CREAT. MATH. INFORM.
Volume **35** (2026), No. 3,
Pages 395-417

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2026.03.07

Existence results for modified $\mathbb{ABC}$ fractional infinite delay equations in Banach spaces via α-dense curves

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ABSTRACT. This paper studies the solvability of initial value problems for modified Atangana-Baleanu-Caputo ($m\mathbb{ABC}$) fractional differential equations with infinite delay in real Banach spaces. The delay is incorporated through a Hale–Kato type phase space, allowing the system dynamics to depend on the entire past history. Both semilinear and neutral fractional systems are investigated. The nonlinear terms are not assumed to be compact or Lipschitz continuous, and the analysis is carried out without employing classical measures of noncompactness. By using the concept of α-dense curves and the associated degree of nondensifiability, sufficient conditions for the existence of at least one solution are established via an appropriate fixed-point theorem. Illustrative examples, including an academic model in an infinite-dimensional Banach space and an application arising from population dynamics with long-term memory, are presented to demonstrate the applicability of the theoretical results.

1. INTRODUCTION

Fractional calculus broadens the usual notions of differentiation and integration by allowing the order of these operations to take arbitrary real or complex values, rather than only integers. This extended framework introduces fractional-order derivatives and integrals that can effectively represent systems with memory, hereditary behavior, and non-local dynamics, which are often beyond the reach of classical integer-order models. Although the concept of fractional operators arose alongside traditional calculus, their practical use in applications was for a long time relatively modest, largely because their physical and geometric interpretations were not fully clarified. Over the last few decades, the intrinsically non-local nature of fractional operators–where the present state of a system is influenced by its entire past evolution–has led to a surge of interest. This characteristic makes fractional calculus an especially powerful tool for modeling processes with memory, hereditary effects, and long-range interactions, and it has proved valuable for describing complex real-world dynamics across many disciplines [2, 13, 16].

Fractional formulations have now been employed in fields as diverse as medicine, engineering, chemistry, physics, and biology, frequently yielding models that capture observed behavior more accurately than their integer-order analogues [19–21]. The rapid expansion of this area has motivated the development of many distinct fractional derivative operators, each crafted to meet particular modeling needs. Among the classical forms, the Caputo derivative, introduced in 1967 [4], remains widely used because it works naturally with standard initial conditions. However, its singular kernel can be restrictive in some applications. To address this issue, Caputo and Fabrizio proposed a derivative with a nonsingular exponential kernel [3]; while this removes the singularity, it is less suitable for systems whose memory does not follow an exponential pattern.

Received: 04.02.2026. In revised form: 20.03.2026. Accepted: 25.05.2026
2020 *Mathematics Subject Classification.* 26A33, 34G20.
Key words and phrases. *Fractional-order, $m\mathbb{ABC}$ fractional derivative, Infinite delay, Existence, Degree of nondensifiability.*
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To further increase modeling flexibility, Atangana and Baleanu introduced a derivative with a nonsingular kernel based on the Mittag–Leffler function, offering a more general description of nonlocal memory effects [1]. Building on these developments, Refai and Baleanu recently defined the modified fractional derivative, or $m\mathbb{ABC}$-derivative, which blends key advantages of both the Caputo and Atangana–Baleanu formulations [17, 18]. This newer operator has demonstrated strong potential for capturing the dynamics of complex systems that cannot be adequately modeled with earlier fractional derivatives.

The $m\mathbb{ABC}$ fractional derivative is chosen in this work for three principal reasons. First, its nonsingular Mittag–Leffler kernel provides an algebraically decaying memory structure that is more faithful to systems with long-term hereditary effects than the exponential kernel of the Caputo–Fabrizio derivative. Second, it admits a proper inverse operator (the $\mathbb{AB}$-fractional integral), making the formulation and analysis of initial value problems rigorous and tractable without requiring fractional resolvent families. Third, to the best of our knowledge, the combination of $m\mathbb{ABC}$ dynamics with infinite delay effects has not previously been studied using densifiability techniques, and filling this gap constitutes the primary motivation of the present work.

Delay differential equations model dynamical systems whose evolution is influenced by their past history and arise naturally in economics, physics, medicine, biology, and ecology. In many realistic situations, the effect of the past cannot be adequately captured by a finite number of discrete delays, since the present dynamics may depend on the cumulative influence of the entire past evolution. Such infinite delay effects are inherent in phenomena involving hereditary mechanisms, long-term memory, gradual maturation, and persistent environmental feedback. Consequently, prescribing the system state at a single time or over a finite interval is often insufficient, necessitating the use of phase spaces that encode the full historical profile of the system [7, 12, 22–24].

The notion of α-dense curves was introduced in the early 1980s [5] as a tool to characterize the geometric structure of bounded subsets in Banach spaces. This concept was initiated by Cherruault et al. [6] and further developed by Mora et al. [14], who subsequently formulated the degree of nondensifiability (DND) as a quantitative measure derived from α-dense curves. In contrast to classical measures of noncompactness, the DND framework captures finer geometric properties of sets and allows the treatment of operators that are not necessarily condensing in the sense of Darbo. Later, Garcia et al. [8–10] established a powerful fixed-point theorem based on the DND, showing that this approach significantly extends the applicability of Darbo-type fixed-point results and their generalizations.

It is worth emphasizing how the DND differs from classical measures of noncompactness (MNCs) such as the Hausdorff MNC χ and the Kuratowski MNC α_K. Classical MNCs quantify how far a bounded set is from being compact by measuring the smallest radius of a finite cover or the smallest diameter of a finite partition. The DND, by contrast, measures how closely a bounded set can be traced by a single continuous curve: $\lambda(\mathcal{K})$ is the infimum of all $\alpha \geq 0$ for which an α-dense curve exists in $\mathcal{K}$. A key distinction is that $\lambda(\mathcal{K}) = 0$ requires both relative compactness and arc-connectedness, whereas $\chi(\mathcal{K}) = 0$ requires only relative compactness. Consequently, operators satisfying $\lambda(\mathcal{T}(\mathcal{K})) \leq h(\lambda(\mathcal{K}))$ with $h \in \mathcal{F}$ (the class of comparison functions defined in Section 2) need not be condensing in the Darbo sense, making the DND-based fixed-point theorem strictly more general than classical Darbo-type results.

Despite these advances, the existing literature predominantly addresses integer- order or classical fractional differential equations, without incorporating modern non-singular fractional operators. In particular, to the best of our knowledge, no results are available for fractional differential equations involving the $m\mathbb{ABC}$ derivative with finite delay using densifiability techniques. This gap in the literature constitutes the primary motivation of the present work, where we combine $m\mathbb{ABC}$ fractional dynamics, finite delay effects, and the degree of nondensifiability to establish new existence results in Banach spaces.

Building upon these insights, we propose the following initial value problem for a fractional $m\mathbb{ABC}$-type differential equation with infinite delay:

$$\begin{cases} ({}^{m\mathbb{ABC}}D_0^{\sigma}\omega)(t) = F(t, \omega_t), & t \in \mathbb{I} = [0, \Theta], \\ \omega(t) = \xi(t), & t \in (-\infty, 0], \end{cases} \tag{1.1}$$

where $0 < \sigma < 1$, $F : \mathbb{I} \times \mathcal{B} \to \mathbb{Z}$ is a given continuous function, and $\xi \in \mathcal{B}$ is the prescribed initial function defined on $(-\infty, 0]$. For any function $\omega : (-\infty, \Theta] \to \mathbb{Z}$ and any $t \in \mathbb{I}$, the history segment $\omega_t \in \mathcal{B}$ is defined by

$$\omega_t(\theta) = \omega(t + \theta), \qquad \theta \in (-\infty, 0]. \tag{1.2}$$

Additionally, we formulate the following initial value problem for a $m\mathbb{ABC}$-fractional neutral differential equation with infinite delay:

$$\begin{cases} {}^{m\mathbb{ABC}}D_0^{\sigma}\big[\omega(t) - G(t, \omega_t)\big] = F(t, \omega_t), & t \in \mathbb{I} = [0, \Theta], \\ \omega(t) = \xi(t), & t \in (-\infty, 0], \end{cases} \tag{1.3}$$

where $G : \mathbb{I} \times \mathcal{B} \to \mathbb{Z}$ is a continuous function, and the remaining functions and parameters are the same as those defined in (1.1).

The subsequent sections of this work are arranged as follows. In Section 2, we recall the necessary preliminaries on Banach spaces, the $m\mathbb{ABC}$ fractional derivative, and the concept of α-dense curves together with the DND. Section 3 is devoted to the statement of the main existence theorems, where sufficient conditions are established using a fixed-point approach based on densifiability techniques. In Section 4, illustrative examples are presented, including an academic example in an infinite-dimensional setting and a fractional logistic equation with finite delay, which demonstrate the applicability of the theoretical results. Finally, Section 5 summarizes the main findings and outlines possible directions for future research.

2. PRELIMINARIES

This section is devoted to recalling the fundamental concepts, notations, and auxiliary results that will be used throughout the paper. We first presents the functional spaces and basic notations required to formulate the problem rigorously. Next, we briefly recall the definition and some essential properties of the $m\mathbb{ABC}$ fractional derivative, which constitutes the main fractional operator in our analysis. Finally, we present the notion of degree of nondensifiability, together with its principal properties and a fixed point result based on α-dense curves, which forms the core analytical tool for establishing the existence of solutions.

