightarrow, \rightsquigarrow, 0)$, the substructures $(A, \rightarrow, 0)$ and $(A, \rightsquigarrow, 0)$ need not be BG-algebras.

3. The main results

This section is the central part of this paper. It consists of four subsections. While in the first subsection we provide some more information about the properties of pseudo BG-algebras, the next tree subsections are dedicated to the determination and description of the properties of sub-algebras, ideals, strong and weak ideals in pseudo BG-algebras.

3.1. A bit more about pseudo BG-algebras.

Proposition 3.1. *Let* $\mathfrak{A}$ *be a pseudo BG-algebra. Then:*

(1.1) $(\forall x \in A)(0 \rightsquigarrow (0 \rightarrow x) = x)$.

(1.2) $(\forall x \in A)(0 \rightarrow (0 \rightsquigarrow x) = x)$.

(2.1) $(\forall x, y \in A)(x \rightarrow y = 0 \implies x = y)$.

(2.2) $(\forall x, y \in A)(x \rightsquigarrow y = 0 \implies x = y)$.

(3.1) $(\forall x \in A)((x \rightarrow (0 \rightsquigarrow x)) \rightsquigarrow x = x)$.

(3.2) $(\forall x \in A)((x \rightsquigarrow (0 \rightarrow x)) \rightarrow x = x)$.

Proof. (1,1): If we put $y = x$ in (pBG_1), we get $(x \rightarrow x) \rightsquigarrow (0 \rightarrow x) = x$. From here it follows that $0 \rightsquigarrow (0 \rightarrow x) = x$ in accordance with (pRe).

(1.2) This statement can be proven analogously to statement (1.1): If we put $y = x$ in (pBG_2), we get $(x \rightsquigarrow x) \rightarrow (0 \rightsquigarrow x) = x$. From here it follows that $0 \rightarrow (0 \rightsquigarrow x) = x$ in accordance with (pRe).

(2.1) Let $x, y \in A$ be such that $x \to y = 0$. Then $(x \to y) \rightsquigarrow (0 \to y) = 0 \rightsquigarrow (0 \to y) = x$ according (pBG$_1$). On the other hand, again according to (pRe) and (pBG$_1$), we have $y \to y = 0$ and $0 \rightsquigarrow (0 \to y) = (y \to y) \rightsquigarrow (0 \to y) = y$. Thus $x = y$.

(2.2) This statement can be proven analogously to statement (2.1).

(3.1) $(x \to (0 \rightsquigarrow x) \rightsquigarrow x = (x \to (0 \rightsquigarrow x)) \rightsquigarrow (0 \to (0 \rightsquigarrow x)) = x$ in accordance with (1.2) and (pBG$_1$).

(3.2) This statement can be proven analogously to statement (3.1). □

Remark 3.1. *Statements (2.1) and (2.2) in the previous proposition have weaker hypotheses than in* [1]*, Properties 3.2.(I). Additionally, our statements (1.1) and (1.2) differ from statement (II) in* [1]*, Properties 3.5.*

In order to relate the pseudo BG-algebra to the pseudo BH-algebra, below we give the determination of the pseudo BH-algebra.

Definition 3.3 ([7], Definition 3.1)**.** *A pseudo BH-algebra is a non-empty set A with a constant* 0 *and two binary operations $\to$ and $\rightsquigarrow$ satisfying the following axioms:*

(pRe) $(\forall x \in A)(x \to x = 0 = x \rightsquigarrow x)$.

(pM) $(\forall x \in A)(x \to 0 = x = x \rightsquigarrow 0)$.

(pAn) $(\forall x, y \in A)((x \to y = 0 \ \wedge \ y \rightsquigarrow x = 0) \implies x = y)$.

The connection between pseudo BG-algebras and pseudo BH-algebras is given by the following proposition.

Theorem 3.1. *Every pseudo BG-algebra is a pseudo BH-algebra.*

Proof. Proof follows immediately from statements (2.1) and (2.2) of Proposition 3.1. □

The converse of Theorem 3.1 need not be true as the following example shows.

