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Colorings of Order Sum Graphs of Groups

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ABSTRACT. The study of graphs emerging from various algebraic structures like groups, rings, vector spaces, etc. is a prominent topic of research in algebraic graph theory. Several graphs have been constructed on groups and one such graph that has been recently defined on finite groups and is studied is the order sum graphs of finite groups. The investigation of different coloring problems in algebraic graphs is an inquisitive topic of research as many algebraic graphs in the literature were constructed to satisfy certain coloring protocols. Therefore, we discuss certain vertex and edge colorings of the order sum graphs of groups, in this article.

1. INTRODUCTION

For terminology in group theory, we refer to [11]. For basic definitions and results in graph theory, see [34] and for further concepts in graph coloring see [5].

The *order of a group* G, denoted by $|G|$, is the number of elements in the group, and if $|G|$ is finite, then we say G is a finite group of order $|G|$. For every $a \in G$, the *order* of a, denoted by $|a|$, is the least positive integer k such that $a^k = e$. Note that, we always denote a finite group of order n by G. A group G is said to be *cyclic* if there exists an element $a \in G$ such that $G = \langle a \rangle$ and a is called the *generator* of G. In view of Cayley's theorem, any cyclic group of order n is isomorphic to the additive group of integer modulo n, denoted by $\mathbb{Z}_n$ and there are $\phi(n)$ generators for the group $\mathbb{Z}_n$, where $\phi(n)$ denotes the *Euler's totient function* that gives the number of relatively prime numbers that are are less than n, for any integer n.

Algebraic graph theory is an emerging research field in which the properties of graphs defined on algebraic structures are studied (*c.f.* [16,18,20,21,23,26,27,31]). In this regard, several graphs have been defined on finite groups, especially based on the properties of the order of elements in the finite group. One among them is the order sum graphs of finite groups, that was introduced in [2] as follows.

Definition 1.1. [2] The *order sum graph* of a finite group G, denoted by $\Gamma_{os}(G)$, is a graph with $V(\Gamma_{os}(G)) = \{v_a : a \in G\}$ and any two distinct vertices $v_x, v_y \in V(\Gamma_{os}(G))$ are adjacent, when $|x| + |y| > |G|$, where $x, y \in G$.

An illustration of the order sum graph associated with a finite group is given in Figure 1.

Graph coloring is the assignment of colors to the entities of a graph such as its vertices or edges, according to certain rules. A *proper vertex coloring* of a graph X is assigning colors to the vertices of X such that no two adjacent vertices receive the same color and and the minimum number of colors used in such a coloring of X is called the *chromatic number* of X, denoted by $\chi(X)$ (see [33]).

In a graph X, if a vertex $v \in V(X)$ is adjacent to all vertices $u \in A$, for some $A \subseteq V(X)$ or $A = \{v\}$, we say that v *dominates* A and A is *dominated* by v. In this context, if $v \notin A$, we say that v *properly dominates* A (ref. [28]).

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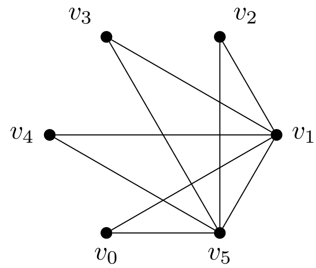

FIGURE 1 The order sum graph of a cyclic group of order 6.

Following the introduction of the order sum graphs of finite groups in [2], different types of domination parameters were discussed for these graphs in [1]. The investigation of different coloring problems in algebraic graphs is an inquisitive topic of research as many algebraic graphs in the literature were constructed to satisfy certain coloring protocols. Also, we see that several coloring and related properties of different algebraic graphs have been studied (*c.f.* [7,10,15,22,24,25,29,32]). Therefore, we discuss certain variants of vertex and edge coloring schemes of the order sum graphs of finite groups, in this article.

2. Vertex Colorings of Order Sum Graphs of Finite Groups

For any finite group G, it was proved that the order sum graph of G is non-empty if and only if $G \cong \mathbb{Z}_n$; $n \geq 2$. By Definition 1.1, it can be seen that vertex corresponding to a generator of G, is a universal vertex of $\Gamma_{os}(G)$. Hence, $\Gamma_{os}(\mathbb{Z}_n) \cong K_n$, when n is prime, and $K_{\phi(n)+1} \subset \Gamma_{os}(\mathbb{Z}_n)$, when n is composite (see [17]).

In view of the structure of the graph $\Gamma_{os}(G)$, we discuss certain vertex coloring schemes that yield us a coloring pattern which is neither the trivial coloring or its chromatic coloring, in this section. In this regard, we first study the equitable coloring of the order sum graphs of finite groups, where an *equitable coloring* of a graph X is a proper vertex coloring of X in which the cardinality of the color classes differ by at most 1 and the minimum number of colors used to obtain an equitable coloring of X is the *equitable chromatic number* of X, denoted by $\chi_e(X)$ (ref. [15]).