2.1. **Functional spaces and notations.**

Let $(\mathbb{Z}, \|\cdot\|)$ be a real Banach space. The family of all bounded and nonempty subsets of $\mathbb{Z}$ is represented by $\mathcal{M}_{\mathbb{Z}}$. The space of all bounded linear operators acting on $\mathbb{Z}$ is represented by $\mathcal{B}(\mathbb{Z})$, which is itself a Banach space when endowed with the operator norm

$$\|\Theta\|_{\mathcal{B}(\mathbb{Z})} = \sup\{\|\Theta\omega\| : \omega \in \mathbb{Z},\ \|\omega\| \le 1\}.$$

The space $L^1(\mathbb{I},\mathbb{Z})$ comprises all strongly measurable functions $\omega : \mathbb{I} \to \mathbb{Z}$ that are Bochner integrable. It constitutes a Banach space endowed with the norm:

$$\|\omega\|_{L^1(\mathbb{I},\mathbb{Z})} = \int_0^{\Theta} \|\omega(t)\|\, dt.$$

We also consider the Banach space $L^\infty(\mathbb{I},\mathbb{Z})$ of all essentially bounded measurable functions from $\mathbb{I}$ into $\mathbb{Z}$, endowed with the norm

$$\|\omega\|_{L^\infty(\mathbb{I},\mathbb{Z})} = \operatorname*{ess\,sup}_{t\in\mathbb{I}} \|\omega(t)\|.$$

The notation $C(\mathbb{I},\mathbb{Z})$ represents the Banach space comprising all continuous $\mathbb{Z}$-valued functions defined on the interval $\mathbb{I}$. This space is equipped with the supremum norm:

$$\|\omega\|_{C(\mathbb{I},\mathbb{Z})} = \sup_{t\in\mathbb{I}} \|\omega(t)\|.$$

Let χ be the space defined by

$$\chi = \{\omega : (-\infty,\Theta] \to \mathbb{Z} :\ \omega|_{\mathbb{I}} \in C(\mathbb{I},\mathbb{Z}) \text{ and } \omega_0 \in \mathcal{B}\},$$

where $\omega|_{\mathbb{I}}$ denotes the restriction of ω to the interval $\mathbb{I} = [0,\Theta]$.

Define the semi-norm $\|\cdot\|_b$ on χ by

$$\|\omega\|_b = \|\omega_0\|_{\mathcal{B}} + \sup\{\, |\omega(t)| :\ 0 \le t \le \Theta\,\}, \qquad \omega \in \chi.$$

Throughout this work, we adopt an axiomatic framework for the infinite-delay phase space, inspired by the classical construction of Hale–Kato [11], but adapted to our fractional setting and notation. Accordingly, let $\mathcal{B}$ be a seminormed linear space endowed with the seminorm $\|\cdot\|_{\mathcal{B}}$, whose elements are $\mathbb{Z}$-valued functions defined on $(-\infty,0]$.

We assume that the phase space $\mathcal{B}$ satisfies the following conditions.

(P1) Let $\omega : (-\infty,\Theta] \to \mathbb{Z}$ be such that $\omega|_{\mathbb{I}} \in C(\mathbb{I},\mathbb{Z})$ and $\omega_0 \in \mathcal{B}$. Then, for every $t \in [0,\Theta)$, the following properties hold:

(i) the history segment ω_t belongs to $\mathcal{B}$;

(ii) there exists a constant $\mathcal{H} > 0$ such that

$$|\omega(t)| \le \mathcal{H}\, \|\omega_t\|_{\mathcal{B}};$$

(iii) there exist two functions $\mathcal{O},\mathcal{P} : \mathbb{R}^+ \to \mathbb{R}^+$, independent of ω, with $\mathcal{O}$ continuous and $\mathcal{P}$ locally bounded, such that

$$\|\omega_t\|_{\mathcal{B}} \le \mathcal{O}(t) \sup\{|\omega(s)| : 0 \le s \le t\} + \mathcal{P}(t)\|\omega_0\|_{\mathcal{B}}.$$

We set

$$\mathcal{P}^* = \sup\{\mathcal{P}(t) : t \in \mathbb{I}\}, \qquad \mathcal{O}^* = \sup\{\mathcal{O}(t) : t \in \mathbb{I}\}.$$

(P2) For every function ω satisfying **(P1)**, the mapping $t \mapsto \omega_t$ is a $\mathcal{B}$-valued continuous function on $[0,\Theta]$.

(P3) The phase space $\mathcal{B}$ is complete with respect to the seminorm $\|\cdot\|_{\mathcal{B}}$.

Remark. Condition **(P1)(ii)** is equivalent to the estimate

$$|\xi(0)| \le \mathcal{H}\, \|\xi\|_{\mathcal{B}}, \qquad \text{for all } \xi \in \mathcal{B}.$$

Example (Admissible phase space). A standard concrete example of a phase space satisfying axioms **(P1)–(P3)** is the weighted supremum space

$$\mathcal{B}_\gamma = \left\{\omega : (-\infty,0] \to \mathbb{Z} : \sup_{\theta\le 0} e^{\gamma\theta}\|\omega(\theta)\| < \infty\right\}, \quad \gamma > 0,$$

equipped with the norm $\|\omega\|_{\mathcal{B}_\gamma} = \sup_{\theta\le 0} e^{\gamma\theta}\|\omega(\theta)\|$. This space satisfies **(P1)–(P3)** with constants $\mathcal{H} = 1$, $\mathcal{O}^* = 1$, $\mathcal{P}^* = 1$, and is used in the academic example of Section 4.1 with $\mathbb{Z} = \mathbb{R}$. Another standard example is the space $C_g = \{\omega : (-\infty, 0] \to \mathbb{Z} : \omega/g \in C((-\infty, 0], \mathbb{Z})\}$ for a continuous nonincreasing weight $g : (-\infty, 0] \to [1, \infty)$, which models polynomial decay of past states.

Definition 2.1. *A mapping $F : \mathbb{I} \times \mathcal{B} \to \mathbb{Z}$ is said to be an L^1–Carathéodory function if it satisfies:*

(i) *for almost every $t \in \mathbb{I}$, the mapping $\omega \mapsto F(t, \xi)$ is continuous on $\mathcal{B}$;*
(ii) *for each $\xi \in \mathcal{B}$, the mapping $t \mapsto F(t, \xi)$ is measurable on $\mathbb{I}$;*
(iii) *for every $r > 0$, there exists a function $h_r \in L^1(\mathbb{I}, \mathbb{R}_+)$ such that*

$$f(t, \xi)\| \le h_r(t), \qquad \text{for all } \|\xi\|_{\mathcal{B}} \le r \text{ and for a.e. } t \in \mathbb{I}.$$

2.2. The $m\mathbb{ABC}$ fractional derivative.

We begin by revisiting the formal definition and the requisite properties of the $m\mathbb{ABC}$ fractional derivative. These mathematical preliminaries establish the framework necessary for the analytical developments throughout the remainder of this work.

Definition 2.2 ([15]). *Let $0 < \sigma < 1$ and let $F \in L^1(\mathbb{I}, \mathbb{Z})$, where $\mathbb{I} = [0, \Theta]$ with $\Theta > 0$. The $m\mathbb{ABC}$ fractional derivative of order σ of the function F is defined by*

$$\begin{aligned}({}^{m\mathbb{ABC}}D_0^\sigma F)(t) = \frac{B(\sigma)}{1-\sigma}\Bigg[&F(t) - E_\sigma(\lambda_\sigma t^\sigma)\,F(0)\\ &- \lambda_\sigma \int_0^t (t-s)^{\sigma-1} E_{\sigma,\sigma}(\lambda_\sigma (t-s)^\sigma)\,F(s)\,ds\Bigg],\end{aligned} \tag{2.4}$$

for $0 < t < \Theta$, where $\lambda_\sigma = -\sigma/(1-\sigma)$ and $B : [0, 1] \to \mathbb{R}_+$ is a normalization function satisfying $B(0) = B(1) = 1$. Here, $E_{\sigma,\beta}$ denotes the two-parameter Mittag–Leffler function defined by

$$E_{\sigma,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n + \beta)}, \qquad \sigma, \beta > 0,\ z \in \mathbb{C}.$$

Remark 2.1. *The two-parameter Mittag–Leffler series $E_{\sigma,\beta}(z) = \sum_{n=0}^{\infty} z^n/\Gamma(\sigma n+\beta)$ converges absolutely for all $z \in \mathbb{C}$ whenever $\sigma > 0$ and $\beta > 0$, since an application of the ratio test together with Stirling's asymptotic formula for $\Gamma(\sigma n + \beta)$ shows that the ratio of successive terms tends to zero. For $\sigma = \beta = 1$, $E_{1,1}(z) = e^z$, recovering the classical exponential function. In the $m\mathbb{ABC}$ setting, $z = \lambda_\sigma t^\sigma$ with $\lambda_\sigma = -\sigma/(1-\sigma) < 0$ for $\sigma \in (0, 1)$, so the series converges for all $t \ge 0$.*

The parameter $\lambda_\sigma = -\sigma/(1-\sigma) < 0$ for $\sigma \in (0, 1)$ ensures that the Mittag–Leffler kernel $E_\sigma(\lambda_\sigma t^\sigma)$ decays monotonically, giving the $m\mathbb{ABC}$ derivative its physically correct dissipative memory character. As $\sigma \to 1^-$, $\lambda_\sigma \to -\infty$ and $E_\sigma(\lambda_\sigma t^\sigma) \to 0$, so the memory term vanishes and the derivative recovers the classical first-order derivative, consistently with the condition $\mathcal{B}(1) = 1$.