Example 3.1. *Let $A = \{0, a, b, c\}$ and let the operation in A be determined as follows*

$\to$	0	a	b	c
0	0	0	0	c
a	a	0	0	0
b	b	b	0	b
c	c	0	a	0

$\rightsquigarrow$	0	a	b	c
0	0	0	0	c
a	a	0	0	c
b	b	b	0	0
c	c	c	0	0

With a little effort it can be proved that $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ is a pseudo BH-algebra (see Example 3.3(2) in [7]*), but it is not a pseudo BG-algebra because, for example, we have $(a \to b) \rightsquigarrow (0 \to b) = 0 \rightsquigarrow 0 = 0 \neq a$.* □

The following proposition gives some important properties of pseudo BG-algebras.

Proposition 3.2. *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ be a pseudo BG-algebra. Then:*

(4) *The right cancellation law holds in any pseudo BG-algebra for both internal operations.*

(5) *The left* 0*-cancellation law holds in any pseudo BG-algebra for both internal operations, in the following sense:*

(5.1) $(\forall x, y \in A)(0 \to x = 0 \to y \implies x = y)$.

(5.2) $(\forall x, y \in A)(0 \rightsquigarrow x = 0 \rightsquigarrow y \implies x = y)$.

Proof. (4.1) Let $x, y, z \in A$ be such that $x \to y = z \to y$. Then

$$x = (x \to y) \rightsquigarrow (0 \to y) = (z \to y) \rightsquigarrow (0 \to y) = z$$

in accordance with (pBG$_1$).

(4.2) The implication $x \rightsquigarrow y = z \rightsquigarrow y \implies x = z$ can be proven analogously to the previous implication.

(5.1) Let $x, y \in A$ be such that $0 \to x = 0 \to y$. Then

$$x = (x \to x) \rightsquigarrow (0 \to x) = (y \to y) \rightsquigarrow (0 \to y) = y$$

with respect to (pBG$_1$).

(5.2) The implication $0 \rightsquigarrow x = 0 \rightsquigarrow y \implies x = y$ can be proven analogously to the previous implication. □

Remark 3.2. *Our statement (4) and its proof in Proposition 3.2 differs from statement (I) in* [1], *Properties 3.5.*

From the previous proposition we obtain some important consequences that allow us to understand more deeply the interconnection of internal operations in a pseudo BG-algebra.

Corollary 3.1. *Let* $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ *be a pseudo BG-algebra. then*

(6) $(\forall x, y \in A)(x \neq y \implies (0 \to x \neq 0 \to y) \wedge (0 \rightsquigarrow x \neq 0 \rightsquigarrow y))$.

(7) $(\forall x, y, z \in A)(x \neq y \implies (x \to z \neq y \to z \wedge x \rightsquigarrow z \neq y \rightsquigarrow z))$.

Proof. The validity of statements (6) and (7) follows from the validity of statements (4) and (5) since they are contrapositions of the latter. □

Example 3.2. *Let* $A = \{0, a, b, c\}$ *and let the operation in* A *be determined as follows*

$\to$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	b	0	c
c	c	a	a	0

$\rightsquigarrow$	0	a	b	c
0	0	c	b	a
a	a	0	c	c
b	b	a	0	b
c	c	b	a	0

With a little more effort, it can be proven that $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$, *is a pseudo BG-algebra.* □

Remark 3.3. *It should be noted here that, in the general case,* $(A, \to, 0)$ *and* $(A, \rightsquigarrow, 0)$ *need not be BG-algebras. Thus, for the sake of illustration, in the Example 3.2, the structure* $(A, \to, 0)$ *is not a BG-algebra because we have* $a = (a \to c) \to (0 \to c) = b \to a = b$ *which is not possible since* $b \neq a$.

Example 3.3. *Let* $A = \{0, a, b, c, d\}$ *and let the operation in* A *be determined as follows*

$\to$	0	a	b	c	d
0	0	c	b	a	d
a	a	0	c	b	a
b	b	b	0	c	b
c	c	a	a	0	c
d	d	d	d	d	0

$\rightsquigarrow$	0	a	b	c	d
0	0	c	b	a	d
a	a	0	c	c	a
b	b	a	0	b	b
c	c	b	a	0	c
d	d	d	d	d	0

With a little more effort, it can be proven that $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$, *is a pseudo BG-algebra.* □

