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Best proximity discs and elliptic discs in normed spaces

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ABSTRACT. As a generalization of classical fixed point theory, the fixed disc and fixed ellipse problems constitute significant geometric questions. In this paper, we provide some new necessary conditions for the existence of best proximity discs and best proximity ellipses associated with a given self-mapping in metric spaces. The obtained results are supported by illustrative examples.

1. INTRODUCTION

In recent years, the geometric structure of fixed point sets of self-mappings has been intensively studied, particularly through the study of fixed figures associated with self-mappings. The major contribution in this direction was achieved by Özgür and Taş [14, 15, 13], who presented a geometric approach to fixed point theory by exploring conditions under which a given circle is completely contained in the fixed point set of a self-mapping. Their results marked the beginning of a study of fixed figures and, having been developed further in subsequent contributions, thereby stimulating extensive research on fixed geometric figures and their variants, including fixed discs, fixed ellipses, and elliptic discs, both in metric and normed spaces [6, 5, 7, 1, 8, 9, 10, 14, 16, 17, 18, 19, 21]. These investigations have also revealed meaningful applications, notably in neural networks and activation functions.

On the other hand, the theory of best proximity points has emerged as an extension of fixed point theory when fixed points may fail to exist, particularly in the presence of disjoint subsets. For cyclic and non-cyclic mappings defined on pairs of sets, best proximity theory aims to identify points (or pairs of points) that realize the minimal possible distance between the domain and its image. Despite the parallel development of fixed figure theory and best proximity theory, the combination of these two perspectives has remained largely unexplored. This gap motivates the following question: under what conditions can a geometric figure, such as a circle, a disc or an elliptic disc, consist entirely of best proximity points of a given self-mapping?

The present paper addresses this question by introducing and studying the notions of best proximity discs and best proximity elliptic discs in normed spaces. We establish new necessary conditions ensuring that a given disc or elliptic disc constitutes a best proximity figure for cyclic and non-cyclic self-mappings. The obtained results extend several known fixed figure theorems by replacing fixed point assumptions with best proximity conditions. Furthermore, the theoretical findings are supported by illustrative examples that demonstrate the scope and sharpness of the proposed conditions.

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2. Preliminaries

Before delving into the main findings, let us recall some fundamental terminology and notations.

Throughout this paper, let $(\mathcal{X}, d)$ be a normed space endowed with the metric distance $d(x,y) = \|x-y\|$, for all x, $y \in X$. Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of the space $(\mathcal{X}, d)$. A self-mapping T on $\mathcal{A} \cup \mathcal{B}$ is said to be cyclic (resp. non-cyclic) if $T(\mathcal{A}) \subseteq \mathcal{B}$ and $T(\mathcal{B}) \subseteq \mathcal{A}$ (resp. $T(\mathcal{A}) \subseteq \mathcal{A}$ and $T(\mathcal{B}) \subseteq \mathcal{B}$). Given a cyclic (resp. non-cyclic) self-mapping T on $\mathcal{A} \cup \mathcal{B}$, one can formulate a minimization problem to find a pair of points $(p,q) \in \mathcal{A} \times \mathcal{B}$, referred to as the best proximity point of T, such that

$$\|p - T(p)\| = \|q - T(q)\| = \operatorname{dist}(A, B)$$

$$(\text{resp. } T(p) = p,\ T(q) = q \text{ and } \|p-q\| = \operatorname{dist}(\mathcal{A}, \mathcal{B}))\,,$$

where $\operatorname{dist}(\mathcal{A}, \mathcal{B}) = \inf\{d(x,y) : (x,y) \in A \times B\}$.
If two sets $\mathcal{A}$ and $\mathcal{B}$ share some properties, we say that the pair $(\mathcal{A}, \mathcal{B})$ satisfies these properties. For instance, we say that $(\mathcal{A}, \mathcal{B})$ is nonempty, closed or convex if $\mathcal{A}$ and $\mathcal{B}$ are.

Throughout this paper, the notations $\mathcal{A}_0$ and $\mathcal{B}_0$ refer to proximal subsets obtained from $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$\mathcal{A}_0 = \{p \in \mathcal{A} : d(p,q) = dist(\mathcal{A}, \mathcal{B}) \text{ for some } q \in \mathcal{B}\}$$

and

$$\mathcal{B}_0 = \{q \in \mathcal{B} : d(p,q) = dist(\mathcal{A}, \mathcal{B}) \text{ for some } p \in \mathcal{A}\}.$$

Let x_0, c_1 and c_2 be three points of a normed space $(\mathcal{X}, \|.\|)$ and consider T as a self-mapping on $\mathcal{X}$. We denote the metric by d associated with $\|.\|$. To begin, let us recall the following definitions:

- the sets $\mathcal{C}_{x_0,r} = \{x \in \mathcal{X} : d(x,x_0) = r\}$ and $\mathcal{D}_{x_0,r} = \{x \in \mathcal{X} : d(x,x_0) \leqslant r\}$ denote the circle and the disc of $\mathcal{X}$ respectively, with center x_0 and radius $r \geqslant 0$;
- the sets $\mathcal{E}_r(c_1,c_2) = \{x \in \mathcal{X} : d(x,c_1) + d(x,c_2) = r\}$ and $\mathcal{EL}_r(c_1,c_2) = \{x \in \mathcal{X} : d(x,c_1) + d(x,c_2) \leqslant r\}$ denote the ellipse and the elliptic disc of $\mathcal{X}$ respectively, with foci c_1 and c_2.

Let $T : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a mapping.

- We define a disc $\mathcal{D}_{x_0,r}$ as the best proximity disc of T if $d(u, Tu) = dist(\mathcal{A}, \mathcal{B})$, for all $u \in \mathcal{D}_{x_0,r}$.
- We define an elliptic disc $\mathcal{EL}_r(c_1,c_2)$ as the best proximity elliptic disc of T if $d(u, Tu) = dist(\mathcal{A}, \mathcal{B})$, for all $u \in \mathcal{EL}_r(c_1,c_2)$.

Definition 2.1. *Let $\mathcal{X}$ be a normed space and let $T : \mathcal{X} \longrightarrow \mathcal{X}$ be a self-mapping. A nonempty subset $F \subset \mathcal{X}$ is said to be a fixed figure associated with the mapping T if*

$$F \subset Fix(T).$$

In particular, when F is a disc, an ellipse, or an elliptic disc defined by metric conditions, it is called a fixed disc, fixed ellipse, or fixed elliptic disc of T, respectively.

3. Main results

In the sequel, it is assumed that $\mathcal{A}$ and $\mathcal{B}$ are disjoint, that is, $\mathcal{A} \cap \mathcal{B} = \emptyset$.