Definition 2.3 ([15]). *Let $0 < \sigma < 1$ and $F \in L^1(\mathbb{I}, \mathbb{Z})$. The fractional integral associated with the $m\mathbb{ABC}$ derivative is defined by*

$$({}^{\mathbb{AB}}I_0^\sigma F)(t) = \frac{1-\sigma}{B(\sigma)}\,F(t) + \frac{\sigma}{B(\sigma)}({}^{RL}I_0^\sigma F)(t), \qquad 0 < t < \Theta, \tag{2.5}$$

where ${}^{RL}I_0^\sigma$ denotes the Riemann–Liouville fractional integral operator given by

$$({}^{RL}I_0^\sigma F)(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1} F(s)\,ds, \qquad t > 0.$$

Theorem 2.1 ([15]). *Let $0 < \sigma < 1$ and $F \in L^1(\mathbb{I}, \mathbb{Z})$. Then, for all $0 < t < \Theta$, the following identity holds:*

$$({}^{\mathbb{AB}}I_0^{\sigma}\, {}^{m\mathbb{ABC}}D_0^{\sigma} F)(t) = F(t) - F(0).$$

Remark 2.2. *The normalization function $B(\sigma)$ satisfying $B(0) = B(1) = 1$ and $B(\sigma) > 0$ for all $\sigma \in (0, 1)$ plays a purely dimensional role: it ensures that the $m\mathbb{ABC}$ derivative reduces to known operators at the boundary values $\sigma = 0$ and $\sigma = 1$, and that the factor $(1 - \sigma)/B(\sigma)$ in the $\mathbb{AB}$-fractional integral is finite and nonzero, guaranteeing well-posedness of the integral representation. Common choices include $B(\sigma) = 1$ (used in our examples) and $B(\sigma) = 1 - \sigma + \sigma/\Gamma(\sigma)$ as suggested in [1]. Throughout this paper, $B(\sigma)$ is treated as a fixed but arbitrary normalization satisfying these conditions.*

The fundamental identity $({}^{\mathbb{AB}}I_0^{\sigma}\, {}^{m\mathbb{ABC}}D_0^{\sigma} f)(t) = f(t) - f(0)$ holds for any $f \in L^1(\mathcal{I}, \mathbb{Z})$, where the proof follows from the definition of the $\mathbb{AB}$-fractional integral as a combination of the identity and the Riemann–Liouville integral, together with the Fubini theorem for Bochner integrals. In the context of system (1.1), *this identity is applied to $f(t) = \omega(t)$ where $t \mapsto F(t, \omega_t) \in L^1(\mathcal{I}, \mathbb{Z})$ is guaranteed by the L^1-Carathéodory condition (H1).*

Definition 2.4. *A function $\omega \in C((-\infty, \Theta], \mathbb{Z})$ is said to be a solution of the governing problem* (1.1) *if it satisfies the following relation:*

$$\omega(t) = \begin{cases} \xi(0) + ({}^{\mathbb{AB}}I_0^{\sigma} F(\cdot, \omega.))(t), & t \in [0, \Theta], \\ \xi(t), & t \in (-\infty, 0]. \end{cases} \tag{2.6}$$

Using the explicit representation of the $\mathbb{AB}$-fractional integral, the solution formula (2.6) can be written equivalently as

$$\omega(t) = \begin{cases} \xi(0) + \dfrac{1-\sigma}{B(\sigma)} F(t, \omega_t) + \dfrac{\sigma}{B(\sigma)\Gamma(\sigma)} \displaystyle\int_0^t (t-s)^{\sigma-1} F(s, \omega_s)\, ds, & t \in [0, \Theta], \\ \xi(t), & t \in (-\infty, 0]. \end{cases} \tag{2.7}$$

Theorem 2.2. *Assume that $F : \mathbb{I} \times \mathcal{B} \to \mathbb{Z}$ is such that $t \mapsto F(t, \omega_t) \in L^1(\mathbb{I}, \mathbb{Z})$. Then a function $\omega \in C((-\infty, \Theta], \mathbb{Z})$ with $\omega_0 \in \mathcal{B}$ is a solution of problem* (1.1) *if and only if it satisfies the integral equation* (2.7)*.*

Remark 2.3. *The Riemann–Liouville convolution $\int_0^t (t-s)^{\sigma-1} F(s, \omega_s)\, ds$ appearing in* (2.7) *is well defined as a Bochner integral for every $t \in \mathcal{I}$, provided $F(\cdot, \omega.) \in L^1(\mathcal{I}, \mathbb{Z})$. This integrability is guaranteed by the L^1-Carathéodory condition (H1), which supplies the dominating function $\mu \in L^1(\mathcal{I}, \mathbb{R}^+)$ satisfying $\|F(t, \omega_t)\| \le \mu(t)\psi(\mu^*)$ for a.e. $t \in \mathcal{I}$. Since $(t-s)^{\sigma-1} \in L^1([0, t])$ for $\sigma \in (0, 1)$, the convolution is integrable by Young's inequality for Bochner integrals.*

Proof. Assume that ω is a solution of problem (1.1). Then, for $t \in [0, \Theta]$, we have

$$({}^{m\mathbb{ABC}}D_0^{\sigma} \omega)(t) = F(t, \omega_t).$$

Applying the $\mathbb{AB}$–fractional integral operator ${}^{\mathbb{AB}}I_0^{\sigma}$ to both sides yields

$$({}^{\mathbb{AB}}I_0^{\sigma}\, {}^{m\mathbb{ABC}}D_0^{\sigma} \omega)(t) = ({}^{\mathbb{AB}}I_0^{\sigma} F(\cdot, \omega.))(t).$$

Using the fundamental identity of the $m\mathbb{ABC}$–type fractional operators,

$$({}^{\mathbb{AB}}I_0^{\sigma}\, {}^{m\mathbb{ABC}}D_0^{\sigma} \omega)(t) = \omega(t) - \omega(0),$$

we obtain

$$\omega(t) = \omega(0) + ({}^{\mathbb{AB}}I_0^{\sigma} F(\cdot, \omega.))(t), \qquad t \in [0, \Theta].$$

Since $\omega(0) = \xi(0)$ and $\omega(t) = \xi(t)$ for $t \in (-\infty, 0]$, it follows that ω satisfies (2.6). The explicit representation (2.7) then follows from the definition of the $\mathbb{AB}$–fractional integral. $\square$

Definition 2.5. *A function $\omega \in C((-\infty,\Theta],\mathbb{Z})$ is said to be a solution of the neutral governing problem* (1.3) *if it satisfies the following relation:*

$$\omega(t) = \begin{cases} \xi(0) - G(0,\xi) + G(t,\omega_t) + ({}^{\mathbb{AB}}I_0^{\sigma} F(\cdot,\omega.))(t), & t \in [0,\Theta], \\ \xi(t), & t \in (-\infty, 0]. \end{cases} \tag{2.8}$$

Using the explicit representation of the $\mathbb{AB}$-fractional integral, the solution formula (2.8) can be written equivalently as

$$\omega(t) = \begin{cases} \xi(0) - G(0,\xi) + G(t,\omega_t) + \dfrac{1-\sigma}{B(\sigma)} F(t,\omega_t) \\ \quad + \dfrac{\sigma}{B(\sigma)\Gamma(\sigma)} \displaystyle\int_0^t (t-s)^{\sigma-1} F(s,\omega_s)\,ds, & t \in [0,\Theta], \\ \xi(t), & t \in (-\infty, 0]. \end{cases} \tag{2.9}$$

Theorem 2.3. *Assume that $F : \mathbb{I} \times \mathcal{B} \to \mathbb{Z}$ is such that $t \mapsto F(t,\omega_t) \in L^1(\mathbb{I},\mathbb{Z})$ and that $G : \mathbb{I} \times \mathcal{B} \to \mathbb{Z}$ is given. Then a function $\omega \in C((-\infty,\Theta],\mathbb{Z})$ with $\omega_0 \in \mathcal{B}$ is a solution of the neutral problem* (1.3) *if and only if it satisfies the integral equation* (2.9).

Proof. Assume that ω is a solution of the neutral problem (1.3). Then, for $t \in [0,\Theta]$,

$${}^{m\mathbb{ABC}}D_0^{\sigma}\big[\omega(t) - G(t,\omega_t)\big] = F(t,\omega_t).$$

Applying the fractional integral operator ${}^{\mathbb{AB}}I_0^{\sigma}$ to both sides and using the fundamental identity

$$({}^{\mathbb{AB}}I_0^{\sigma}\, {}^{m\mathbb{ABC}}D_0^{\sigma} u)(t) = u(t) - u(0),$$

we obtain

$$\omega(t) - G(t,\omega_t) = \omega(0) - G(0,\xi) + ({}^{\mathbb{AB}}I_0^{\sigma} F(\cdot,\omega.))(t), \qquad t \in [0,\Theta].$$

Since $\omega(0) = \xi(0)$ and $\omega(t) = \xi(t)$ for $t \in (-\infty,0]$, this yields (2.8). The explicit form (2.9) follows directly from the definition of the $\mathbb{AB}$-fractional integral. □

2.3. **Degree of nondensifiability.**

In this part, we revisit the concept of densifiability along with its related DND. These notions serve as a cornerstone for the analytical framework used to establish our existence results.

Definition 2.6 ([14])**.** *Let $\alpha \geq 0$ and let $\mathcal{K} \in \mathcal{M}_{\mathbb{Z}}$. A continuous mapping $\gamma : [0,1] \to \mathbb{Z}$ is called an α-dense curve in $\mathcal{K}$ if the subsequent axioms hold:*

(1) $\gamma([0,1]) \subset \mathcal{K}$;
(2) for every $\omega \in \mathcal{K}$ $\exists$ $\bar{\tau} \in [0,1]$ in a way that

$$\|\omega - \gamma(\bar{\tau})\| \leq \alpha.$$

The set $\mathcal{K}$ is said to be densifiable if $\exists$ some $\alpha > 0$ such that $\mathcal{K}$ admits an α-dense curve.

Definition 2.7 ([10])**.** *For $\alpha > 0$ and $\mathcal{K} \in \mathcal{M}_{\mathbb{Z}}$, denote by $\mathcal{D}_{\alpha}(\mathcal{K})$ the family of all α-dense curves in $\mathcal{K}$. The DND is the mapping $\lambda : \mathcal{M}_{\mathbb{Z}} \to \mathbb{R}_+$ defined by*

$$\lambda(\mathcal{K}) = \inf\{\alpha \geq 0 : \mathcal{D}_{\alpha}(\mathcal{K}) \neq \varnothing\}.$$

Remark 2.4. *The DND $\lambda(\mathcal{K})$ captures how closely the set $\mathcal{K}$ can be traced by a single continuous curve, whereas traditional noncompactness measures (like Hausdorff's) rely on finite coverings by small sets, the DND focuses on the existence of an α-dense curve — a fundamentally different approach to measuring geometric complexity.*

Example. *Let $\mathbb{Z} = \mathbb{R}$ and $\mathcal{K} = [0, 1]$. The curve $\gamma(t) = t$ for $t \in [0, 1]$ is 0-dense in $\mathcal{K}$ (since $\gamma([0, 1]) = \mathcal{K}$ exactly), and hence $\lambda(\mathcal{K}) = 0$. In contrast, consider $\mathcal{K} = \{0\} \cup \{1/n : n \in \mathbb{N}\}$. This set is bounded and compact in $\mathbb{R}$, but it is not arc-connected. Nevertheless, for any $\alpha > 0$, we can construct an α-dense curve: define $\gamma_\alpha(t)$ to traverse $\mathcal{K}$ in finite time while staying within α of every point. Thus $\mathcal{K}$ is densifiable with $\lambda(\mathcal{K}) = 0$ as well, despite not being connected. This illustrates that densifiability is a weaker condition than arc-connectedness.*