Theorem 2.1. *For a finite group G of order n,*

$$\chi_e(\Gamma_{os}(G)) = \begin{cases} 1, & \text{when } G \text{ is acyclic;} \\ \phi(n) + \lceil \frac{n-\phi(n)}{2} \rceil, & \text{otherwise.} \end{cases}$$

Proof. When G is acyclic, it is immediate that $\chi_e(\Gamma_{os}(G)) = 1$, as the order sum graph of G in this case is empty. When G is a cyclic group of order n, all the vertices corresponding to the generators must be given unique colors in any proper coloring of $\Gamma_{os}(G)$, as they are universal vertices. Hence, the remaining $n - \phi(n)$ vertices corresponding to the non-generators of G must be colored such that the cardinality of each color class is at most 2. As $\phi(n)$ is always even, $n - \phi(n)$ is even, when n is even and in this case, exactly two vertices corresponding to the non-generators of G are colored using one color. When n is odd, $n - \phi(n)$ is odd and hence, we need $\frac{n-\phi(n)-1}{2} + 1$ colors to color the vertices corresponding to the non-generators of G. Therefore, $\chi_e(\Gamma_{os}(G)) \geq \phi(n) + \lceil \frac{n-\phi(n)}{2} \rceil$. The assignment of color to two vertices of $\Gamma_{os}(G)$ corresponding to the non-generators of G is possible as all vertices corresponding to the non-generators of G form an independent set in $\Gamma_{os}(G)$; thereby proving the result. $\square$

An $L(2,1)$-*coloring* of a graph X is the assignment of non-negative integers as colors, to the vertices of X such that the colors assigned to adjacent vertices differ by at least 2, and the colors assigned to vertices at distance 2 differ by at least 1. For an $L(2,1)$-coloring c of X, the *c-cap* of X, $\lambda_c(X) = \max\{c(v) : v \in V(X)\}$ and *L-cap* of X, $\lambda_L(X) = \min\{\lambda_c(X)\}$, where the minimum is taken over all possible $L(2,1)$-colorings c of X (*c.f.* [5]).

Theorem 2.2. *For a finite group G of order n,*

$$\lambda_L(\Gamma_{os}(G)) = \begin{cases} 1, & \textit{when } G \textit{ is acyclic;} \\ n+\phi(n)-1, & \textit{otherwise.} \end{cases}$$

Proof. When G is acyclic, $\Gamma_{os}(G)$ is empty and hence assigning the same color to all the vertices of the graph $\Gamma_{os}(G)$ is its $L(2,1)$-coloring. When $\Gamma_{os}(G)$ is connected, the graph $\Gamma_{os}(G)$ has diameter at most 2, and hence all n vertices must be assigned different colors in any $L(2,1)$-coloring of $\Gamma_{os}(G)$. Here, consider an $L(2,1)$-coloring of $\Gamma_{os}(G)$ in which all the vertices in the graph $\Gamma_{os}(G)$ that corresponds to the non-generators of the group G are assigned consecutive integers from 0 to $n-\phi(n)-1$, as they are at distance 2, from every other vertex corresponding to a non-generator of G. As the vertices corresponding to the generators of G are universal vertices in $\Gamma_{os}(G)$, they cannot be assigned consecutive colors in any $L(2,1)$-coloring of $\Gamma_{os}(G)$. Therefore, these vertices are given the colors from $n-\phi(n)+1$ to $n-\phi(n)-1+2\phi(n)$, which gives the required $L(2,1)$-cap of $\Gamma_{os}(G)$. □

A proper vertex coloring of a graph X such that two colors i and j are assigned to $u, v \in V(X)$, when $d(u,v) + |i-j| \geq 1+r$, for some fixed $r \leq diam(X)$, is called *r-radio coloring* of X. The *value* of a radio coloring c of X, $rc_r(c) = \max\{c(v) : v \in V(X)\}$ and the *r-radio chromatic number* of X, $rc_r(X) = \min\{rc_r(c)\}$, where the minimum is taken over all possible r-radio colorings c of X (*c.f.* [5]). When $r = diam(X)$ and when $r = diam(X)-1$, the colorings are respectively called the *radio* and *antipodal colorings* of X, and the corresponding chromatic numbers are called the *radio* and *antipodal numbers* of X, denoted by $rn(X)$ and $an(X)$, respectively (see [5]). Since the diameter of $\Gamma_{os}(G)$ is 2 or ∞, we obtain the following proposition.

Proposition 2.1. *For any G, $rn(\Gamma_{os}(G)) = \lambda_L(\Gamma_{os}(G))$ and $an(\Gamma_{os}(G)) = \chi(\Gamma_{os}(G))$.*

A proper vertex coloring of a graph X is said to be an *exact coloring* of X if every pair of colors appear on exactly one pair of adjacent vertices and the maximum number of colors used to obtain an exact coloring of X is the *exact chromatic number* of X (see [15]).