Theorem 3.1. *Consider two nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ of a normed space $\mathcal{X}$ and a non-cyclic mapping $T : \mathcal{A} \cup \mathcal{B} \longrightarrow \mathcal{A} \cup \mathcal{B}$. Let $\mathcal{D}_{x_0,r_1}$ and $\mathcal{D}_{y_0,r_2}$ be two discs in $\mathcal{A}$ and $\mathcal{B}$, respectively, such that $\mathcal{A}_0 \cap \mathcal{C}_{x_0,r_1} \neq \emptyset$ and $\mathcal{B}_0 \cap \mathcal{C}_{y_0,r_2} \neq \emptyset$.*
Denote $\Delta_{x,y} := \max\{d(x,Tx), d(y,Ty)\}$ and

$$f_\beta(x,y) := (2-\beta)[d(x,x_0)+d(y,y_0)] + \beta[d(x_0,Tx)+d(y_0,Ty)],$$

for every $(x,y) \in \mathcal{A}\times\mathcal{B}$ *and every* $\beta \geq 0$. *If there exists a real number* $\beta \in [0,\frac{1}{2})$ *such that:*

(C_1) $\forall (x,y)\in\mathcal{A}\times\mathcal{B},\ \Delta_{x,y} > 0 \Longrightarrow \Delta_{x,y} \leqslant f_\beta(x,y) - 2(r_1+r_2)$,

(C_2) *for every* $(x,y) \in \mathcal{C}_{x_0,r_1}\times\mathcal{C}_{y_0,r_2}$ *and* $(u,v)\in\mathcal{A}\times\mathcal{B}$,

$$d(x,v) = d(u,y) = dist(\mathcal{A},\mathcal{B}) \Longrightarrow \begin{cases} \max\{d(u,x_0),d(v,y_0)\} \leqslant \frac{1}{4}f_\beta(x,y) \\ d(u,x_0)\geqslant r_1 \text{ and } d(v,y_0)\geqslant r_2, \end{cases}$$

then

(i) $\mathcal{D}_{x_0,r_1}$ *and* $\mathcal{D}_{y_0,r_2}$ *are fixed discs of* T;

(ii) $dist(\mathcal{C}_{x_0,r_1},\mathcal{C}_{y_0,r_2}) = dist(\mathcal{D}_{x_0,r_1},\mathcal{D}_{y_0,r_2}) = dist(\mathcal{A},\mathcal{B})$.

Proof.

(i) Let $(x,y)\in\mathcal{D}_{x_0,r_1}\times\mathcal{D}_{y_0,r_2}$ and suppose that $\Delta_{x,y}>0$. Then:

$$\begin{aligned}\Delta_{x,y} &\leqslant \beta(d(x_0,Tx)+d(y_0,Ty)) - \beta(r_1+r_2)\\ &\leqslant \beta\left(d(x_0,x)+d(x,Tx)+d(y_0,y)\right)+d(y,Ty)) - \beta(r_1+r_2)\\ &\leqslant 2\beta\max\{d(x,Tx),d(y,Ty)\}.\end{aligned}$$

Hence, $(1-2\beta)\Delta_{x,y}\leqslant 0$, which is a contradiction since $0\leq\beta<\frac{1}{2}$. Thus, $\Delta_{x,y}=0$ and, consequently, $Tx=x$ and $Ty=y$.

(ii) Since $\mathcal{A}_0\cap\mathcal{C}_{x_0,r_1}\neq\emptyset$ and $\mathcal{B}_0\cap\mathcal{C}_{y_0,r_2}\neq\emptyset$, there exist $(x,y)\in\mathcal{C}_{x_0,r_1}\times\mathcal{C}_{y_0,r_2}$ and $(u,v)\in\mathcal{A}\times\mathcal{B}$ such that $d(x,v)=dist(\mathcal{A},\mathcal{B})$ and $d(y,u)=dist(\mathcal{A},\mathcal{B})$. Thus, according to (i), one has $Tx=x$ and $Ty=y$. As a consequence,

$$\begin{aligned}\frac{1}{4}f_\beta(x,y) &= \frac{2-\beta}{4}[d(x,x_0)+d(y,y_0)] + \frac{\beta}{4}[d(x_0,x)+d(y_0,y)]\\ &\leqslant \frac{1}{2}(r_1+r_2).\end{aligned}$$

Hence, from the condition (C_2), one has $\max\{d(u,x_0),d(v,y_0)\}\leqslant\frac{1}{2}(r_1+r_2)$. Also, the condition (C_2) implies that $d(u,x_0)\geqslant r_1$ and $d(v,y_0)\geqslant r_2$.
If $d(u,x_0)>r_1$ or $d(v,y_0)>r_2$, then we will have

$$\frac{r_1+r_2}{2}\leqslant\max\{r_1,r_2\}<\max\{d(u,x_0),d(v,y_0)\}\leqslant\frac{1}{2}(r_1+r_2),$$

which is a contradiction. Hence, $d(u,x_0)=r_1$ and $d(v,y_0)=r_2$, which means that $u\in\mathcal{C}_{x_0,r_1}$ and $v\in\mathcal{C}_{y_0,r_2}$. Therefore,

$$dist(A,B)\leqslant dist(\mathcal{D}_{x_0,r_1},\mathcal{D}_{y_0,r_2})\leqslant dist(\mathcal{C}_{x_0,r_1},\mathcal{C}_{y_0,r_2})\leqslant d(x,v) = dist(\mathcal{A},\mathcal{B}).$$

□

Remark 3.1. *Based on the reasoning provided in* (*i*) *of Theorem 3.1, we can establish that, for every* $(x,y)\in\mathcal{C}_{x_0,r_1}\times\mathcal{C}_{y_0,r_2}$ *and* $(u,v)\in\mathcal{A}\times\mathcal{B}$, *we have*

$$\begin{cases} d(x,v) = dist(\mathcal{A},\mathcal{B})\\ d(y,u) = dist(\mathcal{A},\mathcal{B})\end{cases} \Longrightarrow (u,v)\in\mathcal{C}_{x_0,r_1}\times\mathcal{C}_{y_0,r_2}.$$

Example 3.1. *Suppose that the space* $\mathbb{R}^2$ *is equipped with the Euclidean norm* $\|.\|_2$. *Let* $x_0 = (-2,1)$, $y_0=(2,1)$ *and* $r_1=r_2=1$. *Consider the subsets* $\mathcal{A}$ *and* $\mathcal{B}$ *of* $\mathbb{R}^2$ *defined by* $\mathcal{A} =$

$\big(\mathcal{D}_{x_0,1} \cup \{x \in \mathbb{R}^2 : 3 \leqslant d(x, x_0) \leqslant 4\}\big) \cap ((-\infty, -1] \times [0, +\infty))$ *and* $\mathcal{B}$ *is the symmetric image of* $\mathcal{A}$ *with respect to the* y*-axis. We define the even mapping* $T : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ *on* $\mathcal{A}$ *as follows:*

$$Tx = \begin{cases} (3\cos(\theta) - 2, 3\sin(\theta) + 1) & ; \text{if } x = (r\cos(\theta) - 2, r\sin(\theta) + 1) \in \mathcal{A}, \\ x & ; \text{if } x \in \mathcal{D}_{x_0,r_1}, \end{cases}$$

where $3 \leqslant r \leqslant 4$ *and* $\theta \in \mathbb{R}$.