Lemma 2.1 ([9, 10])**.** *Let $\mathcal{K}_1, \mathcal{K}_2 \in \mathcal{M}_{\mathbb{Z}}$. The following assertions are established for the DND:*

(a) If $\mathcal{K}_1$ is nonempty, bounded, and arc-connected, then $\lambda(\mathcal{K}_1) = 0$ if and only if $\mathcal{K}_1$ is relatively compact in $\mathbb{Z}$.
(b) $\lambda(\overline{\mathcal{K}_1}) = \lambda(\mathcal{K}_1)$.
(c) $\lambda(\psi\mathcal{K}_1) = |\psi|\, \lambda(\mathcal{K}_1)$ for all $\psi \in \mathbb{R}$.
(d) $\lambda(\omega + \mathcal{K}_1) = \lambda(\mathcal{K}_1)$ for all $\omega \in \mathbb{Z}$.
(e) $\lambda(\operatorname{co}\mathcal{K}_1) \leq \lambda(\mathcal{K}_1)$ and $\lambda(\operatorname{co}(\mathcal{K}_1 \cup \mathcal{K}_2)) \leq \max\big\{\lambda(\operatorname{co}\mathcal{K}_1), \lambda(\operatorname{co}\mathcal{K}_2)\big\}$, where co denotes the convex hull.
(f) $\lambda(\mathcal{K}_1 + \mathcal{K}_2) \leq \lambda(\mathcal{K}_1) + \lambda(\mathcal{K}_2)$.

Remark 2.5. *We provide a brief proof sketch of property (f). Let $\varepsilon > 0$. By definition of the DND, there exist an α_1-dense curve γ_1 in $\mathcal{K}_1$ and an α_2-dense curve γ_2 in $\mathcal{K}_2$ with $\alpha_i \leq \lambda(\mathcal{K}_i) + \varepsilon$. The curve $\gamma(\tau) = \gamma_1(\tau) + \gamma_2(\tau)$ maps $[0, 1]$ continuously into $\mathcal{K}_1 + \mathcal{K}_2$. For any $\omega_1 + \omega_2 \in \mathcal{K}_1 + \mathcal{K}_2$, there exist $\tau_1, \tau_2 \in [0, 1]$ with $\|\omega_i - \gamma_i(\tau_i)\| \leq \alpha_i$. Choosing $\tau = \tau_1$ (or constructing a suitable reparametrization) and applying the triangle inequality gives*

$$\|(\omega_1 + \omega_2) - \gamma(\tau)\| \leq \|\omega_1 - \gamma_1(\tau)\| + \|\omega_2 - \gamma_2(\tau)\| \leq \alpha_1 + \alpha_2.$$

Hence γ is $(\alpha_1 + \alpha_2)$-dense in $\mathcal{K}_1 + \mathcal{K}_2$, giving $\lambda(\mathcal{K}_1 + \mathcal{K}_2) \leq \lambda(\mathcal{K}_1) + \lambda(\mathcal{K}_2) + 2\varepsilon$. Letting $\varepsilon \to 0$ completes the argument.

We introduce the class of comparison functions

$$\mathcal{F} = \Big\{h : \mathbb{R}_+ \to \mathbb{R}_+ : h \text{ is non-decreasing and } \lim_{n\to\infty} h^n(\rho) = 0 \;\forall\, \rho \geq 0\Big\},$$

where $n \in \mathbb{N}$ and h^n denotes the n-fold composition of h.

Remark 2.6. *The class $\mathcal{F}$ extends the classical notion of comparison functions used in contraction mapping theory. To understand its properties, let $\rho_0 \geq 0$ and define the sequence $\{\rho_n\}_{n\geq 0}$ by*

$$\rho_{n+1} = h(\rho_n), \qquad n \geq 0.$$

Since h is non-decreasing, if $\rho_1 \leq \rho_0$, then by induction $\rho_{n+1} \leq \rho_n$ for all n, so the sequence is monotone decreasing. In fact, the condition $\lim_{n\to\infty} h^n(\rho) = 0$ implies that for any $\rho > 0$, there exists an integer $N = N(\rho) \in \mathbb{N}$ such that $h^N(\rho) < \rho$. Once we have $h^N(\rho) < \rho$, the monotonicity of h ensures that the sequence $\rho_n = h^n(\rho)$ satisfies $h(\rho_n) = \rho_{n+1} \leq \rho_n$ for all $n \geq N$.

The sequence $\{\rho_n\}$ is bounded below by 0, so it converges to some $\ell \geq 0$. Taking limits in the recurrence relation and using continuity (if assumed) or the monotonicity of h, we obtain

$$\ell = \lim_{n\to\infty} h(\rho_n) = h(\ell).$$

Now, if $\ell > 0$, then by the defining property of $\mathcal{F}$,

$$0 < \ell = h(\ell) = h^2(\ell) = \cdots = \lim_{n\to\infty} h^n(\ell) = 0,$$

a contradiction. Hence $\ell = 0$, which gives

$$\lim_{n\to\infty} h^n(\rho_0) = 0 \qquad \textit{for every } \rho_0 \geq 0.$$

Examples:

1. For the nonlinear comparison function

$$h(\rho) = \frac{\rho}{1+\rho},$$

we have $0 < h(\rho) < \rho$ *for all* $\rho > 0$. *By induction,*

$$h^n(\rho) = \frac{\rho}{1+n\rho}, \qquad n \in \mathbb{N},$$

so clearly $\lim\limits_{n\to\infty} h^n(\rho) = 0$.

2. For the linear case $h(\rho) = c\rho$ *with* $0 < c < 1$, *we obtain*

$$h^n(\rho) = c^n\rho \longrightarrow 0.$$

3. Note that a function in $\mathcal{F}$ *need not satisfy* $h(\rho) < \rho$ *for all* $\rho > 0$. *For instance,*

$$h(\rho) = \begin{cases} \rho & \text{if } 0 \le \rho \le 1, \\ \dfrac{\rho}{2} & \text{if } \rho > 1, \end{cases}$$

belongs to $\mathcal{F}$ *but has* $h(\rho) = \rho$ *on* $[0, 1]$.

4. ***Logarithmic example.*** *The function* $h(\rho) = \ln(1+\rho)$ *for* $\rho \ge 0$ *belongs to* $\mathcal{F}$. *Indeed,* h *is non-decreasing and* $h(\rho) < \rho$ *for all* $\rho > 0$. *Setting* $\rho_0 > 0$, *the iterates satisfy* $\rho_{n+1} = \ln(1+\rho_n)$. *Since* $\ln(1+\rho) < \rho$, *the sequence* $\{\rho_n\}$ *is strictly decreasing and bounded below by* 0, *converging to a limit* $\ell \ge 0$ *satisfying* $\ell = \ln(1+\ell)$, *which forces* $\ell = 0$. *Hence* $\lim_{n\to\infty} h^n(\rho_0) = 0$ *for every* $\rho_0 \ge 0$. *This function appears naturally in the verification of hypothesis (H2) in Example 4.1, where the nonlinearity involves* $\ln(1+|\omega|)$.

In the context of the DND, if an operator $\mathcal{T}$ *satisfies*

$$\lambda(\mathcal{T}(\mathcal{K})) \le h(\lambda(\mathcal{K})) \qquad \text{with } h \in \mathcal{F},$$

then by iteration

$$\lambda(\mathcal{T}^n(\mathcal{K})) \le h^n(\lambda(\mathcal{K})) \longrightarrow 0 \quad \text{as } n \to \infty.$$

This forces the iterates $\mathcal{T}^n(\mathcal{K})$ *to become arbitrarily "curve-like" (i.e.,* $\lambda \to 0$*), which can replace classical compactness or contraction assumptions in fixed-point theorems. The DND-based approach thus provides a flexible alternative to traditional compactness requirements.*

Theorem 2.4 ([8]). *Consider a nonempty, closed, convex, and bounded subset* $\widehat{\mathcal{K}}$ *within a Banach space* $\mathbb{Z}$. *Let* $\mathcal{T} : \widehat{\mathcal{K}} \to \widehat{\mathcal{K}}$ *be a continuous mapping. Suppose there exists a function* $h \in \mathcal{F}$ *satisfying the condition:*

$$\lambda(\mathcal{T}(\mathcal{K})) \le h(\lambda(\mathcal{K})) \tag{2.10}$$

for every nonempty subset $\mathcal{K} \subseteq \widehat{\mathcal{K}}$. *Then, the operator* $\mathcal{T}$ *possesses at least one fixed point in* $\widehat{\mathcal{K}}$.

Lemma 2.2 ([8]). *Let* $\mathcal{K} \subset C(\mathbb{I}, \mathbb{Z})$ *be nonempty and bounded. Then*

$$\sup_{t\in\mathbb{I}} \lambda(\mathcal{K}(t)) \le \lambda(\mathcal{K}),$$

where $\mathcal{K}(t) = \{\omega(t) : \omega \in \mathcal{K}\}$.

3. Existence results

This section is devoted to establishing the existence of solutions for the $m\mathbb{ABC}$-fractional system with infinite delay (1.1). Our approach does not rely on compactness arguments or classical Lipschitz conditions; instead, it is based on the DND framework.

Throughout this section, let

$$F : \mathbb{I} \times \mathcal{B} \to \mathbb{Z}$$

be the nonlinear term appearing in problem (1.1). The following hypotheses are imposed.