Let $\{(A_i, \to_i, \rightsquigarrow_i 0_i) : i \in I\}$ be a family of pseudo BG-algebras. If on the set

$$\prod_{i \in I} A_i = \{f : I \longrightarrow \cup_{i \in I} A_i \mid (\forall i \in I)(f(i) \in A_i)\},$$

we define the operation $\odot$ as follows

$$(\forall f, g \in \prod_{i \in I} A_i)(\forall \in I)((f \odot g)(i) =: f(i) \to_i g(i))$$

and the operation $\otimes$ as follows

$$(\forall f, g \in \prod_{i\in I} A_i)(\forall \in I)((f \otimes g)(i) =: f(i) \rightsquigarrow_i g(i)),$$

we created the structure $(\prod_{i\in I} A_i, \odot, \otimes, f_0)$, where f_0 was chosen as follows

$$(\forall i \in I)(f_0(i) =: 0_i).$$

Before we start working with direct products of BG-algebras, we say that the operations, determined in this way, are well-defined in the following sense: in the following sense: If for $f, g, x, y \in \prod_{i\in I} A_i$ are $f = x$ and $g = y$, then

$$f \odot g = x \odot y \text{ and } f \otimes g = x \otimes y.$$

If a priori we accept conditions that ensure the existence of a non-empty direct product, we can prove the following theorem.

Theorem 3.2. *The direct product of any family of pseudo BG-algebras, determined as above, is a pseudo BG-algebra.*

Proof. By direct verification, it can be proved that this structure satisfies the axioms of a pseudo BG-algebra:

Let $f, g \in \prod_{i\in I} A_i$ be arbitrary elements and $i \in I$. Then, we have:

(pRe) $(f \odot f)(i) = f(i) \rightarrow_i f(i) = 0_i = f_0(i)$.

$(f \otimes f)(i) = f(i) \rightsquigarrow_i f(i) = 0_i = f_0(i)$.

(pM) $(f \odot f_0)(i) = f(i) \rightarrow_i f_0(i) = f(i) \rightarrow_i 0_i = f(i)$.

$(f \otimes f_0)(i) = f(i) \rightsquigarrow_i f_0(i) = f(i) \rightsquigarrow_i 0_i = f(i)$.

Considering that

$((f \odot g) \otimes (f_0 \odot g))(i) = (f(i) \rightarrow_i g(i)) \rightsquigarrow_i (f_0(i) \rightarrow_i g(i)) = f(i),$ and

$((f \otimes g) \odot (f_0 \odot g))(i) = (f(i) \rightsquigarrow_i g(i)) \rightarrow_i (f_0(i) \rightsquigarrow_i g(i)) = f(i),$

we have that (pBG_1) and (pBG_2) are valid formulas for the observed structure.

Therefore, the structure $(\prod_{i\in I} A_i, \odot, \otimes, f_0)$ is a pseudo BG-algebra. □

Example 3.4. *Let* $\mathfrak{A} =: (A, \rightarrow, \rightsquigarrow, 0)$ *be as in Example 3.2. The structure* $\mathfrak{A}\times\mathfrak{A} =: (A, \odot, \otimes, (0, 0))$, *where the operations* $\odot$ *and* $\otimes$ *are defined as follows*

$$(\forall x, y, u, v \in A)((x, y) \odot (u, v) =: (x \rightarrow u, y \rightarrow v) \text{ and}$$

$$(\forall x, y, u, v \in A)((x, y) \otimes (u, v) =: (x \rightsquigarrow u, y \rightsquigarrow v)),$$

is a pseudo BG-algebra, according to the previous theorem. □

3.2. **Sub-algebras and ideals.** The concepts of subalgebras and ideals in a pseudo BG-algebra are introduced by the standard way:

Definition 3.4 ([1], point 2.6, pp. 74)**.** *Let* $\mathfrak{A} =: (A, \rightarrow, \rightsquigarrow, 0)$ *be a pseudo BG-algebra. The non-empty subset* S *of* A *is called a sub-algebra in* $\mathfrak{A}$ *if the following holds:*

(S1.1) $(\forall x, y \in A)((x \in S \wedge y \in S) \implies x \rightarrow y \in S)$.

(S1.2) $(\forall x, y \in A)((x \in S \wedge y \in S) \implies x \rightsquigarrow y \in S)$.