Proposition 2.2. *The graph $\Gamma_{os}(G)$ admits an exact coloring if and only of $|G|$ is prime.*

Proof. When G is acyclic, it is immediate that the graph $\Gamma_{os}(G)$ does not admit an exact coloring, as $\Gamma_{os}(G)$ is empty. If G is cyclic, the order sum graph of G has at least one universal vertex, say v, and hence all the other $n-1$ vertices of $\Gamma_{os}(G)$ must be given distinct colors so that all edges with v as an end point have unique colors, in its other end point. As all the vertices of $\Gamma_{os}(G)$ are given distinct colors, they have to be adjacent in $\Gamma_{os}(G)$, so that this coloring forms an exact coloring of $\Gamma_{os}(G)$, which happens if and only if $\Gamma_{os}(G)$ is complete; that is, if and only if $|G|$ is prime. □

Corollary 2.1. *The exact chromatic number of the order sum graph of a group of prime order n is n.*

An *r-dominator coloring* of a graph X is a proper vertex coloring of X in which every vertex of X dominates at least r color classes and the *r-dominator chromatic number* of X, denoted by $\chi_d^{(r)}(X)$, is the minimum number of colors used to obtain an r-dominator coloring of X, for a given r (see [19,28]).

Theorem 2.3. *For any finite group G of order n,*

$$\chi_d^{(r)}(\Gamma_{os}(G)) = \begin{cases} \phi(n), & \text{when } G \cong \mathbb{Z}_n, \text{ where } n \text{ is composite, and } 1 \leq r \leq \delta(\Gamma_{os}(G)); \\ n, & \text{otherwise.} \end{cases}$$

Proof. When G is acyclic, the only possible r value for which an r-dominator coloring of $\Gamma_{os}(G)$ exists is $r = 1$. In this coloring, every vertex of $\Gamma_{os}(G)$ is assigned a unique color, so that it dominates its own color class. Hence, $\chi_d^{(r)}(\Gamma_{os}(G)) = n$, for all $1 \leq r \leq \delta(\Gamma_{os}(G)) + 1$. When G is cyclic, we assign a unique color to all $\phi(n)$ universal vertices of $\Gamma_{os}(G)$, and hence any vertex of $\Gamma_{os}(G)$ dominates at least $\phi(n)$ color classes, in any of its proper coloring. Here, when n is prime, $\Gamma_{os}(G) \cong K_n$ and hence $\chi_d^{(r)}(\Gamma_{os}(G)) = n$, for any $1 \leq r \leq \delta(\Gamma_{os}(G)) + 1$, as $\phi(n) = n - 1$. When n is composite, the $n - \phi(n)$ non-universal vertices of $\Gamma_{os}(G)$ form an independent set and hence, in any optimal chromatic coloring, all these vertices are assigned the same color. Hence, in this case, $\chi_d^{(r)}(\Gamma_{os}(G)) = \phi(n)$, for $1 \leq r \leq \delta(\Gamma_{os}(G))$, and when $r = \delta(\Gamma_{os}(G)) + 1$, we must assign a unique color to all the $n - \phi(n)$ non-universal vertices of $\Gamma_{os}(G)$, as these vertices must dominate their own color class to dominate the $(\phi(n) + 1)$-th color class. Hence, $\chi_d^{(r)}(\Gamma_{os}(G)) = n$, when $r = \phi(n) + 1$. □

A proper vertex coloring of a graph X in which every vertex of X dominates (resp. properly dominates) exactly r color classes is called an *efficient r-dominator coloring* (resp. *efficient open r-dominator coloring*) of X and the minimum number of colors used to obtain the respective colorings of X is *efficient r-dominator chromatic number* (resp. *efficient open r-dominator chromatic number*) of G, denoted by $\chi_{effd}^{(r)}(X)$ (resp. $\chi_{efftd}^{(r)}(X)$) (see [28]).

Theorem 2.4. *For any finite group G of order n, the graph $\Gamma_{os}(G)$ admits an efficient r-dominator coloring if and only if $r = \delta(\Gamma_{os}(G)) + 1$ and G is either acyclic or n is prime.*

Proof. When $\Gamma_{os}(G)$ is an empty graph or complete graph, its $(\delta(\Gamma_{os}(G)) + 1)$-dominator coloring is efficient as every vertex dominates exactly one color class in the former case and n color classes in the latter case. Note that these are the values of $\delta(\Gamma_{os}(G)) + 1$, in the respective cases. In the former case, as if $\delta(\Gamma_{os}(G)) = 0$, and in the latter case, as $\phi(n)$ colors are used in any r-dominator coloring of $K_{\phi(n)}$, for all $1 \leq r \leq \delta(\Gamma_{os}(G)) + 1$, it cannot be an efficient r-dominator coloring of $\Gamma_{os}(G)$, for any $r < \delta(\Gamma_{os}(G)) + 1$.

When $\Gamma_{os}(G)$ is a graph with diameter 2, all the universal vertices of $\Gamma_{os}(G)$ dominate all $\phi(n) + 1$ color classes, in any r-dominator coloring of $\Gamma_{os}(G)$, for any $1 \leq r \leq \phi(n)$. Also, when $r = \phi(n) + 1$, all the universal vertices of $\Gamma_{os}(G)$ dominate $n - 1$ color classes, which is efficient if and only if n is prime; leading to a contradiction as $diam(\Gamma_{os}(G)) = 1$, when n is prime. Hence the result. □

Corollary 2.2. *The order sum graph $\Gamma_{os}(\mathbb{Z}_n)$ does not admit an efficient r-dominator coloring for any $1 \leq r \leq \delta(\Gamma_{os}(\mathbb{Z}_n))$, when n is composite.*