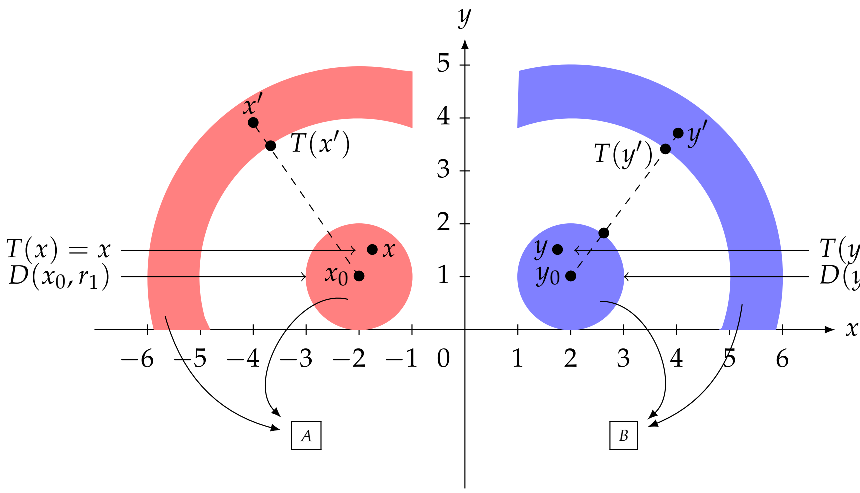

FIGURE 1. The geometric figures of Example 3.1

We have $\mathcal{A}_0 = \{(-1, 1)\} \cup (\{x = (x_1, x_2) \in \mathbb{R}^2 : 3 \leqslant d(x, x_0) \leqslant 4 \text{ and } x_1 = -1\})$*,* B *is its symmetry with respect to the* y*-axis,* $dist(\mathcal{A}, \mathcal{B}) = 2$ *and* T *is non-cyclic.*
Firstly, $Tx = x$ *and* $Ty = y$*, for every* $(x, y) \in \mathcal{D}_{x_0,r_1} \times \mathcal{D}_{y_0,r_2}$*. Let* $\beta \in [0, \frac{1}{2})$.

- *Verification of the condition* (C_1).
  Let $(x, y) \in A \times B$ *such that* $\max\{d(x, Tx), d(y, Ty)\} > 0$*. Without loss of generality, we can suppose that* $\max\{d(x, Tx), d(y, Ty)\} = d(x, Tx)$*. In this case, one has*

$$\begin{aligned} f_\beta(x, y) - 2(r_1 + r_2) &\geqslant (2 - \beta)d(x_0, x) + \beta d(x_0, Tx) - 4 \\ &= (2 - \beta)(d(x_0, Tx) + d(Tx, x)) + \beta d(x_0, Tx) - 4 \\ &\geqslant (2 - \beta)d(Tx, x) \\ &> d(Tx, x), \text{ because } 2 - \beta > 1. \end{aligned}$$

- *Verification of the condition* (C_2).
  Let $(x, y) \in \mathcal{C}_{x_0,r_1} \times \mathcal{C}_{y_0,r_2}$ *and* $(u, v) \in \mathcal{A} \times \mathcal{B}$ *such that* $d(x, v) = dist(\mathcal{A}, \mathcal{B})$ *and* $d(y, u) = dist(\mathcal{A}, \mathcal{B})$*. Note that one possible case is* $u = x = (-1, 1)$ *and* $v = y = (1, 1)$*. Hence,*

$$\begin{cases} d(u, x_0) = d(v, y_0) = 1 \\ \max\{d(u, x_0), d(v, y_0)\} \leq \dfrac{1}{4} f_\beta(x, y) = 2. \end{cases}$$

Therefore, all assumptions of Theorem 3.1 are satisfied for the mapping T *and the discs* $\mathcal{D}_{x_0,r_1}$ *and* $\mathcal{D}_{y_0,r_2}$*. Consequently, the discs* $\mathcal{D}_{x_0,r_1}$ *and* $\mathcal{D}_{y_0,r_2}$ *are fixed discs of* T*, and* $dist\left(\mathcal{D}_{x_0,r_1}, \mathcal{D}_{y_0,r_2}\right) = dist(\mathcal{A}, \mathcal{B})$.

Example 3.2. *Equip $\mathbb{R}^2$ with the metric d defined by $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Let $\mathcal{A} = \mathcal{D}_{x_0,1} \cup ([-3,-1] \times [4,6])$ and $\mathcal{B} = \mathcal{D}_{y_0,1} \cup ([1,3] \times [4,6])$, where $x_0 = (-2,1)$, $y_0 = (2,1)$, and consider the even mapping $T : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ defined on $\mathcal{A}$ by:*

$$Tx = \begin{cases} x \text{ if } x \in \mathcal{D}_{x_0,1} \\ (x_1, 4) \text{ if } x = (x_1, x_2) \in \mathcal{A} \setminus \mathcal{D}_{x_0,1}. \end{cases}$$

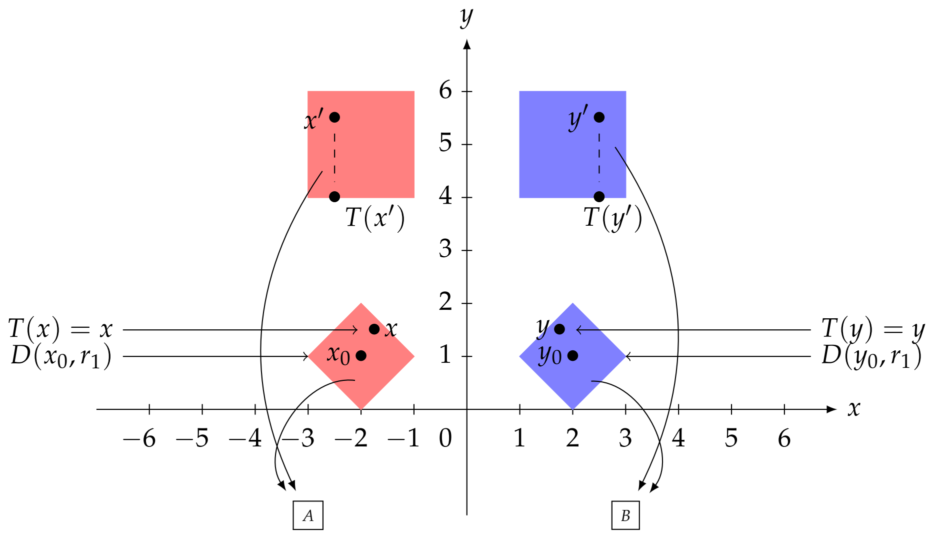

FIGURE 2. The geometric figures of Example 3.2

It is clear that T is non-cyclic and dist$(\mathcal{A}, \mathcal{B}) = 2$. Moreover,

$$\mathcal{A}_0 = \{(-1,1)\} \cup (\{-1\} \times [4,6]) \text{ and } \mathcal{B}_0 = \{(1,1)\} \cup (\{1\} \times [4,6]).$$

Let $\beta \in [0, \frac{1}{2})$.

- *Verification of the condition (C_1).*
  Let $(x,y) \in \mathcal{A} \times \mathcal{B}$ such that $\max\{d(x,Tx), d(y,Ty)\} > 0$. Without loss of generality, we can suppose that $\max\{d(x,Tx), d(y,Ty)\} = d(x,Tx)$. In this case, we have $d(x,Tx) = x_2 - 4$ and

$$\begin{aligned} f_\beta(x,y) - 2(r_1 + r_2) &\geqslant (2-\beta) d((-2,1),(x_1,x_2)) + \beta d((-2,1),(x_1,4)) - 4 \\ &= (2-\beta)(x_2 - 1) + 2|x_1 + 2| + 3\beta - 4 \\ &\geqslant (2-\beta)x_2 + 4\beta - 6 \\ &= x_2 - 4 + (1-\beta)(x_2 - 2) + 2\beta \\ &\geqslant x_2 - 4 \text{ , because } x_2 \geq 4 \\ &= \max\{d(x,Tx), d(y,Ty)\}. \end{aligned}$$

- *Verification of the condition (C_2).*
  Note that the elements $(x,y) \in \mathcal{C}_{x_0,1} \times \mathcal{C}_{y_0,1}$ and $(u,v) \in \mathcal{A} \times \mathcal{B}$, where $u = x = (-1,1)$ and $v = y = (1,1)$, are the unique elements satisfying the relation $d(x,v) =$

$d(y,u) = dist(\mathcal{A},\mathcal{B})$. *Thus,* $\max\{d(u,x_0), d(v,y_0)\} = \dfrac{1}{4}f_\beta(x,y) = 1$ *and therefore* $\max\{d(u,x_0), d(v,y_0)\} \leqslant \dfrac{1}{4}f_\beta(x,y)$.