(H1) The mapping F satisfies the L^1–Carathéodory conditions. Moreover, there exist a function $\mu \in L^1(\mathbb{I}, \mathbb{R}_+)$ and a continuous, nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|F(t,\xi)\| \le \mu(t)\,\psi(\|\xi\|_{\mathcal{B}}), \qquad \text{for a.e. } t \in \mathbb{I},\ \xi \in \mathcal{B}.$$

(H2) There exist a function $K \in L^\infty(\mathbb{I}, \mathbb{R}_+)$ and a comparison function $h \in \mathcal{F}$ such that, for every nonempty bounded and convex subset $\mathcal{K} \subset \mathcal{B}$,

$$\lambda(F(t,\mathcal{K})) \le K(t) \sup_{t\in(-\infty,0]} h(\lambda(\mathcal{K}(t))), \qquad \text{for a.e. } t \in \mathbb{I}.$$

(H3) There exists a constant $r^* > 0$ such that

$$r^* \ge \left(\frac{1-\sigma}{B(\sigma)} + \frac{\sigma\,\Theta^\sigma}{B(\sigma)\Gamma(\sigma+1)}\right)\psi(\mu^*)\,\|\mu\|_{L^1},$$

where

$$\mu^* = 2\big(\mathcal{P}^* + \mathcal{H}\mathcal{O}^*\big)\|\xi\|_{\mathcal{B}} + 2\mathcal{O}^* r^*.$$

(H4) The inequality

$$\left(\frac{1-\sigma}{B(\sigma)} + \frac{\sigma\,\Theta^\sigma}{B(\sigma)\Gamma(\sigma+1)}\right)\|K\|_\infty < 1$$

is satisfied.

Theorem 3.5. *Under assumptions* (H1)–(H4)*, problem* (1.1) *admits at least one solution* $\omega \in C((-\infty,\Theta],\mathbb{Z})$ *with* $\omega_0 \in \mathcal{B}$.

Lemma 3.1 (Key Constants)**.** *The following constants play a central role in the proof.*

(i) *For any* $z \in \widehat{\mathcal{K}}$ *and* $t \in \mathcal{I}$*, the history norm satisfies* $\|z_t + \beta_t\|_{\mathcal{B}} \le \mu^*$*, where*

$$\mu^* = 2(\mathcal{P}^* + \mathcal{H}\mathcal{O}^*)\|\xi\|_{\mathcal{B}} + 2\mathcal{O}^* r^*,$$

and $\mathcal{H}$*,* $\mathcal{O}^*$*,* $\mathcal{P}^*$ *are the phase-space constants from axiom* ***(P1)****.*

(ii) *The* $\mathbb{AB}$*-fractional integral acting on a function bounded in* L^∞ *produces the coefficient*

$$C_\Theta = \frac{1-\sigma}{B(\sigma)} + \frac{\sigma\Theta^\sigma}{B(\sigma)\Gamma(\sigma+1)},$$

which is the contraction constant appearing in hypotheses (H3) and (H4).

Proof of (i): *Apply axiom* ***(P1)(iii)*** *to* z *and* β *separately, using* $z_0 = 0$, $\beta_0 = \xi$, *and* $\sup_{0\le t\le\Theta}|\beta(t)| \le \mathcal{H}\|\xi\|_{\mathcal{B}}$. Proof of (ii): *Evaluate* $\int_0^t (t-s)^{\sigma-1}\,ds = t^\sigma/\sigma \le \Theta^\sigma/\sigma$ *and apply the identity* $\Gamma(\sigma+1) = \sigma\Gamma(\sigma)$.

Proof. Define the operator $\mathcal{T} : \chi \to \chi$ by

$$(\mathcal{T}\omega)(t) = \begin{cases} \xi(0) + \dfrac{1-\sigma}{B(\sigma)}F(t,\omega_t) + \dfrac{\sigma}{B(\sigma)\Gamma(\sigma)}\displaystyle\int_0^t (t-s)^{\sigma-1}F(s,\omega_s)\,ds, & t \in \mathbb{I},\\ \xi(t), & t \in (-\infty,0]. \end{cases}$$

Let

$$\beta(t) = \begin{cases} \xi(t), & t \in (-\infty,0],\\ \xi(0), & t \in \mathbb{I}, \end{cases} \qquad \omega(t) = z(t) + \beta(t).$$

Then $z_0 = 0$ and $z \in \chi_0$. Define $\mathcal{T}_e : \chi_0 \to \chi_0$ by

$$(\mathcal{T}_e z)(t) = \begin{cases} \dfrac{1-\sigma}{B(\sigma)}F(t, z_t+\beta_t) + \dfrac{\sigma}{B(\sigma)\Gamma(\sigma)}\displaystyle\int_0^t (t-s)^{\sigma-1}F(s, z_s+\beta_s)\,ds, & t \in \mathbb{I},\\ 0, & t \in (-\infty,0]. \end{cases}$$

Let

$$\widehat{\mathcal{K}} = \{z \in \chi_0 : \|z\|_b \le r^*\}.$$

Step 1: Invariance of $\widehat{\mathcal{K}}$.

Let $z \in \widehat{\mathcal{K}}$, that is,

$$\|z\|_b \le r^*.$$

For $t \in \mathbb{I} = [0, \Theta]$, by the definition of the operator $\mathcal{T}_e$, we have

$$\begin{aligned}\|(\mathcal{T}_e z)(t)\| &= \left\| \frac{1-\sigma}{B(\sigma)}\, F(t, z_t + \beta_t) + \frac{\sigma}{B(\sigma)\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} F(s, z_s + \beta_s)\, ds \right\| \\ &\le \frac{1-\sigma}{B(\sigma)}\, \|F(t, z_t + \beta_t)\| + \frac{\sigma}{B(\sigma)\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \|F(s, z_s + \beta_s)\|\, ds.\end{aligned}$$

By assumption (H1), for a.e. $t \in \mathbb{I}$,

$$\|F(t, z_t + \beta_t)\| \le \mu(t)\, \psi(\|z_t + \beta_t\|_{\mathcal{B}})\,.$$

Next, we estimate the history norm $\|z_t + \beta_t\|_{\mathcal{B}}$. By the axioms of the phase space $\mathcal{B}$, we obtain for $s \in (-\infty, 0]$,

$$\begin{aligned}\|z(s) + \beta(s)\|_b &\le 2\big(\|z\|_b + \|\beta\|_b\big) \\ &\le 2\Big(\mathcal{O}^* \sup\{|z(t)| : 0 \le s \le t\} + \mathcal{P}^* \|z_0\|_{\mathcal{B}}\Big) \\ &\quad + 2\Big(\mathcal{O}^* \sup\{|\beta(t)| : 0 \le s \le t\} + \mathcal{P}^* \|\beta_0\|_{\mathcal{B}}\Big).\end{aligned}$$

Since $z_0 = 0$ and $\beta_0 = \xi$, it follows that

$$\|z(s) + \beta(s)\|_b \le 2\mathcal{P}^* \|\xi\|_{\mathcal{B}} + 2\mathcal{O}^* \Big(\sup\{|z(t)| : 0 \le s \le t\} + \sup\{|\beta(t)| : 0 \le s \le t\}\Big).$$

Using the fact that $z \in \widehat{\mathcal{K}}$ and the definition of β, we have

$$\sup_{0 \le t \le \Theta} |z(t)| \le r^*, \qquad \sup_{0 \le t \le \Theta} |\beta(t)| \le \|\xi(0)\| \le \mathcal{H} \|\xi\|_{\mathcal{B}}.$$

Therefore,

$$\begin{aligned}\|z(s) + \beta(s)\|_b &\le 2\Big(\mathcal{P}^* \|\xi\|_{\mathcal{B}} + \mathcal{O}^* \big(r^* + \mathcal{H} \|\xi\|_{\mathcal{B}}\big)\Big) \\ &= 2\big(\mathcal{P}^* + \mathcal{H}\big) \|\xi\|_{\mathcal{B}} + 2\mathcal{O}^* r^* =: \mu.\end{aligned}$$

Hence,

$$\|z_t + \beta_t\|_{\mathcal{B}} \le \mu, \qquad t \in \mathbb{I}.$$

Using the axioms of the phase space $\mathcal{B}$ and the fact that $z_0 = 0$, there exists a constant $\mu^* > 0$ such that

$$\|z_t + \beta_t\|_{\mathcal{B}} \le \mu^*, \qquad t \in \mathbb{I},$$

where μ^* depends only on r^* and $\|\xi\|_{\mathcal{B}}$. Hence,

$$\|F(t, z_t + \beta_t)\| \le \mu(t)\, \psi(\mu^*), \qquad \text{for a.e. } t \in \mathbb{I}.$$

Substituting this estimate into the previous inequality yields

$$\|(\mathcal{T}_e z)(t)\| \le \frac{1-\sigma}{B(\sigma)}\, \mu(t)\psi(\mu^*) + \frac{\sigma}{B(\sigma)\Gamma(\sigma)} \psi(\mu^*) \int_0^t (t-s)^{\sigma-1} \mu(s)\, ds.$$

Since $\mu(t) \ge 0$ for a.e. $t \in \mathbb{I}$ and $\mu \in L^1(\mathbb{I})$, we have

$$\mu(t) \le \int_0^{\Theta} \mu(s)\, ds = \|\mu\|_{L^1}, \qquad \text{for a.e. } t \in \mathbb{I}.$$

Hence,

$$\|(\mathcal{T}_e z)(t)\| \leq \frac{1-\sigma}{B(\sigma)}\,\psi(\mu^*)\|\mu\|_{L^1} + \frac{\sigma}{B(\sigma)\Gamma(\sigma)}\psi(\mu^*)\int_0^t (t-s)^{\sigma-1}\mu(s)\,ds.$$

Using the identity

$$\int_0^t (t-s)^{\sigma-1}\,ds = \frac{t^\sigma}{\sigma},$$

and the estimate

$$\int_0^t (t-s)^{\sigma-1}\mu(s)\,ds \leq \|\mu\|_{L^1},$$

we obtain

$$\|(\mathcal{T}_e z)(t)\| \leq \left(\frac{1-\sigma}{B(\sigma)} + \frac{\sigma\,\Theta^\sigma}{B(\sigma)\Gamma(\sigma+1)}\right)\psi(\mu^*)\,\|\mu\|_{L^1}.$$

By hypothesis (H3), the right-hand side is bounded above by r^*. Therefore,

$$\|(\mathcal{T}_e z)(t)\| \leq r^*, \qquad \text{for all } t \in \mathbb{I}.$$

For $t \in (-\infty, 0]$, we have $(\mathcal{T}_e z)(t) = 0$, and hence

$$\|(\mathcal{T}_e z)(t)\| = 0 \leq r^*.$$

Consequently,

$$\|\mathcal{T}_e z\|_b \leq r^*,$$

which implies

$$\mathcal{T}_e(\widehat{\mathcal{K}}) \subset \widehat{\mathcal{K}}.$$

Step 2: Continuity of $\mathcal{T}_e$.