The family of all sub-algebras in the pseudo BG-algebra $\mathfrak{A}$ *is denoted by* $\mathfrak{S}(A)$.

Due to the presence of condition (S1.1) (that is, condition (S1.2)), one can immediately deduce the presence of condition

(S0): $0 \in S$.

Indeed, since S is a nonempty subset of A, there exists at least some $x \in A$ such that $x \in S$. Then, we get $x \to x = 0 \in S$, according to (S1.1) and (pRe).

Definition 3.5 ([1], point 2.7, pp. 74)**.** *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ be a pseudo BG-algebra. The non-empty subset J of A is called an ideal in $\mathfrak{A}$ if the following holds:*

(J0) $\quad 0 \in J$.

(J1.1) $(\forall x, y \in A)((x \to y \in J \,\wedge\, y \in J) \implies x \in J)$.

(J1.2) $(\forall x, y \in A)((x \rightsquigarrow y \in S \,\wedge\, y \in J) \implies x \in J)$.

The family of all ideals in the pseudo BG-algebra $\mathfrak{A}$ is denoted by $\mathfrak{J}(A)$.

Proposition 3.3. *Every sub-algebra in a pseudo BG-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$. This means $\mathfrak{S}(A) \subseteq \mathfrak{J}(A)$.*

Proof. Let S be a sub-algebra in $\mathfrak{A}$ and let $x, u \in A$ be such that $x \to y \in S$, $x \rightsquigarrow y \in S$ and $y \in S$. Then $0 \to y \in S$ and $0 \rightsquigarrow y \in S$ by (S0) and with respect to (S1.1) and (S1.2). Thus $x = (x \to y) \rightsquigarrow (0 \to y) \in S$ and $x = (x \rightsquigarrow y) \to (0 \rightsquigarrow y) \in S$ again according to (S1.1), (S1.2), (pBG_1) and (pBG_2). Hence, S is an ideal in $\mathfrak{A}$. □

To establish the connection between ideals and sub-algebras in a pseudo BG-algebra, we need the following determination:

An ideal J in a pseudo BG-algebra $\mathfrak{A}$ is said to be closed if

(Jc1) $(\forall y \in A)(y \in J \implies 0 \to y \in J)$ and

(Jc2) $(\forall y \in A)(y \in J \implies 0 \rightsquigarrow y \in J)$.

Proposition 3.4. *A closed ideal in a pseudo BG-algebra $\mathfrak{A}$ is a sub-algebra in $\mathfrak{A}$.*

Proof. Let J be a closed in a pseudo BG-algebra $\mathfrak{A}$ and let $x, y \in A$ be such that $x \in J$ and $y \in J$. Then $0 \to y \in J$ and $0 \rightsquigarrow y \in J$ by (Jc1) and (Jc2) since J is a closed ideal in $\mathfrak{A}$. Then $(x \to y) \rightsquigarrow (0 \to y) = x \in J$ and $(x \rightsquigarrow y) \to (0 \rightsquigarrow y) = x \in J$ in accordance with (pBG_1) and (pBG_2). From here we get $x \to y \in J$ and $x \rightsquigarrow y \in J$ according to (J1.1) and (J1.2). Thus, J is a sub-algebra in $\mathfrak{A}$. □

Example 3.5. *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ be a pseudo BG-algebra as in Example 3.2. Subsets $S_0 =: \{0\}$ and $S_2 =: \{0, b\}$ are sub-algebras in $\mathfrak{A}$. Subsets $J_0 =; \{0\}$ and $J_2 =; \{0, b\}$ are ideals in $\mathfrak{A}$.* □

Remark 3.4. *Let J be an ideal in a pseudo BG-algebra $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$. In the general case,*

$$(\forall x, y \in A)(x \rightsquigarrow y \in J \implies x \to y \in J)$$

does not hold, as the previous example shows. Namely, for the the pseudo BG-algebra and the ideal J_2 in Example 3.5, we have $b \rightsquigarrow c = b \in J_2$ but $b \to c = c \notin J_2$.