Hence, all assumptions of Theorem 3.1 are satisfied for the mapping T *and the discs* $\mathcal{D}_{x_0,1}$ *and* $\mathcal{D}_{y_0,1}$. *Therefore, the discs* $\mathcal{D}_{x_0,1}$ *and* $\mathcal{D}_{y_0,1}$ *are fixed discs of* T, *and*

$$dist\left(\mathcal{D}_{x_0,r_1}, \mathcal{D}_{y_0,r_2}\right) = dist(\mathcal{A},\mathcal{B}).$$

Remark 3.2. *We give an example where the mapping* $T : \mathcal{A}\cup\mathcal{B} \to \mathcal{A}\cup\mathcal{B}$ *satisfies the condition* $(C2)$ *but does not satisfy the condition* $(C1)$, *and the result* (i) *of Theorem 3.1 does not hold for* T.

Example 3.3. *Suppose that the space* $\mathbb{R}$ *is equipped with the standard absolute value distance metric* $|\cdot|$. *Let* $\mathcal{A} = \mathcal{D}_{x_0,\frac{1}{2}} \cup [-3,-2)$ *and* $\mathcal{B} = \mathcal{D}_{y_0,\frac{1}{2}} \cup (2,3])$, *where* $x_0 = -\frac{3}{2}$, $y_0 = \frac{3}{2}$, *and consider the even mapping* $T : \mathcal{A}\cup\mathcal{B} \to \mathcal{A}\cup\mathcal{B}$ *defined on* $\mathcal{A}$ *by:*

$$Tx = \begin{cases} x \text{ if } x \in \mathcal{D}_{x_0,\frac{1}{2}} \setminus \{-2\} \\ -1 \text{ if } x \in [-3,-2]. \end{cases}$$

It is clear that T *is non-cyclic and* $dist(\mathcal{A},\mathcal{B}) = 2$, $\mathcal{A}_0 = \{-1\}$ *and* $\mathcal{B}_0 = \{1\}$. *Let* $\beta \in [0,\frac{1}{2})$ *and consider* $(x,y) \in \mathcal{D}_{x_0,\frac{1}{2}} \times \mathcal{D}_{x_0,\frac{1}{2}}$ *and* $(u,v) \in \mathcal{A}\times\mathcal{B}$ *such that* $d(u,y) = d(v,x) = dist(\mathcal{A},\mathcal{B})$. *Clearly,* $x = u = -1$ *and* $y = v = 1$. *Then,* $d(x_0,u) = d(y_0,v) = \frac{1}{2}$ *and* $\dfrac{1}{4}f_\beta(x,y) = \frac{1}{2} = \max\{d(x_0,u), d(y_0,v)\}$. *Hence,* T *satisfies the condition* $(C2)$. *However,* T *does not fix the discs* $\mathcal{D}_{x_0,\frac{1}{2}}$ *and* $\mathcal{D}_{y_0,\frac{1}{2}}$.

Theorem 3.2. *Consider two nonempty subsets* $\mathcal{A}$ *and* $\mathcal{B}$ *of a normed space* $\mathcal{X}$ *and a non-cyclic mapping* $T : \mathcal{A}\cup\mathcal{B} \to \mathcal{A}\cup\mathcal{B}$. *Let* $\mathcal{EL}_{r_1}(c_1,c_2)$ *and* $\mathcal{EL}_{r_2}(c_1',c_2')$ *be two elliptic discs in* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively such that* $\mathcal{A}_0 \cap \mathcal{E}_{r_1}(c_1,c_2) \neq \emptyset$ *and* $\mathcal{B}_0 \cap \mathcal{E}_{r_2}(c_1',c_2') \neq \emptyset$, *and denote* $\Delta_{x,y} := \max\{d(x,Tx), d(y,Ty)\}$, $\Delta'_{u,v} := \max\{d(u,c_1)+d(u,c_2), d(v,c_1')+d(v,c_2')\}$ *and*

$$\begin{aligned} g_\beta(x,y) :=& (2-\beta)\left[d(x,c_1)+d(x,c_2)+d(y,c_1')+d(y,c_2')\right] \\ &+ \beta\left[d(c_1,Tx)+d(c_2,Tx)+d(c_1',Ty)+d(c_2',Ty)\right], \end{aligned}$$

for every $(x,y) \in \mathcal{A}\times\mathcal{B}$ *and every* $\beta > 0$. *If there exists a real number* $\beta \in (0,\frac{1}{4})$ *such that*

(C_1) $\forall (x,y) \in \mathcal{A}\times\mathcal{B}$, $\Delta_{x,y} > 0 \Rightarrow \Delta_{x,y} \leqslant g_\beta(x,y) - 2(r_1+r_2)$,
(C_2) *for all* $(x,y) \in \mathcal{E}_{r_1}(c_1,c_2) \times \mathcal{E}_{r_2}(c_1',c_2')$ *and* $(u,v) \in \mathcal{A}\times\mathcal{B}$,

$$d(x,v) = d(y,u) = dist(\mathcal{A},\mathcal{B}) \Longrightarrow \begin{cases} \Delta'_{u,v} \leqslant \frac{1}{4}g_\beta(x,y) \\ d(u,c_1)+d(u,c_2) \geqslant r_1 \\ d(v,c_1')+d(v,c_2') \geqslant r_2 \end{cases},$$

then

(i) *the elliptic discs* $\mathcal{EL}_{r_1}(c_1,c_2)$ *and* $\mathcal{EL}_{r_2}(c_1',c_2')$ *are fixed elliptic discs of* T,
(ii) $dist(\mathcal{E}_{r_1}(c_1,c_2), \mathcal{E}_{r_2}(c_1',c_2')) = dist(\mathcal{EL}_{r_1}(c_1,c_2), \mathcal{EL}_{r_2}(c_1',c_2')) = dist(\mathcal{A},\mathcal{B})$.

Proof.

(i) Let $(x,y) \in \mathcal{EL}_{r_1}(c_1,c_2) \times \mathcal{EL}_{r_2}(c'_1,c'_2)$ and assume that $\Delta_{x,y} > 0$. By the conditions (C_1), one has

$$\begin{aligned}\Delta_{x,y} &\leqslant \beta\left[(d(c_1,Tx)+d(c_2,Tx))+(d(c'_1,Ty)+d(c'_2,Ty))\right]-\beta(r_1+r_2)\\ &\leqslant \beta\left[(d(c_1,x)+d(c_2,x))+(d(c'_1,y)+d(c'_2,y))\right]\\ &\quad +2\beta(d(x,Tx)+d(y,Ty))-\beta(r_1+r_2)\\ &\leqslant 4\beta\max\{d(x,Tx),d(y,Ty)\}.\end{aligned}$$

Then, $(1-4\beta)\max\{d(x,Tx),d(y,Ty)\} \leqslant 0$, which is a contradiction, because $0<\beta<\frac{1}{4}$. Thus, $Tx = x$ and $Ty = y$.