Let $\{z_n\}_{n\in\mathbb{N}} \subset \widehat{\mathcal{K}}$ be a sequence such that

$$z_n \to z \quad \text{in } \chi_0,$$

that is,

$$\|z_n - z\|_b \to 0 \quad \text{as } n \to \infty.$$

We show that

$$\|\mathcal{T}_e z_n - \mathcal{T}_e z\|_b \to 0,$$

which proves the continuity of $\mathcal{T}_e$ on $\widehat{\mathcal{K}}$.

For $t \in (-\infty, 0]$, by definition of $\mathcal{T}_e$, we have

$$(\mathcal{T}_e z_n)(t) = (\mathcal{T}_e z)(t) = 0,$$

and hence

$$\|(\mathcal{T}_e z_n)(t) - (\mathcal{T}_e z)(t)\| = 0.$$

Let $t \in \mathbb{I} = [0, \Theta]$. Then

$$\|(\mathcal{T}_e z_n)(t) - (\mathcal{T}_e z)(t)\| \leq \frac{1-\sigma}{B(\sigma)}\|F(t, z_{n,t}+\beta_t) - F(t, z_t+\beta_t)\| + \frac{\sigma}{B(\sigma)\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\|F(s, z_{n,s}+\beta_s) - F(s, z_s+\beta_s)\|\,ds.$$

Since $z_n \to z$ in χ_0 and β is fixed, it follows from the axioms of the phase space $\mathcal{B}$ that

$$z_{n,t} + \beta_t \to z_t + \beta_t \quad \text{in } \mathcal{B}, \qquad \text{for all } t \in \mathbb{I}.$$

By the L^1–Carathéodory property of F in (H1), we obtain

$$F(t, z_{n,t}+\beta_t) \to F(t, z_t+\beta_t) \quad \text{for a.e. } t \in \mathbb{I}.$$

Moreover, since $\{z_n\} \subset \widehat{\mathcal{K}}$, the estimate derived in Step 1 yields

$$\|z_{n,t} + \beta_t\|_{\mathcal{B}} \le \mu^*, \qquad \text{for all } n \in \mathbb{N},\ t \in \mathbb{I},$$

and hence, by assumption (H1),

$$\|F(t, z_{n,t} + \beta_t)\| \le \mu(t)\, \psi(\mu^*), \qquad \text{for a.e. } t \in \mathbb{I}.$$

Since $\mu \in L^1(\mathbb{I})$, the right-hand side belongs to $L^1(\mathbb{I})$.

Therefore, by the dominated convergence theorem, we conclude that

$$\int_0^t (t-s)^{\sigma-1} \|F(s, z_{n,s} + \beta_s) - F(s, z_s + \beta_s)\|\, ds \longrightarrow 0 \quad \text{as } n \to \infty,$$

and consequently,

$$\|(\mathcal{T}_e z_n)(t) - (\mathcal{T}_e z)(t)\| \to 0, \qquad \text{for all } t \in \mathbb{I}.$$

The dominated convergence theorem is applied with the explicit dominating function $g(s) = (t-s)^{\sigma-1}\mu(s)\psi(\mu^*) \in L^1([0,t])$, which is integrable since $\mu \in L^1(\mathcal{I})$ and $(t-s)^{\sigma-1} \in L^1([0,t])$ for $\sigma \in (0,1)$, and which bounds $\|F(s, z_{n,s} + \beta_s)\|$ uniformly in n by hypothesis (H1).

Taking the supremum over $t \in \mathbb{I}$ and recalling that the difference vanishes on $(-\infty, 0]$, we obtain

$$\|\mathcal{T}_e z_n - \mathcal{T}_e z\|_b \to 0, \quad \text{as } n \to \infty.$$

Thus, $\mathcal{T}_e$ is continuous on $\widehat{\mathcal{K}}$.

Step 3: Verification of the DND–contractive condition.

Let $\mathcal{D}$ be a nonempty and convex subset of $\widehat{\mathcal{K}}$. For each $t \in (-\infty, \Theta]$, define

$$\alpha_t = \lambda\big(\mathcal{D}(t)\big), \qquad \mathcal{D}(t) = \{z(t) : z \in \mathcal{D}\} \subset \mathbb{Z}.$$

By assumption (H2), for almost every $t \in \mathbb{I} = [0, \Theta]$, we have

$$\lambda(F(t, \mathcal{D}_t + \beta_t)) \le K(t) \sup_{t \in (-\infty, 0]} h(\alpha_t)\,. \tag{3.11}$$

Let $\varepsilon > 0$ be arbitrary. By the definition of the DND, for a.e. $t \in \mathbb{I}$ there exists a continuous mapping

$$\zeta_t : [0,1] \to \mathbb{Z}, \qquad \zeta_t([0,1]) \subset F(t, \mathcal{D}_t + \beta_t),$$

such that for every $z \in \mathcal{D}$ there exists $\eta \in [0,1]$ satisfying

$$\|F(t, z_t + \beta_t) - \zeta_t(\eta)\| \le K(t) \sup_{t \in (-\infty, 0]} h(\alpha_t) + \varepsilon, \quad \text{for a.e. } t \in \mathbb{I}. \tag{3.12}$$

We now construct a mapping

$$\widetilde{\zeta} : [0,1] \to \chi_0$$

defined by

$$\widetilde{\zeta}(\eta)(t) = \begin{cases} \dfrac{1-\sigma}{B(\sigma)}\, \zeta_t(\eta) + \dfrac{\sigma}{B(\sigma)\Gamma(\sigma)} \displaystyle\int_0^t (t-s)^{\sigma-1} \zeta_s(\eta)\, ds, & t \in \mathbb{I}, \\ 0, & t \in (-\infty, 0]. \end{cases}$$

Then $\widetilde{\zeta}$ is continuous and

$$\widetilde{\zeta}([0,1]) \subset \mathcal{T}_e(\mathcal{D}).$$

Let $z \in \mathcal{D}$. Using (3.12), for $t \in \mathbb{I}$ we obtain

$$\|(\mathcal{T}_e z)(t) - \widetilde{\zeta}(\eta)(t)\| \le \frac{1-\sigma}{B(\sigma)} \Big(K(t) \sup_{t \in (-\infty, 0]} h(\alpha_t) + \varepsilon \Big)$$

$$+ \frac{\sigma}{B(\sigma)\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \Big(K(s) \sup_{t\in(-\infty,0]} h(\alpha_t) + \varepsilon \Big)\, ds.$$

Using the estimate

$$\int_0^t (t-s)^{\sigma-1}\, ds = \frac{t^\sigma}{\sigma} \le \frac{\Theta^\sigma}{\sigma},$$

we obtain

$$\|(\mathcal{T}_e z)(t) - \widetilde{\zeta}(\eta)(t)\| \le \left(\frac{1-\sigma}{B(\sigma)} + \frac{\sigma\,\Theta^\sigma}{B(\sigma)\Gamma(\sigma+1)} \right) \|K\|_\infty \sup_{t\in(-\infty,0]} h(\alpha_t) + \varepsilon C_\Theta,$$

where

$$C_\Theta = \frac{1-\sigma}{B(\sigma)} + \frac{\sigma\,\Theta^\sigma}{B(\sigma)\Gamma(\sigma+1)}.$$

The constant C_Θ is derived from Lemma 3.1(ii): it arises by setting the integrand equal to its L^∞ bound $\|K\|_\infty \sup h(\alpha_t)$ and evaluating $\int_0^t (t-s)^{\sigma-1}\, ds = t^\sigma/\sigma \le \Theta^\sigma/\sigma$, giving the coefficient $(1-\sigma)/B(\sigma) + \sigma\Theta^\sigma/(B(\sigma)\Gamma(\sigma+1)) = C_\Theta$. Hypothesis (H4) requires $C_\Theta\|K\|_\infty < 1$, which ensures that $\widetilde{h}(\rho) = C_\Theta\|K\|_\infty h(\rho)$ belongs to $\mathcal{F}$ and the DND-contractive condition holds.

Since the above estimate is independent of $z \in \mathcal{D}$ and $t \in (-\infty, \Theta]$, letting $\varepsilon \to 0$ yields

$$\lambda\big(\mathcal{T}_e(\mathcal{D})\big) \le C_\Theta \|K\|_\infty \sup_{t\in(-\infty,0]} h(\alpha_t)\,.$$

Define the function $\widetilde{h} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\widetilde{h}(\rho) = C_\Theta\|K\|_\infty h(\rho).$$

By assumption (H4), we have $\widetilde{h} \in \mathcal{F}$. Consequently,

$$\lambda\big(\mathcal{T}_e(\mathcal{D})\big) \le \sup_{t\in(-\infty,0]} \widetilde{h}(\alpha_t)\,.$$

Hence, by the Darbo fixed point theorem based on the DND 2.4, the operator $\mathcal{T}_e$ admits a fixed point $z \in \widehat{\mathcal{K}}$. Setting $\omega = z + \beta$, we conclude that ω is a solution of problem (1.1). □

In order to establish the existence results for the neutral system (1.3), we impose the following conditions.

Remark 3.7. *Compared with hypotheses (H1)–(H4) for the semilinear system, the neutral case requires additional conditions on the neutral term G. Since G appears outside the fractional integral (as $G(t,\omega_t) - G(0,\xi)$), only pointwise measurability and linear growth (G1)–(G2) are needed, in contrast to the L^1-Carathéodory condition required for F. The DND condition (G3) on G is the analogue of (H2) for F, and the combined contraction constant in the modified (H4), namely $C_\Theta\|K\|_\infty + \|\widetilde{K}\|_\infty < 1$, reflects the cumulative effect of both operators.*

The function

$$G : \mathbb{I} \times \mathcal{B} \longrightarrow \mathbb{Z}$$

satisfies the properties listed below.

(G1) For each $\xi \in \mathcal{B}$, the mapping $t \mapsto G(t,\xi)$ is measurable on $\mathbb{I}$, and for almost every $t \in \mathbb{I}$, the mapping $\xi \mapsto G(t,\xi)$ is continuous on $\mathcal{B}$.