Example 3.6. *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ be a pseudo BG-algebra as in Example 3.3. Subsets $S_0 =: \{0\}$ and $S_2 =: \{0, b\}$, $S_4 =: \{0, d\}$, $S_9 =: \{0, b, d\}$ and $S_{11} =: \{0, a, b, c\}$ are sub-algebras in $\mathfrak{A}$. Subsets $J_0 =: \{0\}$ and $J_2 =: \{0, b\}$, $J_4 =: \{0, d\}$, $J_9 =: \{0, b, d\}$ and $J_{11} =: \{0, a, b, c\}$ are idals in $\mathfrak{A}$.* □

Since the families $\mathfrak{S}(A)$ and $\mathfrak{J}(A)$ are not empty, we can prove:

Theorem 3.3. *The family $\mathfrak{S}(A)/\mathfrak{J}(A)$ forms a complete lattice.*

Proof. Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ be a pseudo BG-algebra. We will demonstrate the proof for the family $\mathfrak{S}(A)$. The proof for the family $\mathfrak{J}(A)$ can be done analogously.

Let $\{S_i : i \in I\}$ be a family of sub-algebras in $\mathfrak{A}$. It is clear that $0 \in \cap_{i \in I} S_i$ holds due to the presence of (J0) for every $i \in I$. Let $x, y \in A$ be such that $x \in \cap_{i \in I} S_i$ and $y \in \cap_{i \in I} S_i$.

This means $x \in S_i$ and $y \in S_i$ for every $i \in I$. Then $x \to y \in S_i$ and $x \rightsquigarrow y \in S_i$ for each $i \in I$ according to (S1.1) and (S1.2). Thus $x \to y \in \cap_{\in I} S_i$ and $x \rightsquigarrow y \in \cap_{\in I} S_i$.

If we denote by $\mathcal{Z}$ the family of all sub-algebras of the algebra $\mathfrak{A}$ that contain $\cup_{i\in I} S_i$, then $\cap\mathcal{Z}$ is a sub-algebra of the algebra $\mathfrak{A}$ that contains $\cup_{i\in I} S_i$, according to the first part of this proof.

If we put $\sqcap_{i\in I} S_i = \cap_{i\in I} S_i$ and $\sqcup_{i\in I} S_i = \cap\mathcal{Z}$, then $(\mathfrak{S}(A), \sqcap, \sqcup)$ is a complete lattice. $\square$

Corollary 3.2. *For any element x of the pseudo BG-algebra A, there exists a minimal sub-algebra/ideal S_x in $\mathfrak{A}$ that contains x.*

Proof. Let $\mathcal{Z}$ be the family of all sub-algebras/ideals of the algebra A that contain the element x. Then, $S_x =: \cap\mathcal{Z}$ is a sub-algebra/ideals in $\mathfrak{A}$ by the previous theorem. If S is a sub-algebra/ideal in A containing x, then $S \in \mathcal{Z}$. Thus, $S_x \subseteq S$. Therefore, S_x is a minimal sub-algebra/ideals in $\mathfrak{A}$ which contains x. $\square$

3.3. **Strong ideal.** The concept of strong ideals in a pseudo BG-algebra was introduced in [1], Definition 2.14.

Definition 3.6. *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ be a pseudo BG-algebra. A non-empty subset J in A is a strong ideal in $\mathfrak{A}$ if, in addition to condition (J0), it also satisfies the following conditions*

(SJ1.1) $(\forall x, y, z \in A)(((x \to y) \rightsquigarrow z \in J \wedge y \in J) \Longrightarrow x \to z \in J)$.

(SJ1.2) $(\forall x, y, z \in A)(((x \rightsquigarrow y) \to z \in J \wedge y \in J) \Longrightarrow x \rightsquigarrow z \in J)$.

We denote the family of all strong ideals in the pseudo BG-algebra $\mathfrak{A}$ by $\mathfrak{J}_s(A)$.