Corollary 2.3. *For any group G such that $\Gamma_{os}(\mathbb{Z}_n)$ admits an efficient $(\delta(\Gamma_{os}(G))+1)$-dominator coloring, $\chi_d^{(r)}(\Gamma_{os}(G)) = \chi_{effd}^{(r)}(\Gamma_{os}(G))$, where $r = \delta(\Gamma_{os}(G)) + 1$.*

Proposition 2.3. *For any finite group G of order n, the graph admits an efficient open r-dominator coloring if and only if $r = \phi(n)$ and G is cyclic.*

Proof. When the group G is acyclic, the vertices of the graph $\Gamma_{os}(G)$ cannot properly dominate any color class and hence it does not admit an open dominator coloring, for any r. When $G \cong \mathbb{Z}_n$ and n is prime, in any proper coloring, every vertex properly dominates $n - 1$ other color classes and hence it becomes efficient open dominator coloring if and only if $r = n-1$. When $G \cong \mathbb{Z}_n$ and n is composite, any optimal chromatic coloring of $\Gamma_{os}(G)$ is a $\chi^{(r)}_{efftd}$-coloring of $\Gamma_{os}(G)$, for any $1 \le r \le \phi(n)$. In any such optimal chromatic coloring of $\Gamma_{os}(G)$, every vertex of $\Gamma_{os}(G)$ open dominates $\phi(n)$ color classes and hence it becomes an efficient open dominator coloring if and only if $r = \phi(n)$; thereby proving the result. □

Corollary 2.4. *For any group $G \cong \mathbb{Z}_n$, $\chi^{(\phi(n))}_{efftd}(\Gamma_{os}(G)) = \chi(\Gamma_{os}(G))$.*

An equitable coloring of X in which every vertex $v \in V(X)$ dominates at least one color class is called an *equitable dominator coloring* of X and the minimum number of colors used to obtain such a coloring of X is called the *equitable dominator chromatic number* of X, denoted by $\chi_{ed}(X)$ (ref. [8,9]). As any non-empty $\Gamma_{os}(G)$ has at least one universal vertex and $\Gamma_{os}(G)$, otherwise is empty, we get the following result.

Proposition 2.4. *For any group G,*

$$\chi_{ed}(\Gamma_{os}(G)) = \begin{cases} \chi_e(\Gamma_{os}(G)), & \text{when } G \text{ is cyclic;} \\ \chi^{(1)}_d(\Gamma_{os}(G)), & \text{otherwise.} \end{cases}$$

A *co-coloring* of a graph X is an improper vertex coloring in which every color class must form an independent set in either X or $\overline{X}$ and the minimum number of colors used to obtain a co-coloring of X is the *co-chromatic number* of X, denoted by $\chi_{co}(X)$ (refer to [30]).

Proposition 2.5. *For a finite group G of order n,*

$$\chi_{co}(\Gamma_{os}(G)) = \begin{cases} 2, & \text{when } G \text{ is cyclic and } n \text{ is composite;} \\ 1, & \text{otherwise.} \end{cases}$$

Proof. For any finite group of order n, let $V(\Gamma_{os}(G)) = \{v_i : 0 \le i \le n-1\}$. When $G \cong \mathbb{Z}_n$, let $c : V(\Gamma_{os}(G)) \to \{c_1, c_2\}$ be a coloring such that for all $1 \le i \le n-1$, $c(v_i) = c_1$, when $\gcd(i, n) = 1$ and $c(v_i) = c_2$, otherwise, and $c(v_0) = c_1$. It is a co-coloring as all vertices corresponding to generators along with the vertex corresponding to the identity of $\mathbb{Z}_n$ form a clique in $\Gamma_{os}(G)$; therefore, an independent set in $\overline{\Gamma}_{os}(G)$ and all the other vertices of $\Gamma_{os}(G)$ corresponding to the non-generators of $\mathbb{Z}_n$ form an independent set in $\Gamma_{os}(G)$. This is an optimal co-coloring of $\Gamma_{os}(G)$, as we need at least two colors in any co-coloring of a non-empty, non-complete graph. When n is prime, the coloring c uses exactly 1 color to the all the vertices of $\Gamma_{os}(\mathbb{Z}_n)$. Also, when G is not cyclic, all the vertices of $\Gamma_{os}(G)$ are given the same color, as $\Gamma_{os}(G) \cong \overline{K}_n$ and hence the result. □

3. Edge Colorings of Order Sum Graphs of Finite Groups

Following the vertex coloring schemes of the order sum graphs of finite groups, we investigate certain edge colorings of the same, in this section. We begin by determining the chromatic index χ' of the graph $\Gamma_{os}(G)$, where the chromatic index $\chi'(X)$ of a graph X is the minimum number of colors used to properly color the edges of X. Note that we consider only the graphs $\Gamma_{os}(G)$, when $G \cong \mathbb{Z}_n$, in this section, as the graph $\Gamma_{os}(G)$ is empty, otherwise.