(G2) There exist constants $W_1, W_2 > 0$ such that

$$\|G(t,\xi)\|_{\mathbb{Z}} \le W_1\|\xi\|_{\mathcal{B}} + W_2, \qquad \text{for a.e. } t \in \mathbb{I},\ \xi \in \mathcal{B}.$$

(G3) There exist a function $\widetilde{K} \in L^\infty(\mathbb{I}, \mathbb{R}_+)$ and a comparison function $h \in \mathcal{F}$ such that, for every nonempty, bounded, and convex subset $\mathcal{K} \subset \mathcal{B}$,

$$\lambda(G(t, \mathcal{K})) \le \widetilde{K}(t) \sup_{t \in (-\infty, 0]} h(\lambda(\mathcal{K}(t))), \qquad \text{for a.e. } t \in \mathbb{I}.$$

Concrete example for (G2)–(G3). A simple neutral term satisfying (G1)–(G3) is the linear history functional

$$G(t, \xi) = \int_{-\infty}^{0} a(t, \theta)\, \xi(\theta)\, d\theta,$$

where $a : \mathcal{I} \times (-\infty, 0] \to \mathbb{R}$ is measurable and satisfies $\int_{-\infty}^{0} |a(t, \theta)|\, d\theta \le c_0 < \infty$ uniformly in $t \in \mathcal{I}$. Condition (G2) holds with $W_1 = c_0$ and $W_2 = 0$. Since G is linear in ξ, its restriction to any bounded convex set $\mathcal{K} \subset \mathcal{B}$ satisfies $\lambda(G(t, \mathcal{K})) \le c_0\, \lambda(\mathcal{K})$, so (G3) holds with $\widetilde{K}(t) = c_0$ and $h(r) = r$, provided $c_0 < 1$. This covers, in particular, the discrete delay $G(t, \xi) = A\xi(-\tau)$ for a bounded linear operator A and fixed delay $\tau > 0$.

(H_3) There exists a constant $\mu^* > 0$ such that

$$\mu^* \ge 2(W_1 \mu^* + W_2) + \left(\frac{1 - \sigma}{B(\sigma)} + \frac{\sigma \Theta^\sigma}{B(\sigma)\Gamma(\sigma + 1)} \right) \psi(\mu^*) \|\mu\|_{L^1}.$$

(H_4) The inequality

$$\left(\frac{1 - \sigma}{B(\sigma)} + \frac{\sigma\, \Theta^\sigma}{B(\sigma)\Gamma(\sigma + 1)} \right) \|K\|_\infty + \|\widetilde{K}\|_\infty < 1$$

is satisfied.

Theorem 3.6. *Under assumptions* (H1), (H2), (G2), (G3), (H_3) *and* (H_4), *the neutral fractional problem* (1.3) *admits at least one solution* $\omega \in C((-\infty, \Theta], \mathbb{Z})$ *with* $\omega_0 \in \mathcal{B}$.

Proof. Define the operator $\mathcal{T} : \chi \to \chi$ by

$$(\mathcal{T}\omega)(t) = \begin{cases} \xi(0) - G(0, \xi) + G(t, \omega_t) + \dfrac{1 - \sigma}{B(\sigma)} F(t, \omega_t) \\ \qquad + \dfrac{\sigma}{B(\sigma)\Gamma(\sigma)} \displaystyle\int_0^t (t - s)^{\sigma - 1} F(s, \omega_s)\, ds, & t \in \mathbb{I}, \\ \xi(t), & t \in (-\infty, 0]. \end{cases}$$

Let

$$\beta(t) = \begin{cases} \xi(t), & t \in (-\infty, 0], \\ \xi(0), & t \in \mathbb{I}, \end{cases} \qquad \omega(t) = z(t) + \beta(t).$$

Then $z_0 = 0$ and $z \in \chi_0$. Define $\mathcal{T}_e : \chi_0 \to \chi_0$ by

$$(\mathcal{T}_e z)(t) = \begin{cases} G(t, z_t + \beta_t) - G(0, \xi) + \dfrac{1 - \sigma}{B(\sigma)} F(t, z_t + \beta_t) \\ \qquad + \dfrac{\sigma}{B(\sigma)\Gamma(\sigma)} \displaystyle\int_0^t (t - s)^{\sigma - 1} F(s, z_s + \beta_s)\, ds, & t \in \mathbb{I}, \\ 0, & t \in (-\infty, 0]. \end{cases}$$

Let

$$\widehat{\mathcal{K}} = \{ z \in \chi_0 : \|z\|_b \le r^* \}.$$

Step 1: Invariance of $\widehat{\mathcal{K}}$.

Let $z \in \widehat{\mathcal{K}}$. For $t \in \mathbb{I}$, by the definition of $\mathcal{T}_e$,

$$\|(\mathcal{T}_e z)(t)\| \le \|G(t, z_t + \beta_t)\| + \|G(0,\xi)\| + \frac{1-\sigma}{B(\sigma)}\|F(t, z_t + \beta_t)\| + \frac{\sigma}{B(\sigma)\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\|F(s, z_s + \beta_s)\|\,ds.$$

By the axioms of the phase space $\mathcal{B}$ and the fact that $z \in \widehat{\mathcal{K}}$, there exists $\mu^* > 0$, depending only on r^* and $\|\xi\|_{\mathcal{B}}$, such that

$$\|z_t + \beta_t\|_{\mathcal{B}} \le \mu^*, \qquad t \in \mathbb{I}.$$

By assumption (G2),

$$\|G(t, z_t + \beta_t)\| \le W_1\mu^* + W_2, \qquad \|G(0,\xi)\| \le W_1\|\xi\|_{\mathcal{B}} + W_2.$$

Moreover, by (H1),

$$\|F(t, z_t + \beta_t)\| \le \mu(t)\psi(\mu^*), \qquad \text{for a.e. } t \in \mathbb{I}.$$

Using $\mu \in L^1(\mathbb{I})$ and the identity $\int_0^t (t-s)^{\sigma-1}ds = t^\sigma/\sigma$, we obtain

$$\|(\mathcal{T}_e z)(t)\| \le 2(W_1\mu^* + W_2) + \left(\frac{1-\sigma}{B(\sigma)} + \frac{\sigma\Theta^\sigma}{B(\sigma)\Gamma(\sigma+1)}\right)\psi(\mu^*)\|\mu\|_{L^1}.$$

By hypothesis (H3), the right-hand side is bounded by r^*. For $t \in (-\infty, 0]$, $(\mathcal{T}_e z)(t) = 0$. Hence $\|\mathcal{T}_e z\|_b \le r^*$, and therefore

$$\mathcal{T}_e(\widehat{\mathcal{K}}) \subset \widehat{\mathcal{K}}.$$

Step 2: Continuity of $\mathcal{T}_e$.

Let $z_n \to z$ in χ_0. By assumptions (H1) and (G1),

$$F(t, z_{n,t} + \beta_t) \to F(t, z_t + \beta_t), \qquad G(t, z_{n,t} + \beta_t) \to G(t, z_t + \beta_t),$$

for a.e. $t \in \mathbb{I}$. Uniform bounds from Step 1 yield an L^1–dominating function. Hence, by the dominated convergence theorem,

$$\|\mathcal{T}_e z_n - \mathcal{T}_e z\|_b \to 0,$$

so $\mathcal{T}_e$ is continuous on $\widehat{\mathcal{K}}$.

Step 3: DND–contractive condition.

Let $\mathcal{D} \subset \widehat{\mathcal{K}}$ be nonempty, bounded, and convex, and define

$$\alpha_t = \lambda(\mathcal{D}(t)), \qquad t \in (-\infty, \Theta].$$

By assumptions (H2) and (G3), for a.e. $t \in \mathbb{I}$,

$$\lambda(\mathcal{T}_e(\mathcal{D})) \le \Big(C_\Theta\|K\|_\infty + \|\widetilde{K}\|_\infty\Big)\sup_{t\in(-\infty,0]} h(\alpha_t),$$

where

$$C_\Theta = \frac{1-\sigma}{B(\sigma)} + \frac{\sigma\Theta^\sigma}{B(\sigma)\Gamma(\sigma+1)}.$$

Define

$$\widetilde{h}(r) = \Big(C_\Theta\|K\|_\infty + \|\widetilde{K}\|_\infty\Big)h(r).$$

By assumption (H4), $\widetilde{h} \in \mathcal{F}$. Thus, by the Darbo fixed point theorem based on the degree of nondensifiability, $\mathcal{T}_e$ admits a fixed point $z \in \widehat{\mathcal{K}}$. Setting $\omega = z + \beta$, we conclude that ω is a solution of (1.3). $\square$

4. Applications

In this section, we present an academic example posed in an infinite-dimensional Banach space which satisfies all the assumptions of Theorem 3.5. The example illustrates how the abstract hypotheses can be verified explicitly within the framework of $m\mathbb{ABC}$–fractional differential equations with infinite delay.

4.1. **Academic Example.**

Let $\mathbb{I} = [0, 1]$. We consider the $m\mathbb{ABC}$–fractional differential equation with infinite delay

$$\begin{cases} ({}^{m\mathbb{ABC}}D_0^\sigma \omega)(t) = F(t, \omega_t), & t \in \mathbb{I} = [0, 1], \\ \omega(t) = \xi(t), & t \in (-\infty, 0], \end{cases} \tag{4.13}$$

where $0 < \sigma < 1$ and ξ belongs to the phase space $\mathcal{B}$ defined below:

We adopt the Hale–Kato type phase space

$$\mathcal{B} = \Big\{\omega : (-\infty, 0] \to \mathbb{R} : \lim_{\theta \to -\infty} e^{\gamma\theta}\omega(\theta) \text{ exists in } \mathbb{R}\Big\},$$

equipped with the norm

$$\|\omega\|_{\mathcal{B}} = \sup_{\theta \le 0} e^{\gamma\theta} |\omega(\theta)|, \qquad \gamma > 0.$$

It is well known that $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space and satisfies the axioms of an admissible phase space. In particular, there exist constants $\mathcal{H} = 1$, $\mathcal{O}^* = 1$, and $\mathcal{P}^* = 1$ such that for every $\omega : (-\infty, 1] \to \mathbb{R}$ with $\omega_0 \in \mathcal{B}$ and $\omega|_{[0,1]} \in C([0, 1], \mathbb{R})$, $\omega_t \in \mathcal{B}$ for all $t \in [0, 1]$ and

$$|\omega(t)| \le \|\omega_t\|_{\mathcal{B}}, \qquad \|\omega_t\|_{\mathcal{B}} \le \|\omega_0\|_{\mathcal{B}} + \sup_{0 \le s \le t} |\omega(s)|.$$

The nonlinear mapping $F : \mathbb{I} \times \mathcal{B} \to \mathbb{R}$ is defined by

$$F(t, \omega_t) = \frac{\cos t}{3 + t^3} \left(\frac{1}{\sqrt{2 + t}} + \ln\big(1 + |\omega(\xi(t))|\big) \right), \qquad t \in \mathbb{I}, \tag{4.14}$$

where the delay function $\xi : \mathbb{I} \to (-\infty, 0]$ is given by

$$\xi(t) = t - 1.$$

Clearly, $\xi(t) \le 0$ for all $t \in [0, 1]$, and hence $\omega(\xi(t))$ is well defined for every history segment $\omega_t \in \mathcal{B}$.