Proposition 3.5 ([1], Proposition 3.15)**.** *Every strong ideal in a pseudo BG-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$.*

Proposition 3.6. *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ be a pseudo BG-algebra. The subset $J_0 =: \{0\}$ is a strong ideal in $\mathfrak{A}$. Therefore, the family $\mathfrak{J}_s(A)$ is not empty.*

Proof. Let $x, y, z \in A$ be such that $(x \to y) \rightsquigarrow z = 0$, $(x \rightsquigarrow y) \to z = 0$ and $y = 0$. Then $x \to z = 0$ and $x \rightsquigarrow z = 0$ by (pM). Thus, $\{0\}$ is a strong ideal in $\mathfrak{A}$. $\square$

Example 3.7. *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ be a pseudo BG-algebra as in Example 3.3. The ideal $J_{11} =: \{0, a, b, c\}$ in $\mathfrak{A}$ is a strong ideal in $\mathfrak{A}$.* $\square$

Not every ideal in a pseudo BG-algebra $\mathfrak{A}$ has to be a strong ideal in $\mathfrak{A}$ as the following example shows.

Example 3.8. *(a) Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ be a pseudo BG-algebra as in Example 3.2. The ideal $J_2 =: \{0, b\}$ is not a strong ideal in $\mathfrak{A}$ because, for example, we have $(c \to b) \rightsquigarrow a = a \rightsquigarrow a = 0 \in J_3$ and $b \in J_3$ but $c \to a = a \notin J_2$.*

(b) Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ be a pseudo BG-algebra as in Example 3.3. The ideal $J_4 = \{0, d\}$ in $\mathfrak{A}$ is not a strong ideal in $\mathfrak{A}$ because, for example, we have $(0 \to d) \rightsquigarrow a = d \rightsquigarrow a = d \in J_4$ and $d \in J_4$ but $0 \to a = b \notin J_4$. $\square$

In what follows, we need the following notion. Let $\mathfrak{A} =: (A, \to_A, \rightsquigarrow_A, 0_A)$ and $\mathfrak{B} =: (B, \to_B, \rightsquigarrow_B, 0_B)$ be pseudo BG-algebras. A mapping $f : A \longrightarrow B$ is called a homomorphism of pseudo BG-Algebras if

(f1.1) $(\forall x, y \in A)(f(x \to_A y) = f(x) \to_B f(y))$.

(f1.2) $(\forall x, y \in A)(f(x \rightsquigarrow_A y) = f(x) \rightsquigarrow_B f(y))$.

We denote the homomorphism f between pseudo BG-algebras $\mathfrak{A}$ and $\mathfrak{B}$ by $f : \mathfrak{A} \longrightarrow \mathfrak{B}$. And, at the same time, it is easy to see that $f(0_A) = 0_B$. Indeed, for arbitrary $x \in A$, we have $f(0_A) = f(x \to_A x) = f(x) \to_A f(x) = 0_B$ by (f1.1).

Theorem 3.4. *Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism between pseudo BG-algebras. If C is a strong ideal in $\mathfrak{B}$, then $f^{-1}(C) = \{x \in A : f(x) \in C\}$ is a strong ideal in $\mathfrak{A}$.*

Proof. Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism between pseudo BG-algebras, C be a strong ideal in a pseudo BG-algebra $\mathfrak{B} =: (B, \rightarrow_B, \rightsquigarrow_B, 0_B)$ and let $x, y, z \in A$ be such that $(x \rightarrow_A y) \rightsquigarrow_A z \in f^{-1}(C)$, $(x \rightsquigarrow_A y) \rightarrow_A z \in f^{-1}(C)$ and $y \in f^{-1}(C)$. This means

$$f((x \rightarrow_A y) \rightsquigarrow_A z) = ... = (f(x) \rightarrow_B f(y)) \rightsquigarrow_B f(z) \in C,$$

$$f((x \rightsquigarrow_A y) \rightarrow_A z) = ... = (f(x) \rightsquigarrow_B f(y)) \rightarrow_B f(z) \in C,$$

and $f(y) \in C$. Then

$$f(x \rightarrow_A z) = f(x) \rightarrow_B f(z) \in C \text{ and } f(x \rightsquigarrow_A z) = f(x) \rightsquigarrow_B f(z) \in C$$

since C is a strong ideal in $\mathfrak{B}$. Hence $x \rightarrow_A z \in f^{-1}(C)$ and $x \rightsquigarrow_A z \in f^{-1}(C)$. So, $f^{-1}(C)$ is a strong ideal in $\mathfrak{A}$. □

Corollary 3.3. *Let $f : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a homomorphism between pseudo BG-algebras. Then $Ker(f) =: \{x \in A : f(x) = 0_B\}$ is a strong ideal in $\mathfrak{A}$.*