Theorem 3.5. *For any group $G \cong \mathbb{Z}_n$; $n \ge 2$,*

$$\chi'(\Gamma_{os}(G)) = \begin{cases} n, & \text{when } n \ge 3 \text{ is a prime or is odd with } \binom{n-\phi(n)}{2} > \frac{n-1}{2}; \\ n-1, & \text{otherwise.} \end{cases}$$

Proof. Let G be a finite cyclic group isomorphic to $\mathbb{Z}_n$. When $n \geq 3$ is prime, we know that $\Gamma_{os}(G) \cong K_n$ and hence, $\chi'(\Gamma_{os}(G)) = n$, as $\chi'(K_n) = n$, when n is odd. When $n = 2$, $\chi'(\Gamma_{os}(G)) = 1$. When n is composite, as $\Gamma_{os}(G) \cong K_{\phi(n)} + (n - \phi(n))K_1$ with $\Delta(\Gamma_{os}(G)) = n - 1$ and owing to the fact that $\chi'(X_1) \leq \chi'(X_2)$, for any subgraph X_1 of a graph X_2, $\chi'(\Gamma_{os}(G)) \leq n - 1$, when n is even and $\chi'(\Gamma_{os}(G)) \leq n$, when n is odd. Hence, by Vizing's theorem, we obtain that $\chi'(\Gamma_{os}(G)) = n - 1$, when n is even.

In [6], it has been proved that if X is a graph of order $2r + 1$ with $\Delta(X) = 2r$, then $\chi'(X) = \Delta(X)$ if and only if $|E(\overline{X})| \geq r$. When n is an odd composite integer, $|E(\overline{\Gamma}_{os}(G))| = \binom{n-\phi(n)}{2}$. Hence, we deduce that $\chi'(\Gamma_{os}(G)) = n$, when $\binom{n-\phi(n)}{2} > \frac{n-1}{2}$ and $n - 1$, when $\binom{n-\phi(n)}{2} \leq \frac{n-1}{2}$; thereby completing the proof. □

A path in an edge-colored graph X is called a *conflict-free path* if there exists a color that is used only on one of the edges of the path. A path in the graph X in which every edge has a unique color is called a *rainbow path* in X and a path P in X is said to be a *proper path*, if each pair of adjacent edges in P have different colors. In all the cases, if the $u - v$ path considered is a geodesic between any two vertices $u, v \in X$, then the *conflict-free path*, *rainbow path* and the *proper path* are called *conflict-free geodesic*, *rainbow geodesic* and *proper geodesic*, respectively (ref. [3,5]).

An edge-colored graph X is called *conflict-free connected* (resp. *rainbow connected* and *proper path connected*) if there exists a conflict-free path (resp. rainbow path and a proper path) in X between each pair of distinct vertices of X and the minimum number of colors used to obtain a conflict-free coloring, (resp. rainbow coloring and proper path coloring) of X is called the *conflict-free connection number*, (resp. *rainbow connection number* and *proper connection number*) of X, denoted by $cfc(X)$, (resp. $rc(X)$ and $pc(X)$) (*c.f.* [3,5]).

In these notions, if the existence of a $u - v$ path is replaced with the existence of a $u - v$ geodesic between any $u, v \in V(X)$, we obtain the notions of *strong conflict-free connection*, *strong rainbow connection*, and *strong proper path connection* and the graphs which admit such an edge coloring are *strong conflict-free connected*, *strongly rainbow connected*, and *strong proper path connected*, respectively. Also, the minimum number of colors required to make a graph X strong conflict-free connected, strong rainbow connected and strong proper path connected is called the *strong conflict-free connection number*, *strong rainbow connection number* and *strong proper connection number*, denoted by $scfc(X)$, $src(X)$, $spc(X)$, respectively (see [3,5]).

Theorem 3.6. *For any group $G \cong \mathbb{Z}_n$; $n \geq 2$,*

$$cfc(\Gamma_{os}(G)) = pc(\Gamma_{os}(G)) = \begin{cases} 1, & \text{when } n \text{ is prime;} \\ 2, & \text{otherwise.} \end{cases}$$

Proof. For a finite group $G \cong \mathbb{Z}_n$; $n \geq 2$, $\Gamma_{os}(G) \cong K_n$, when n is prime, and hence coloring all the edges with the same color gives a conflict free and a proper path between any two vertices of a K_n. When n is a composite integer, $\Gamma_{os}(G)$ contains a $K_{\phi(n)}$ induced by the vertices that correspond to the generators of $\mathbb{Z}_n$ and the other $n - \phi(n)$ vertices of $\Gamma_{os}(G)$ are adjacent only to the vertices of this $K_{\phi(n)}$. In this case, color the edges in $K_{\phi(n)}$ using a color, say c_1, and the other $\phi(n)(n - \phi(n))$ edges that have only one end vertex in $K_{\phi(n)}$ using a different color, say c_2. In this coloring, between every pair of vertices in the $K_{\phi(n)}$, and between a vertex corresponding to a generator of G and non-generator of G, we have a path of length 1, which is conflict-free and proper. Now, between any two vertices, say v_1 and v_2, in $\Gamma_{os}(G)$ that correspond to two non-generators of G, we take a path $v_1 - u_1 - u_2 - v_2$, where u_1 and u_2 are vertices of $\Gamma_{os}(G)$ that correspond to the generators of G. In this path, the edges v_1u_1 and u_2v_2 are colored with c_2 and u_1u_2

is colored with c_1, in the above mentioned coloring, thereby making the path a conflict-free as well as a proper path of length 3, with two colors. This is an optimal coloring as $diam(\Gamma_{os}(G)) = 2$, when n is composite. $\square$