(ii) As $\mathcal{A}_0 \cap \mathcal{E}_{r_1}(c_1,c_2) \neq \emptyset$ and $\mathcal{B}_0 \cap \mathcal{E}_{r_1}(c'_1,c'_2) \neq \emptyset$, there exist $(u,v) \in \mathcal{A}\times\mathcal{B}$ and $(x,y) \in \mathcal{E}_{r_1}(c_1,c_2)\times\mathcal{E}_{r_1}(c'_1,c'_2)$ for which $d(x,v) = d(y,u) = dist(\mathcal{A},\mathcal{B})$. Hence, $Tx = x$ and $Ty = y$. According to the condition (C_2), one has

$$\begin{aligned}\Delta_{u,v} &\leqslant \frac{1}{4}g_\beta(x,y)\\ &= \frac{1}{2}\left[(d(c_1,x)+d(c_2,x))+(d(c'_1,y)+d(c'_2,y))\right]\\ &\leqslant \frac{1}{2}(r_1+r_2).\end{aligned}$$

Also, (C_2) implies that $d(u,c_1)+d(u,c_2) \geqslant r_1$ and $d(v,c'_1)+d(v,c'_2) \geqslant r_2$. If $d(u,c_1)+d(u,c_2) > r_1$ or $d(v,c'_1)+d(v,c'_2) > r_2$, then we will have

$$\frac{r_1+r_2}{2} \leqslant \max\{r_1,r_2\} < \Delta'_{u,v} \leqslant \frac{1}{2}(r_1+r_2),$$

which is a contradiction. Hence, $d(u,c_1)+d(u,c_2) = r_1$ and $d(v,c'_1)+d(v,c'_2) = r_2$, which means that $u \in \mathcal{EL}_{r_1}(c_1,c_2)$ and $v \in \mathcal{EL}_{r_2}(c'_1,c'_2)$, and

$$dist(\mathcal{A},\mathcal{B}) = dist(\mathcal{EL}_{r_1}(c_1,c_2),\mathcal{EL}_{r_2}(c'_1,c'_2)) = dist(\mathcal{E}_{r_1}(c_1,c_2),\mathcal{E}_{r_2}(c'_1,c'_2)).$$

□

Remark 3.3. *In the proof (i) of Theorem 3.2, we have shown that, for all $(x,y) \in \mathcal{E}_{r_1}(c_1,c_2)\times\mathcal{E}_{r_2}(c'_1,c'_2)$ and $(u,v) \in \mathcal{A}\times\mathcal{B}$, one has*

$$\begin{cases} d(x,v) = dist(\mathcal{A},\mathcal{B})\\ d(y,u) = dist(\mathcal{A},\mathcal{B})\end{cases} \Longrightarrow (u,v) \in \mathcal{E}_{r_1}(c_1,c_2)\times\mathcal{E}_{r_2}(c'_1,c'_2).$$

Example 3.4. *Let $\mathcal{A} = \{x \in \mathbb{Q} : x \leqslant -6\} \cup [-\frac{9}{2},-3]$, $\mathcal{B} = \{x \in \mathbb{Q} : x \geqslant \frac{5}{2}\} \cup [0,\frac{3}{2}]$ and consider the elliptic discs of $\mathbb{R}$ defined by*

$$\mathcal{EL}_{r_1}(c_1,c_2) = \{x \in \mathcal{A} : |x-c_1|+|x-c_2| \leqslant r_1\} \text{ and}$$
$$\mathcal{EL}_{r_2}(c'_1,c'_2) = \{x \in \mathcal{B} : |x-c'_1|+|x-c'_2| \leqslant r_2\},$$

where $c_1 = -\frac{9}{2}$, $c_2 = -3$, $c'_1 = 0$, $c'_2 = \frac{3}{2}$, and $r_1 = r_2 = \frac{3}{2}$. It is obvious that $\mathcal{EL}_{r_1}(c_1,c_2) = \mathcal{E}_{r_1}(c_1,c_2) = [-\frac{9}{2},-3]$ and $\mathcal{EL}_{r_2}(c'_1,c'_2) = \mathcal{E}_{r_2}(c'_1,c'_2) = [0,\frac{3}{2}]$. Define the self-mapping $T : \mathcal{A}\cup\mathcal{B} \to \mathcal{A}\cup\mathcal{B}$ as follows:

$$Tx = \begin{cases} x & , \text{ if } x \in [-\frac{9}{2},-3]\cup[0,\frac{3}{2}],\\ 2x & , \text{ if } x \in (\mathcal{A}\cup\mathcal{B})\setminus([-\frac{9}{2},-3]\cup[0,\frac{3}{2}]).\end{cases}$$

It is easy to see that T is non-cyclic, $\mathcal{A}_0 = \{-3\}$, $\mathcal{B}_0 = \{0\}$, and $dist(\mathcal{A},\mathcal{B}) = 3$. Moreover, if $(x,y) \in \mathcal{EL}_{r_1}(c_1,c_2)\times\mathcal{EL}_{r_2}(c'_1,c'_2)$, then $Tx = x$ and $Ty = y$. Let $\beta \in [0,\frac{1}{4})$.

- *Verification of the condition (C_1).*

Consider $(x,y) \in \mathcal{A} \times \mathcal{B}$ *with* $\max\{d(x,Tx), d(y,Ty)\} > 0$, *where* $d(a,b) = |a-b|$, *for all* $a,b \in \mathbb{R}$. *Observe that, if* $x \in \mathcal{A}$ *then*

$$\begin{aligned}3 &\leqslant 2\left[d(x,c_1) + d(x,c_1)\right] \\ &\leqslant 2\left[d(x,c_1) + d(x,c_2)\right] + \beta\left[d(c_1,Tx) + d(c_2,Tx) - d(c_1,x) - d(c_2,x)\right] \\ &= (2-\beta)\left[d(x,c_1) + d(x,c_2)\right] + \beta\left[d(c_1,Tx) + d(c_2,Tx)\right].\end{aligned}$$

Similarly,

$$3 \leqslant (2-\beta)\left[d(y,c_1') + d(y,c_2')\right] + \beta\left[d(c_1',Ty) + d(c_2',Ty)\right].$$

Case 1: *If* $\max\{|x-Tx|, |y-Ty|\} = |x-Tx|$.
In this case, we have $\max\{|x-Tx|, |y-Ty|\} = |x-Tx| = -x \geqslant 6$ *and*

$$\begin{aligned}g_\beta(x,y) - 2(r_1+r_2) &\geqslant (2-\beta)(|x+\frac{9}{2}| + |x+3|) + \beta(|2x+\frac{9}{2}| + |2x+3|) - 3 \\ &= (-4-2\beta)x - 18 \\ &\geqslant -x = \max\{d(x,Tx), d(y,Ty)\}.\end{aligned}$$

Case 2: *If* $\max\{|x-Tx|, |y-Ty|\} = |y-Ty|$.
In this situation, we have $\max\{|x-Tx|, |y-Ty|\} = |y-Ty| = y \geqslant \frac{5}{2}$ *and:*

$$\begin{aligned}g_\beta(x,y) - 2(r_1+r_2) &\geqslant (2-\beta)(|y| + |y-\frac{3}{2}|) + \beta(2y + |2y-\frac{3}{2}|) - 3 \\ &= (4+2\beta)y - 3 \\ &\geqslant y = \max\{d(x,Tx), d(y,Ty)\}.\end{aligned}$$