Proof. According to Proposition 3.6, the subset $\{0_B\}$ is a strong ideal in a pseudo BG-algebra $\mathfrak{B}$. Therefore $Ker(f) = f^{-1}(\{0_B\})$ is a strong ideal in $\mathfrak{A}$ according to the previous theorem. □

Further on, we have:

Theorem 3.5. *Let $\{(A_i, \rightarrow_i, \rightsquigarrow_i, 0_i) : i \in I\}$ be a family of pseudo BG-algebras, K be a subset of I and let J_i be a strong ideal in $(A_i, \rightarrow_i, \rightsquigarrow_i, 0_i)$ for each $i \in K$. Then $\prod_{i \in I} T_i$, where $T_i = J_i$ for $i \in K$ and $T_i = A_i$ for $i \in I \backslash K$, is a strong ideal in the pseudo BG-algebra $(\prod_{i \in I} A_i, \odot, \otimes, f_0)$.*

Proof. First, it is clear that $f_0 \in \prod_{i \in I} T_i$.

If $K = \emptyset$, then $\prod_{i \in I} T_i = \prod_{i \in I} A_i$, so $\prod_{i \in I} T_i$ is certainly a strong ideal in $\prod_{i \in I} A_i$. Assume, therefore, that $K \neq \emptyset$.

Let $x, y, z \in \prod_{i \in I} A_i$ be such that $(x \odot y) \otimes z \in \prod_{i \in I} T_i$, $(x \otimes y) \odot z \in \prod_{i \in I} T_i$ and $y \in \prod_{i \in I} T_i$. This means $(x(i) \rightarrow_i y(i)) \rightsquigarrow_i z(i) \in J_i$, $(x(i) \rightsquigarrow_i y(i)) \rightarrow_i z(i) \in J_i$ and $y(i) \in J_i$ for each $i \in K$. Then $(x \odot z)(i) = x(i) \rightarrow_i z(i) \in J_i$, $(x \otimes z)(i) = x(i) \rightsquigarrow_i z(i) \in J_i$ since J_i is a strong ideal in $(A_i, \rightarrow_i, \rightsquigarrow_i, 0_i)$ for each $i \in K$. Hence $x \odot z \in \prod_{i \in I} T_i$ and $x \otimes z \in \prod_{i \in I} T_i$.

As shown, $\prod_{i \in I} T_i$ is a strong ideal in $(\prod_{i \in I} A_i, \odot, \otimes, f_0)$. □

Using an analogous procedure to the procedure described in the proof of Theorem 3.2, it can be proved:

Theorem 3.6. *The family $\mathfrak{J}_s(A)$ forms a complete lattice.*

3.4. **Weak ideal.** We introduce the concept of a weak ideal in a pseudo BG-algebra by the following determination:

Definition 3.7. *Let $\mathfrak{A} =: (A, \rightarrow, \rightsquigarrow, 0)$ be a pseudo BG-algebra. A non-empty subset J in A is a weak ideal in $\mathfrak{A}$ if holds*

(WJ1.1) $(\forall x, y, z \in A)((x \rightarrow (y \rightsquigarrow z) \wedge y \in J) \Longrightarrow x \rightarrow z \in J)$.

(WJ1.2) $(\forall x, y, z \in A)((x \rightsquigarrow (y \rightarrow z) \wedge y \in J) \Longrightarrow x \rightsquigarrow z \in J)$.

We denote the family of all weak ideals in the pseudo BG-algebra $\mathfrak{A}$ by $\mathfrak{J}_w(A)$.

Let us show, first, that the weak ideal satisfies the condition (J0).

Proposition 3.7. *Any weak ideal J in a pseudo BG-algebra $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ satisfies the condition* (J0).