Theorem 3.7. *For any group $G \cong \mathbb{Z}_n$; $n \geq 2$,*

$$src(\Gamma_{os}(G)) = \begin{cases} 1, & \text{when } n \text{ is a prime;} \\ 2, & \text{when } n = 2^r,\ r > 1; \\ \lceil \sqrt[n-\phi(n)]{\phi(n)} \rceil & 3 \cdot 5 \cdot 7 \nmid n, \text{ or } 3 \cdot 5 \cdot 11 \nmid n \\ \lceil \sqrt[\phi(n)]{n-\phi(n)} \rceil, & \text{otherwise.} \end{cases}$$

Proof. Consider the order sum graph $\Gamma_{os}(G)$ of a finite group G of order n with $V(\Gamma_{os}(G)) = \{u_i : 1 \leq i \leq \phi(n)\} \cup \{v_j : 1 \leq j \leq n - \phi(n)\}$ such that u_i's and v_j's are vertices that correspond to the generators and non-generators of G, respectively. In the graph $\Gamma_{os}(G)$, $d(u_{i_1}, u_{i_2}) = d(u_i, v_j) = 1$, for any distinct $1 \leq i, i_1, i_2 \leq \phi(n)$, $1 \leq j \leq n - \phi(n)$. Hence, the assignment of colors in any manner to these edges will give a rainbow geodesic between these vertices. When n is prime, there exists exactly one non-generator of G and hence $\Gamma_{os}(G) \cong K_n$ implying $src(\Gamma_{os}(G)) = 1$, in this case.

When n is a composite number, there exists at least two non-generators of G and we know that $d(v_{j_1}, v_{j_2}) = 2$, for all $1 \leq j_1, j_2 \leq n - \phi(n)$, and hence $src(\Gamma_{os}(G)) \geq 2$, in this case. As there exists a $K_{\phi(n), n-\phi(n)}$ as a subgraph induced by the set of edges $\{u_i v_j : 1 \leq i \leq \phi(n), 1 \leq j \leq n - \phi(n)\}$ of $\Gamma_{os}(G)$, obtaining a strong rainbow coloring of this $K_{\phi(n), n-\phi(n)}$ gives a rainbow geodesic between any two vertices corresponding to the non-generators of G. This coloring also gives a rainbow geodesic between the vertices u_i and v_j, for any $1 \leq i \leq \phi(n)$, $1 \leq j \leq n - \phi(n)$.

Here, it can be observed that we do not need additional colors apart from the ones used in an optimal strong rainbow coloring of the edge induced subgraph $K_{\phi(n), n-\phi(n)}$ of $\Gamma_{os}(G)$ to color the remaining edges of $\Gamma_{os}(G)$, as they are the edges of a complete graph $K_{\phi(n)}$ induced by the u_i's and coloring them using one of the colors assigned to the edge $u_i v_j$ gives a rainbow geodesic between any two u_i's. Therefore, $src(\Gamma_{os}(\mathbb{Z}_n)) = src(K_{\phi(n), n-\phi(n)})$, for any group of composite order.

In [4], it has been proved that $1 \leq s \leq t$, $src(K_{s,t}) = \lceil \sqrt[s]{t} \rceil$. When n is composite, we know that $2 \leq \phi(n) \leq n-2$, and hence $src(\Gamma_{os}(G)) = src(K_{\phi(n), n-\phi(n)}) = \lceil \sqrt[\phi(n)]{n-\phi(n)} \rceil$ or $src(K_{n-\phi(n), \phi(n)}) = \lceil \sqrt[n-\phi(n)]{\phi(n)} \rceil$, based on the values of $\phi(n)$ and $n - \phi(n)$.

For all even composite n, $\phi(n) \leq n - \phi(n)$, and only when $n = 2^r$, for some $r \geq 2$, $n - \phi(n) = \phi(n) = \frac{n}{2}$. Therefore, as $\lim\limits_{n \to \infty} \lceil \sqrt[\frac{n}{2}]{\frac{n}{2}} \rceil = 2$, we get $src(\Gamma_{os}(\mathbb{Z}_n)) = 2$, when $n = 2^r$, for some $r \geq 2$; and for all other even integers, $src(\Gamma_{os}(\mathbb{Z}_n)) = \lceil \sqrt[\phi(n)]{n-\phi(n)} \rceil$.

When n is odd composite, it has been proved in [17], that $\phi(n) < n - \phi(n)$ if and only if n does not contain $3, 5, 7$, or $3, 5, 11$, simultaneously as its prime factors. Hence, in this case, $src(\Gamma_{os}(\mathbb{Z}_n)) = \lceil \sqrt[\phi(n)]{n-\phi(n)} \rceil$, and for all other odd composite integers, we get $src(\Gamma_{os}(\mathbb{Z}_n)) = \lceil \sqrt[n-\phi(n)]{\phi(n)} \rceil$, thereby completing the proof. $\square$

Owing to the fact that $diam(\Gamma_{os}(G)) \leq 2$, and based on Theorem 3.7, we deduce the following corollaries.