• *Let us ensure the validity of the condition* (C_2).
Consider $(u,v) \in \mathcal{A} \times \mathcal{B}$ *and* $(x,y) \in \mathcal{E}_{r_1}(c_1,c_2) \times \mathcal{E}_{r_2}(c_1',c_2')$ *such that*

$$d(x,v) = dist(\mathcal{A},\mathcal{B}) = 3 \text{ and } d(y,u) = dist(\mathcal{A},\mathcal{B}) = 3.$$

There exists a unique couple (u,v) *for which* $u = x = -3$ *and* $v = y = 0$. *Therefore,* $\Delta_{u,v} = \max\{d(u,c_1) + d(u,c_2), d(v,c_1') + d(v,c_2')\} = \frac{3}{2} = \frac{1}{4}g_\beta(x,y)$. *Furthermore,* $d(u,c_1) + d(u,c_2) = \frac{3}{2}$ *and* $d(v,c_1') + d(v,c_2') = \frac{3}{2}$.
Thus, in accordance with Theorem 3.2, the elliptic discs $\mathcal{EL}_{r_1}(c_1,c_2)$ *and* $\mathcal{EL}_{r_2}(c_1',c_2')$ *are fixed elliptic discs of* T, *and*

$$dist(\mathcal{E}_{r_1}(c_1,c_2), \mathcal{E}_{r_2}(c_1',c_2')) = dist(\mathcal{EL}_{r_1}(c_1,c_2), \mathcal{EL}_{r_2}(c_1',c_2')) = dist(\mathcal{A},\mathcal{B}) = 3.$$

Theorem 3.3. *Consider two nonempty subsets* $\mathcal{A}$ *and* $\mathcal{B}$ *of a normed space* $\mathcal{X}$ *and a cyclic mapping* $T : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$. *Let* $\mathcal{D}_{x_0,r_1}$ *and* $\mathcal{D}_{y_0,r_2}$ *be two discs in* $\mathcal{A}$ *and* B *respectively. Denote* $\Delta_{x,y,u,v} = \max\{d(u,x), d(v,y)\}$ *and* $f_\beta(x,y,u,v) = (2-\beta)[d(x,x_0) + d(y,y_0)] + \beta[d(u,x_0) + d(v,y_0)]$. *If there exists a real number* $\beta \in [0, \frac{1}{2})$ *such that*

(C1) *for every* $(u,v) \in \mathcal{A} \times \mathcal{B}$ *and* $(x,y) \in \mathcal{A} \times \mathcal{B}$,

$$\left(\begin{cases} d(u,Tx) = dist(\mathcal{A},\mathcal{B}) \\ d(v,Ty) = dist(\mathcal{A},\mathcal{B}) \\ \Delta_{x,y,u,v} > 0 \end{cases}\right) \Longrightarrow \Delta_{x,y,u,v} \leqslant f_\beta(x,y,u,v) - 2(r_1+r_2),$$

(C2) $T(\mathcal{D}_{x_0,r_1}) \subseteq \mathcal{B}_0$ *and* $T(\mathcal{D}_{y_0,r_2}) \subseteq \mathcal{A}_0$,

then, the discs $\mathcal{D}_{x_0,r_1}$ *and* $\mathcal{D}_{y_0,r_2}$ *are best proximity discs of* T, *that is, for any* $x \in \mathcal{D}_{x_0,r_1}$ *and* $y \in \mathcal{D}_{y_0,r_2}$, $d(x,Tx) = d(y,Ty) = dist(\mathcal{A},\mathcal{B})$.

Proof. Let $(x,y) \in \mathcal{D}_{x_0,r_1} \times \mathcal{D}_{y_0,r_2}$. According to the condition $(C2)$, we have $T(\mathcal{D}_{x_0,r_1}) \subseteq \mathcal{B}_0$ and $T(\mathcal{D}_{y_0,r_2}) \subseteq \mathcal{A}_0$. Hence, there exist $u \in \mathcal{A}$ and $v \in \mathcal{B}$ such that $d(u,Tx) = dist(\mathcal{A},\mathcal{B})$ and $d(v,Ty) = dist(\mathcal{A},\mathcal{B})$. Suppose that $\Delta_{x,y,u,v} > 0$. In this situation, from the condition $(C1)$, we get

$$\begin{aligned}\Delta_{x,y,u,v} &\leqslant f_\beta(x,y,u,v) - 2(r_1+r_2) \leqslant -\beta(r_1+r_2) + \beta[d(u,x_0)+d(v,y_0)]\\ &\leqslant \beta(d(u,x)+d(v,y)) \leqslant 2\beta \max\{d(u,x),d(v,y)\}.\end{aligned}$$

Then,

$$(1-2\beta)\max\{d(u,x),d(v,y)\} \leqslant 0,$$

which contradicts our assumption. Thus, $u = x$ and $v = y$. As a conclusion, $d(x,Tx) = dist(\mathcal{A},\mathcal{B}) = d(y,Ty)$. □

Remark 3.4. *If the condition $(C2)$ is not satisfied, for example $T(\mathcal{D}_{x_0,r_1}) \not\subseteq \mathcal{B}$, then there exists $x \in \mathcal{D}_{x_0,r_1}$ such that $Tx \notin \mathcal{B}$, therefore $d(x,Tx) \neq dist(\mathcal{A},\mathcal{B})$.*

Proposition 3.1. *Consider two nonempty convex subsets $\mathcal{A}$ and $\mathcal{B}$ of a normed space $\mathcal{X}$ such that $\mathcal{A}_0$ is nonempty.*

(i) If two circles $\mathcal{C}_{x_0,r_1}$ and $\mathcal{C}_{y_0,r_2}$ are included in $\mathcal{A}_0$ and $\mathcal{B}_0$ respectively, then $\mathcal{D}_{x_0,r_1} \subseteq \mathcal{A}_0$ and $\mathcal{D}_{y_0,r_2} \subseteq \mathcal{B}_0$.

(ii) If, in addition, T is cyclic relatively nonexpansive on $\mathcal{A}\cup\mathcal{B}$, i.e., $T\mathcal{A} \subseteq \mathcal{B}$, $T\mathcal{B} \subseteq \mathcal{A}$, and $d(Tx,Ty) \leqslant d(x,y)$ for all $(x,y) \in \mathcal{A}\cup\mathcal{B}$, then $T(\mathcal{D}_{x_0,r_1}) \subseteq \mathcal{B}_0$ and $T(\mathcal{D}_{y_0,r_2}) \subseteq \mathcal{A}_0$.

Proof. (i) • It is obvious, since $\mathcal{A}_0$ is nonempty, that, also, $\mathcal{B}_0$ is nonempty. Moreover, $\mathcal{A}_0$ and $\mathcal{B}_0$ are convex. Indeed, given $\lambda \in [0,1]$ and $(a,a') \in \mathcal{A}_0^2$, there exists $(b,b') \in \mathcal{B}_0^2$ so that $d(a,b) = dist(\mathcal{A},\mathcal{B})$ and $d(a',b') = dist(\mathcal{A},\mathcal{B})$. Since $\mathcal{A}$ and $\mathcal{B}$ are convex, we have $a'' := \lambda.a + (1-\lambda).a' \in \mathcal{A}$, $b'' := \lambda.b + (1-\lambda).b' \in \mathcal{B}$. Also,

$$dist(a'',b'') \leqslant \lambda\, d(a,b) + (1-\lambda)\, d(a',b') = dist(\mathcal{A},\mathcal{B}),$$

which means that $\lambda.a + (1-\lambda).a' \in \mathcal{A}_0$. As a consequence, $\mathcal{A}_0$ is convex. Similarly, $\mathcal{B}_0$ is also convex.