Proof. Since J is a nonempty subset of A, there exists at least some $y \in A$ such that $y \in J$. If we put $x = y$ and $z = y$ in (WJ1.1), we get $y \to (y \rightsquigarrow y) = y \to 0 = y \in J$ and $y \in J$ in accordance with (pRe) and (pM). From here, according to (WJ1.1), we have $y \to y = 0 \in J$ with respect to (pRe). □

Proposition 3.8. *Every weak ideal in a pseudo BG-algebra $\mathfrak{A}$ is an ideal in $\mathfrak{A}$. This means $\mathfrak{J}_w(A) \subseteq \mathfrak{J}(A)$.*

Proof. Let J be a weak ideal in a pseudo BG-algebra $\mathfrak{A}$. J satisfies the condition (J0), according to Proposition 3.7. If we put $z = 0$ in (WJ1.1) and (WJ1.2), we get (J1.1) and (J1.2) respectively. Therefore, J is an ideal in $\mathfrak{A}$. □

Proposition 3.9. *Every weak ideal in a pseudo BG-algebra is closed.*

Proof. Let J be a weak ideal in a pseudo BG-algebra $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$. If we put $x = 0$ and $z = y$ in (WJ1.1) and (WJ1.2), with respect to (pRe) we get that $0 \to (y \rightsquigarrow y) = 0 \to 0 = 0 \in J$, $0 \rightsquigarrow (y \to y) = 0 \to 0 = 0 \in J$ and $y \in J$ implies $0 \to y \in J$ and $0 \rightsquigarrow 0 \in J$. This means that the weak ideal J is closed. □

Proposition 3.10. *Every weak ideal in a pseudo BG-algebra $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ is a sub-algebra in $\mathfrak{A}$. This means $\mathfrak{J}_w(A) \subseteq \mathfrak{S}(A)$.*

Proof. Let J be a weak ideal in $\mathfrak{A}$. Then J is a closed ideal in $\mathfrak{A}$ by Proposition 3.7 and Proposition 3.6. Let $x, y \in A$ be arbitrary elements such that $x \in J$ and $y \in J$. Then $(x \to y) \rightsquigarrow (0 \to y) = x \in J$, $(x \rightsquigarrow y) \to (0 \rightsquigarrow y) = x \in J$, $0 \to y \in J$ and $0 \rightsquigarrow y \in J$ according to (pBG1.1), (pBG1.2), (Jc1) and (Jc1.2). Thus $x \to y \in J$ and $x \rightsquigarrow y \in J$ with the application (J1.1) and (J1.2(. So, J is a sub-algebra in $\mathfrak{A}$. □

At the end of this subsection, we state the following statement. Using an analogous procedure to the procedure described in the proof of Theorem 3.2, it can be proved:

Theorem 3.7. *The family $\mathfrak{J}_w(A)$ forms a complete lattice.*

Example 3.9. *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ be a pseudo BG-algebra as in Example 3.3. The ideal $J_{11} =: \{0, a, b, c\}$ in $\mathfrak{A}$ is a weak ideal in $\mathfrak{A}$.* □

Example 3.10. *Let $\mathfrak{A} =: (A, \to, \rightsquigarrow, 0)$ be a pseudo BG-algebra as in Example 3.2. The sub-algebra/ideal S_0 in $\mathfrak{A}$ is not a weak ideal in $\mathfrak{A}$ because, for example, we have $a \to (0 \rightsquigarrow c) = a \rightsquigarrow a = 0 \in S_0$ and $0 \in S_0$ but $a \to c = b \notin S_0$.* □

Remark 3.5. *If we relate Proposition 3.6 and Example 3.10, we can conclude that the families $\mathfrak{J}_s(A)$ and $\mathfrak{J}_w(A)$ are mutually distinct.*

4. Final comments

BG-algebra, as an algebraic structure, introduced in 2008 by C. B. Kim and H. S. Kim, is sandwiched between B-algebra and BH-algebra. This class of logical algebras, since its introduction into mathematics, has been studied by several researchers, including this author (see, [12, 13, 14]). While in the article [12], the internal architecture of BG-algebras was considered, in the articles [13, 14] the author was interested in several types of ideals and filters in such logical algebras. In this article, the author considers the pseudo extension of BG-algebra, introduced in [1] by S. A. Bajalan, et al., analyzing the properties of pseudo BG-algebra and finding new properties of both pseudo BG-algebra itself and its substructures. The material presented in this paper, complementing the article [1], increasing our knowledge of BG-algebras, opens the door for further and deeper research into BG-algebras and even into the study of pseudo-BG-algebras. So, for example, one could find more examples of pseudo BG-algebras, study more types of ideals in pseudo BG-algebras, and the like.

Acknowledgments

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