Corollary 3.5. *For any group $G \cong \mathbb{Z}_n$; $n \geq 2$, $scfc(\Gamma_{os}(G)) = spc(\Gamma_{os}(G)) = src(\Gamma_{os}(G))$.*

Corollary 3.6. *For any group $G \cong \mathbb{Z}_n$; $n \geq 2$,*

$$rc(\Gamma_{os}(G)) = \begin{cases} 1, & \text{when } n \text{ is an odd prime;} \\ 2, & \text{when } n = 2^r,\ r > 1; \\ \min\{4, src(\Gamma_{os}(G))\}, & \text{otherwise.} \end{cases}$$

The *proper rainbow connection number* (resp. *strong proper rainbow connection number*) of a connected graph X, denoted by $prc(X)$ (resp. $sprc(X)$), is the minimum number of colors needed to properly color the edges of X to make X rainbow connected (resp. strongly rainbow connected) and such an edge coloring of X is called a *proper rainbow connection coloring* (resp. *strong proper rainbow connection coloring*) of X (ref. [12]).

Theorem 3.8. *For any group $G \cong \mathbb{Z}_n$; $n \geq 2$, $prc(\Gamma_{os}(G)) = sprc(\Gamma_{os}(G)) = \chi'(\Gamma_{os}(G))$.*

Proof. Consider the order sum graph $\Gamma_{os}(G)$ of a finite group $G \cong \mathbb{Z}_n$, $n \geq 2$. In view of the definition of proper rainbow connection coloring and strong proper rainbow connection coloring of graphs, $\chi'(\Gamma_{os}(G)) \leq prc(\Gamma_{os}(G)) \leq sprc(\Gamma_{os}(G))$. Here, when n is a prime, as the order sum graph of G is K_n, the result is immediate. When $n \geq 3$ is composite, $\Gamma_{os}(G) \cong K_{\phi(n)} + (n - \phi(n))K_1$. Consider a proper edge coloring of $\Gamma_{os}(G)$. Here, we must obtain a geodesic of length 2 only between two vertices of $\Gamma_{os}(G)$ corresponding to the $n - \phi(n)$ non-generators of G, as the geodesic between any other pair of vertices in $\Gamma_{os}(G)$ is 1.

Consider a path $v_i - u_1 - v_j$, for any $1 \leq i \neq j \leq n - \phi(n)$, where u_1 is a vertex corresponding to a generator of G and v_i, v_j are two distinct vertices of $\Gamma_{os}(G)$ corresponding to non-generators of G. In an optimal proper edge coloring of $\Gamma_{os}(G)$, all edges incident to u_1 are given distinct color and hence, a rainbow path, which is also a geodesic is obtained between any two distinct vertices of $\Gamma_{os}(G)$ corresponding to non-generators of G; thereby, yielding a required optimal proper rainbow connection and strong proper rainbow connection coloring. □

A path in an edge-colored graph X is a *monochromatic path* if all the edges of the path are colored using the same color and the X is called a *monochromatically connected graph*, if any two vertices of X are connected by a monochromatic path. The *monochromatic connection number* of X, denoted by $mc(X)$, is the maximum number of colors that are used to make X monochromatically connected (see [14]).

Theorem 3.9. *For any group $G \cong \mathbb{Z}_n$; $n \geq 2$,*

$$mc(\Gamma_{os}(G)) = \begin{cases} m, & \text{when } n \text{ is a prime;} \\ m - n + \phi(n), & \text{otherwise,} \end{cases}$$

where $m = |E(\Gamma_{os}(G))| = \binom{\phi(n)}{2} + \phi(n)(n - \phi(n))$.

Proof. For a finite group $G \cong \mathbb{Z}_n$; $n \geq 2$, $mc(\Gamma_{os}(G)) = m$, when n is prime, as $\Gamma_{os}(G) \cong K_n$, in this case, and coloring each edge using a distinct color gives a monochromatic path of length 1 between any two vertices of the order sum graph of G. When n is composite, we deduce that $mc(\Gamma_{os}(G)) \leq m - n + \phi(n) + 1$, owing to the facts that for a graph X which is not k-connected, $mc(X) \leq |E(X)| - |V(X)| + k$, and $\Gamma_{os}(G)$ is not $\phi(n) + 1$ connected.

Let $V(\Gamma_{os}(G)) = \{u_i : 1 \leq i \leq \phi(n)\} \cup \{v_j : 1 \leq i \leq n - \phi(n)\}$, where u_i's and v_j's are vertices that correspond to the generators and non-generators of G, respectively. Consider an edge coloring of $\Gamma_{os}(G)$ such that all the edges $u_1 v_j$; $1 \leq j \leq n - \phi(n)$, are given a color, say c_1, and all the remaining $m - n - \phi(n)$ edges of $\Gamma_{os}(G)$ are given distinct colors $c_2, c_3, \ldots, c_{m-n-\phi(n)+1}$, where $m = |E(\Gamma_{os}(G))| = \binom{\phi(n)}{2} + \phi(n)(n - \phi(n))$. In this edge coloring that uses $m - n - \phi(n) + 1$ colors, between any two vertices v_{j_1}, v_{j_2}, there is a monochromatic path of length 2 and between two vertices u_{i_1}, u_{i_2}, and u_i, v_j, there is a monochromatic path of length 1, and hence yielding the result. □