• Suppose that $\mathcal{C}_{x_0,r_1} \subseteq \mathcal{A}_0$ and $\mathcal{C}_{y_0,r_2} \subseteq \mathcal{B}_0$. Let $(x,y) \in (\mathcal{D}_{x_0,r_1})^{\text{o}} \times (\mathcal{D}_{y_0,r_2})^{\text{o}}$. Assume that $x \neq x_0$ and consider the vector line (Δ) passing through x and x_0. This line intersects the boundary $Fr(\mathcal{D}_{x_0,r_1}) = \mathcal{C}_{x_0,r_1}$ in two diametrically opposed elements u_1 and u_2. Therefore, since $\mathcal{D}_{x_0,r_1}$ is convex, the segment $\{\lambda.u_1 + (1-\lambda).u_2 : \lambda \in [0,1]\}$ is contained in the disc $\mathcal{D}_{x_0,r_1}$. Hence, there exists $\eta \in (0,1)$ such that $x = \eta.u_1 + (1-\eta).u_2$. Similarly, we can write $y = \eta'.v_1 + (1-\eta').v_2$ where $\eta' \in (0,1)$ and v_1, v_2 are two elements of the circle $\mathcal{C}_{y_0,r_2}$. As $\mathcal{C}_{x_0,r_1} \subseteq \mathcal{A}_0$, there exists $(b_1,b_2) \in (\mathcal{C}_{y_0,r_2})^2$ such that $d(u_1,b_1) = dist(\mathcal{A},\mathcal{B})$ and $d(u_2,b_2) = dist(\mathcal{A},\mathcal{B})$. Therefore,

$$d(x, \lambda.b_1 + (1-\lambda).b_2) \leqslant \lambda\, d(u_1,b_1) + (1-\lambda)\, d(u_2,b_2) = dist(\mathcal{A},\mathcal{B}).$$

Thus, $x \in \mathcal{A}_0$. Similarly, $y \in \mathcal{B}_0$.

For the case $x = x_0$ and $y = y_0$, it suffices to consider two diametrically opposed elements $(a_1,a_2) \in (\mathcal{C}_{x_0,r_1})^2$ and $(d_1,d_2) \in (\mathcal{C}_{y_0,r_0})^2$ such that $x = \frac{1}{2}(a_1+a_2)$ and $y = \frac{1}{2}(d_1+d_2)$. Consequently, $x = x_0 \in \mathcal{A}_0$

(ii) As T is relatively nonexpansive cyclic on $\mathcal{A}\cup\mathcal{B}$, therefore $T\mathcal{A}_0 \subseteq \mathcal{B}_0$ and $T\mathcal{B}_0 \subseteq \mathcal{A}_0$. Thus, according to (i), we have

$$T(\mathcal{D}_{x_0,r_1}) \subseteq T(\mathcal{A}_0) \subseteq \mathcal{B}_0 \text{ and } T(\mathcal{D}_{y_0,r_2}) \subseteq T(\mathcal{B}_0) \subseteq \mathcal{A}_0.$$

□

Remark 3.5. *Assuming that $\mathcal{A}$ and $\mathcal{B}$ are convex, we obtain the same result of Theorem 3.3 if we replace (C_2) by the two following conditions:*

$(C2^{'})$ *T is cyclic relatively nonexpansive on $\mathcal{A} \cup \mathcal{B}$,*

$(C3^{'})$ *$\mathcal{C}_{x_0,r_1} \subseteq \mathcal{A}_0$ and $\mathcal{C}_{y_0,r_2} \subseteq \mathcal{B}_0$.*

Theorem 3.4. *Consider two nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ of a normed space $\mathcal{X}$ and a cyclic mapping $T : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$. Let $\mathcal{EL}_{r_1}(c_1, c_2)$ and $\mathcal{EL}_{r2}(c_1', c_2')$ be two elliptic discs in $\mathcal{A}$ and $\mathcal{B}$ respectively, and denote $\Delta_{x,y,u,v} = \max\{d(u,x), d(v,y)\}$ and*

$$g_\beta(x,y,u,v) = (2-\beta)[d(x,c_1) + d(x,c_2) + d(y,c_1') + d(y,c_2')] + \beta[d(u,c_1) + d(u,c_2) + d(v,c_1') + d(v,c_2')].$$

If there exists a real number $\beta \in [0, \frac{1}{4})$ such that:

(C1) *for any $(u,v) \in \mathcal{A} \times \mathcal{B}$ and $(x,y) \in \mathcal{A} \times \mathcal{B}$,*

$$\left(\begin{cases} d(u,Tx) = dist(\mathcal{A},\mathcal{B}) \\ d(v,Ty) = dist(\mathcal{A},\mathcal{B}) \\ \Delta_{x,y,u,v} > 0 \end{cases} \right) \Longrightarrow \Delta_{x,y,u,v} \leqslant g_\beta(x,y,u,v) - 2(r_1 + r_2);$$

(C2) *$T(\mathcal{EL}_{r_1}(c_1,c_2)) \subseteq \mathcal{B}_0$ and $T(\mathcal{EL}_{r_2}(c_1',c_2')) \subseteq \mathcal{A}_0$,*

then, the elliptic discs $\mathcal{EL}_{r_1}(c_1,c_2)$ and $\mathcal{EL}_{r_2}(c_1',c_2')$ are best proximity elliptic discs of T.

Proof. Let $(x,y) \in \mathcal{EL}_{r_1}(c_1,c_2) \times \mathcal{EL}_{r_2}(c_1',c_2')$. According to the conditions (C_2), there exists $(u,v) \in \mathcal{A} \times \mathcal{B}$ such that $d(u,Tx) = dist(\mathcal{A},\mathcal{B})$ and $d(v,Ty) = dist(\mathcal{A},\mathcal{B})$. Suppose that $\Delta_{x,y,u,v} > 0$. In this case, one has

$$\begin{aligned} \Delta_{x,y,u,v} &\leqslant \beta[(d(u,c_1) + d(u,c_2)) + (d(v,c_1') + d(v,c_2'))] - \beta(r_1 + r_2) \\ &\leqslant 4\beta \max\{d(u,x), d(v,y)\} - \beta(r_1 + r_2) \\ &\quad + \beta[(d(x,c_1) + d(x,c_2)) + (d(y,c_1') + d(y,c_2'))] \\ &\leqslant 4\beta \max\{d(u,x), d(v,y)\}. \end{aligned}$$

Hence, $(1 - 4\beta)\max\{d(u,x), d(v,y)\} \leqslant 0$ and then $\max\{d(u,x), d(v,y)\} = 0$, which contradicts the assumption $\max\{d(u,x), d(v,y)\} > 0$, since $0 < \beta < \frac{1}{4}$. Thus, $u = x$ and $v = y$. Hence, $d(x,Tx) = dist(\mathcal{A},\mathcal{B}) = d(y,Ty)$. □