A set R of edges in a non-trivial, connected, edge-colored graph X is a *rainbow edge cut* (resp. *monochromatic edge cut*) of X if R is an edge cut in which no two edges (resp.

all edges) of R are colored the same. Any such rainbow edge cut (resp. monochromatic edge cut) R in X is called a $u-v$ *rainbow cut* (resp. $u-v$ *monochromatic edge cut*) in X if u and v belong to different components of $X-R$, for some $u,v \in V(X)$. An edge-coloring of X is a *rainbow edge disconnection coloring* (resp. *monochromatic edge disconnection coloring*) if there exists a $u-v$ rainbow edge cut (resp. monochromatic edge cut) in X, for all $u,v \in V(X)$ and the minimum (resp. maximum) number of colors required to obtain a rainbow disconnection coloring (resp. monochromatic edge disconnection coloring) of X is the *rainbow edge disconnection number* (resp. *monochromatic edge disconnection number*) of X, denoted by $rd(X)$ (resp. $md(X)$). (ref. [5] (resp. [13])).

Theorem 3.10. *For any finite group $G \cong \mathbb{Z}_n$; $n \geq 2$, $rd(\Gamma_{os}(G)) = n-1$ and $md(\Gamma_{os}(G)) = 1$.*

Proof. The order sum graph $\Gamma_{os}(G)$ of a finite group $G \cong \mathbb{Z}_n$; $n \geq 2$, has at least two vertices, say u_1 and u_2, of degree $n-1$. The u_1-u_2 edge cut in $\Gamma_{os}(G)$ must consist of all the edges adjacent to either u_1 or u_2, from which we deduce that $rd(\Gamma_{os}(G)) \geq n-1$. By the definition of a rainbow disconnection coloring, it is evident that $rd(X) \leq n-1$, for any connected non-trivial graph of order n. As $rd(K_n) = n-1$, and $\Gamma_{os}(G) \subseteq K_n$, for any n, we deduce that $rd(\Gamma_{os}(G)) = n-1$, for any $n \geq 2$.

By the above mentioned arguments, we must color all the edges adjacent to the $\phi(n)$ universal vertices $u_1, u_2, \ldots, u_{\phi(n)}$, of the graph $\Gamma_{os}(G)$ for group $G \cong \mathbb{Z}_n$, using the same color, to obtain a monochromatic edge cut between any two universal vertices u_i, u_j of $\Gamma_{os}(G)$. Since any edge of $\Gamma_{os}(G)$ has at least one of its end points as a universal vertex, we get $md(\Gamma_{os}(G)) = 1$. □

Following the discussions on different edge colorings of the order sum graphs of finite groups, we conclude the section with the following result on the total coloring of order sum graphs of finite groups, where a *total coloring* of a graph X is the assignment of colors to both the vertices and the edges of X and the minimum number of colors used in such an assignment is called the *total chromatic number* of X, denoted by $\chi_{tot}(X)$ (see [30]).

Theorem 3.11. *For any group $G \cong \mathbb{Z}_n$; $n \geq 3$, $\chi_{tot}(\Gamma_{os}(G)) = n$.*

Proof. By the definition of total coloring of graphs, we can deduce that $\Delta(\Gamma_{os}(G)) + 1 \leq \chi_{tot}(\Gamma_{os}(G))$, for any group $G \cong \mathbb{Z}_n$; $n \geq 2$. When $G \cong \mathbb{Z}_n$; where $n \geq 3$ is odd, the graph $\Gamma_{os}(G) \subseteq K_n$. As $\chi'_{tot}(K_n) = \chi_{tot}(K_n) = n$ if and only if n is odd, we get $\chi_{tot}(\Gamma_{os}(G)) = n$, when n is odd.

It has been proven in [6] that $\chi_{tot}(K_s + tK_1) = \Delta(K_s + tK_1) + 2$, for any $s, t \geq 1$, if and only if $s+t$ is even and $\binom{t}{2} < \lceil \frac{s}{2} \rceil$. Based on this, we can deduce that $\chi_{tot}(\Gamma_{os}(G)) = n+1$ if and only if n is even and $\binom{n-\phi(n)}{2} < \frac{\phi(n)}{2}$. When $n \geq 3$ is even, $1 \leq \phi(n) \leq \frac{n}{2}$ and hence $\frac{n}{2} \leq n - \phi(n) \leq n-2$. In both cases, the best and the worst, the values of $\phi(n)$, we get $\binom{n-\phi(n)}{2} < \frac{\phi(n)}{2}$ if and only if $n < 4$ is even. Hence, we get that $\chi_{tot}(\Gamma_{os}(G)) = n$, for all even $n \geq 3$. □

Corollary 3.7. *For a group $G \cong \mathbb{Z}_n$, $\chi_{tot}(\Gamma_{os}(G)) = n+1$ if and only if $n = 2$.*

4. Conclusion

In this article, we examined various vertex and edge coloring schemes on the order sum graphs of finite groups that have been introduced in the literature. This increases the interest to investigate some improper vertex colorings in the literature, prompting a wide scope of further study on the graph such as investigating various improper vertex and edge colorings, determining their topological indices, etc.

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