Example 3.5. *Equip $\mathbb{R}^3$ with the Euclidean norm $\|.\|_2$ and consider the subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{R}^3$ defined by:*

$$\mathcal{A} = \{(x_1,y_1,z_1) \in \mathbb{R}^3 : x_1 + z_1 \geqslant 160 \text{ and } y_1 = -1\} \cup \mathcal{EL}_{r_1}(c_1,c_2)$$

$$\mathcal{B} = \{(x_2,y_2,z_2) \in \mathbb{R}^3 : x_2 + z_2 \geqslant 180 \text{ and } y_2 = 1\} \cup \mathcal{EL}_{r_2}(c_1',c_2'),$$

with $c_1 = (3 + 2\sqrt{3}, -1, -1)$, $c_2 = (3 - 2\sqrt{3}, -1, -1)$, $c_1' = (3 + 2\sqrt{3}, 1, -1)$, $c_2' = (3 - 2\sqrt{3}, 1, -1)$, and $r_1 = r_2 = 8$. We have

$$\begin{aligned} \mathcal{EL}_{r_1}(c_1,c_2) &= \{u = (x_1,y_1,z_1) \in \mathbb{R}^3 : d(c_1,u) + d(c_2,u) \leqslant 8 \text{ and } y_1 = -1\} \\ &= \{u = (x_1,-1,z_1) \in \mathbb{R}^3 : \sqrt{(x_1 - 3 - 2\sqrt{3})^2 + (z_1+1)^2} \\ &\quad + \sqrt{(x_1 - 3 + 2\sqrt{3})^2 + (z_1+1)^2} \leqslant 8\} \\ &= \{u = (x_1,-1,z_1) \in \mathbb{R}^3 : \frac{(x_1-3)^2}{16} + \frac{(z_1+1)^2}{4} \leqslant 1\} \end{aligned}$$

and,

$$\begin{aligned}\mathcal{EL}_{r_2}(c_1', c_2') &= \{v = (x_2, y_2, z_2) \in \mathbb{R}^3 : d(c_1', v) + d(c_2', v) \leqslant 8 \text{ and } y_1 = 1\} \\ &= \{v = (x_2, 1, z_2) \in \mathbb{R}^3 : \frac{(x_2-3)^2}{16} + \frac{(z_2+1)^2}{4} \leqslant 1\}\end{aligned}$$

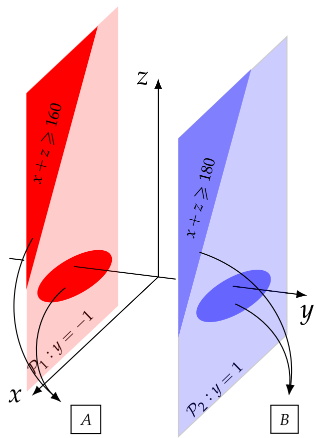

FIGURE 3. The geometric figures of Example 3.5

Define the self-mapping $T : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ *as follows:*

$$Tu = \begin{cases} (3 + 4r\cos\theta, 1, -1 + 2r\sin\theta) & , \text{if } u = (3 + 4r\cos\theta, -1, -1 + 2r\sin\theta), \\ & \qquad \theta \in \mathbb{R} \text{ and } r \geqslant 0, \\ (x_1, 1, z_1) + (8\sqrt{2}, 0, 8\sqrt{2}) & , \text{if } u = (x_1, -1, z_1) \in \mathcal{A} \setminus \mathcal{EL}_{r_1}(c_1, c_2) \end{cases}$$

$$Tv = \begin{cases} (3 + 4r\cos\theta, -1, -1 + 2r\sin\theta) & , \text{if } v = (3 + 4r\cos\theta, 1, -1 + 2r\sin\theta), \\ & \qquad \theta \in \mathbb{R} \text{ and } r \geqslant 0, \\ (x_2, -1, z_2) + (8\sqrt{2}, 0, 8\sqrt{2}) & , \text{if } v = (x_2, 1, z_2) \in \mathcal{B} \setminus \mathcal{EL}_{r_2}(c_1', c_2') \end{cases}$$

Clearly, T *is cyclic,* $\mathcal{A}_0 = \mathcal{A} \setminus \{(x_1, -1, z_1) \in \mathbb{R}^3 : 160 \leqslant x_1 + z_1 < 180\}$, $\mathcal{B}_0 = \mathcal{B}$ *and* $dist(\mathcal{A}, \mathcal{B}) = 2$. *Set* $\beta \in [0, \frac{1}{4})$.

- *Let us justify the condition* (C_1).
  Consider $(x, y) \in \mathcal{A} \times \mathcal{B}$ *and* $(u, v) \in \mathcal{A} \times \mathcal{B}$ *such that*

$$\begin{cases} d(u, Tx) = dist(\mathcal{A}, \mathcal{B}) \\ d(v, Ty) = dist(\mathcal{A}, \mathcal{B}) \\ \max\{d(u, x), d(v, y)\} > 0 \end{cases}$$

  Assume that $\max\{d(u, x), d(v, y)\} = d(u, x)$. *Note that if* $x \in \mathcal{EL}_{r_1}(c_1, c_2)$, *then since* $d(u, Tx) = dist(\mathcal{A}, \mathcal{B})$ *implies that* $x = u$, *which is a contradiction. Hence,* $x \notin \mathcal{EL}_{r_1}(c_1, c_2)$. *If we denote the coordinates of* x *by* $(x_1, -1, z_1)$, *then* $x_1 + z_1 \geqslant 160$, *and therefore* $x_1 \geqslant 80$ *or* $z_1 \geqslant 80$. *In both cases, we have* $d(x, c_1) + d(x, c_2) \geqslant 40$. *Observe also that, in this case, we have* $d(u, x) = 16$. *Therefore,*

$$\begin{aligned} g_\beta(x, y, u, v) - 2(r_1 + r_2) &\geqslant (2 - \beta)(d(x, c_1) + d(x, c_2) + d(y, c_1') + d(y, c_2')) - 32 \\ &\geqslant 40(2 - \beta) - 32 \\ &\geqslant 16 = \max\{d(u, x), d(v, y)\}. \end{aligned}$$

Similarly, if $\max\{d(u,x), d(v,y)\} = d(v,y)$, *then*

$$g_\beta(x,y,u,v) - 2(r_1 + r_2) \geqslant \max\{d(u,x), d(v,y)\}.$$

- *The condition* (C_2) *is immediate, since* $T(\mathcal{EL}_{r_1}(c_1,c_2)) = \mathcal{EL}_{r_2}(c_1', c_2') \subseteq \mathcal{A}_0$ *and* $T(\mathcal{EL}_{r_2}(c_1', c_2')) = \mathcal{EL}_{r_1}(c_1,c_2) \subseteq \mathcal{B}_0$.

Thus, by Theorem 3.4, for any $(x,y) \in \mathcal{EL}_{r_1}(c_1,c_2) \times \mathcal{EL}_{r_2}(c_1', c_2')$,

$$d(x, Tx) = dist(\mathcal{A}, \mathcal{B}) = d(y, Ty)$$

4. Conclusions

In this paper, we studied best proximity discs and best proximity elliptic discs for self-mappings in metric spaces using the fixed figure approach. The main results of this work extend the geometric framework introduced by Özgür and Taş by moving from fixed figure to best proximity type geometric structures.

The main novelty lies in establishing new necessary conditions for the existence of best proximity discs and best proximity elliptic discs, allowing the analysis of mappings that do not necessarily admit fixed points. In contrast to existing results that focus on fixed discs or fixed elliptic discs, our approach guarantees the invariance of entire geometric sets attaining the minimal distance between two subsets. Several examples are provided to illustrate the applicability and nontriviality of the obtained results.

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Declarations.

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