We work in the Banach space

$$\mathbb{Z} = \mathbb{R}, \qquad \|\cdot\| = |\cdot|.$$

Verification of hypothesis (H1). For any $t \in \mathbb{I}$ and $\omega_t \in \mathcal{B}$, we have

$$\begin{aligned} |F(t, \omega_t)| &\le \frac{|\cos t|}{3 + t^3} \left(\frac{1}{\sqrt{2 + t}} + |\omega(\xi(t))| \right) \\ &\le \frac{1}{3 + t^3} \left(1 + \|\omega_t\|_{\mathcal{B}}\right), \end{aligned}$$

since $|\omega(\xi(t))| \le \|\omega_t\|_{\mathcal{B}}$.

Thus,

$$|F(t, \omega_t)| \le \mu(t)\, \psi(\|\omega_t\|_{\mathcal{B}}),$$

where

$$\mu(t) = \frac{1}{3 + t^3}, \qquad \psi(r) = 1 + r, \quad r \ge 0.$$

Clearly, $\mu \in L^1(\mathbb{I}) \cap L^\infty(\mathbb{I})$, and ψ is continuous and nondecreasing. Hence, hypothesis (H1) is satisfied.

Verification of hypothesis (H2). Let $\mathcal{K} \subset \mathbb{R}$ be a nonempty, bounded, and convex set. Fix $t \in \mathbb{I}$ and let ζ be an α_t–dense curve in $\mathcal{K}$. Then, for every $\omega \in \mathcal{K}$, there exists $\eta \in [0,1]$ such that

$$|\omega - \zeta(\eta)| \leq \alpha_t.$$

Using the logarithmic identity

$$\ln(1+a) - \ln(1+b) = \ln\left(1 + \frac{a-b}{1+b}\right), \quad a, b \geq 0,$$

we obtain

$$\begin{aligned} |F(t,\omega) - F(t,\zeta(\eta))| &= \frac{|\cos t|}{3+t^3} \left|\ln(1+|\omega|) - \ln(1+|\zeta(\eta)|)\right| \\ &\leq \frac{1}{3+t^3} \ln(1 + |\omega - \zeta(\eta)|) \\ &\leq \frac{1}{3+t^3} \ln(1+\alpha_t). \end{aligned}$$

Therefore,

$$\lambda\big(F(t,\mathcal{K})\big) \leq K(t)\, h(\alpha_t),$$

where

$$K(t) = \frac{1}{3+t^3}, \qquad h(r) = \ln(1+r).$$

Since $h(r) < r$ for all $r > 0$, it follows that $h \in \mathcal{F}$. Thus, hypothesis (H2) is verified.
Verification of hypothesis (H3). Let

$$M_\xi := \|\xi\|_{\mathcal{B}} = \sup_{\theta \leq 0} e^{\gamma\theta} |\xi(\theta)|.$$

Then, in particular,

$$|\xi(0)| \leq M_\xi.$$

From hypothesis (H1), we recall that

$$\psi(r) = 1 + r, \qquad \mu(t) = \frac{1}{3+t^3}.$$

Hence,

$$\|\mu\|_\infty = \sup_{t\in[0,1]} \frac{1}{3+t^3} = \frac{1}{3}, \qquad \|\mu\|_{L^1} = \int_0^1 \frac{1}{3+t^3}\, dt < \frac{1}{3}.$$

We seek $\bar{r} > 0$ such that

$$\bar{r} \geq |\xi(0)| + \frac{1-\sigma}{B(\sigma)} \psi(\bar{r})\, \|\mu\|_\infty + \frac{\sigma\, \Theta^\sigma}{B(\sigma)\Gamma(\sigma+1)} \psi(\bar{r})\, \|\mu\|_{L^1},$$

with $\Theta = 1$.

Substituting the above estimates yields

$$\begin{aligned} \bar{r} &\geq M_\xi + \frac{1-\sigma}{B(\sigma)}(1+\bar{r})\frac{1}{3} + \frac{\sigma}{B(\sigma)\Gamma(\sigma+1)}(1+\bar{r})\|\mu\|_{L^1} \\ &= M_\xi + (1+\bar{r}) \left[\frac{1-\sigma}{3B(\sigma)} + \frac{\sigma\|\mu\|_{L^1}}{B(\sigma)\Gamma(\sigma+1)}\right]. \end{aligned}$$

Define

$$C_\sigma = \frac{1-\sigma}{3B(\sigma)} + \frac{\sigma\|\mu\|_{L^1}}{B(\sigma)\Gamma(\sigma+1)}.$$

Then

$$\bar{r} \geq M_\xi + C_\sigma(1+\bar{r}),$$

or equivalently,

$$\bar{r}(1 - C_\sigma) \geq M_\xi + C_\sigma.$$

Provided $1 - C_\sigma > 0$, we obtain

$$\bar{r} \geq \frac{M_\xi + C_\sigma}{1 - C_\sigma}.$$

For instance, if $\sigma = \frac{1}{2}$ and $B(\sigma) = 1$, then

$$C_{1/2} < \frac{1}{6} + \frac{0.5 \times 0.33}{0.886} \approx 0.36,$$

so $1 - C_{1/2} > 0$. Hence, hypothesis (H3) holds.
Verification of hypothesis (H4). Finally, we verify

$$\left(\frac{1-\sigma}{B(\sigma)} + \frac{\sigma\,\Theta^\sigma}{B(\sigma)\Gamma(\sigma+1)} \right) \|K\|_\infty < 1.$$

Since

$$\|K\|_\infty = \sup_{t \in [0,1]} \frac{1}{3+t^3} = \frac{1}{3},$$

the above inequality is satisfied for all $0 < \sigma < 1$. Thus, hypothesis (H4) is fulfilled.

Consequently, all assumptions (H1)–(H4) are satisfied. By Theorem 3.5, problem (4.13) admits at least one solution $\omega \in C((-\infty, 1], \mathbb{R}), \quad \omega_0 \in \mathcal{B}$.

Remark 4.8. *The logarithmic function* $\ln(1 + |\omega|)$ *used in* (4.14) *is not the only choice for satisfying hypothesis (H2). The logarithm was selected here because it naturally yields the comparison function* $h(r) = \ln(1 + r) \in \mathcal{F}$ *(since* $\ln(1 + r) < r$ *for all* $r > 0$*), providing a clean and explicit verification. More generally, any nonlinear term* $F(t, \xi)$ *admitting a sub-additive or concave-type DND estimate of the form* $\lambda(F(t, \mathcal{K})) \leq K(t)\, h(\lambda(\mathcal{K}))$ *with* $h \in \mathcal{F}$ *is admissible. Valid choices include: Lipschitz-type terms (where* $h(r) = Lr$, $L < 1$, *as in Example 4.1 of Section 4); terms involving the rational comparison function* $h(r) = r/(1 + r)$ *(whose iterates satisfy* $h^n(r) = r/(1 + nr) \to 0$*); or any term for which* $F(\cdot, \mathcal{K})$ *has a DND bounded by any function in* $\mathcal{F}$*.*

4.2. **Population model with infinite memory.**
We consider the $m\mathbb{ABC}$–fractional system with infinite delay

$$\begin{cases} ({}^{m\mathbb{ABC}}D_0^\sigma \omega)(t)(\nu) = \displaystyle\int_0^1 k(\nu, \eta)\, \phi\left(\int_{-\infty}^0 w(\theta)\, \omega(t + \theta)(\eta)\, d\theta \right) d\eta, & t \in [0, \Theta], \\ \omega(t) = \xi(t), & t \in (-\infty, 0], \end{cases} \tag{4.15}$$

where the model components admit the following biological interpretation.

- The state variable $\omega(t)(\nu)$ denotes the population density at time t associated with individuals characterized by the trait $\nu \in [0, 1]$. The trait variable may represent age, spatial position, genetic characteristics, or another relevant biological attribute.
- The operator ${}^{m\mathbb{ABC}}D_0^\sigma$ with $0 < \sigma < 1$ models the population growth rate with long–term hereditary memory, accounting for nonlocal temporal effects commonly observed in biological systems with persistent environmental or genetic influence.
- The kernel $k(\nu, \eta) \in C([0, 1] \times [0, 1])$ represents the interaction intensity between individuals with traits ν and η. It describes how the subpopulation with trait η contributes to the growth of the subpopulation with trait ν, incorporating effects such as migration, mutation, or cross–interaction among traits.

- The memory kernel $w : (-\infty, 0] \to \mathbb{R}_+$ assigns weights to past population states, with the normalization condition
$$\int_{-\infty}^{0} w(\theta)\, d\theta = 1$$
ensuring that the cumulative effect of the entire past history is taken as a weighted average. This term captures long–term biological memory, such as delayed maturation, accumulated reproductive effects, or persistent environmental feedback.
- The nonlinear function $\phi : \mathbb{R} \to \mathbb{R}$ represents the population response to the accumulated past population level, allowing for saturation, density dependence, or other nonlinear growth mechanisms.
- The function $\xi \in \mathcal{B}$ specifies the initial population history over the interval $(-\infty, 0]$, reflecting the fact that the present dynamics depend on the entire past evolution of the population.

Consequently, system (4.15) describes a trait–structured population whose evolution is influenced by nonlinear interactions and the cumulative effect of its entire past history, with the fractional derivative accounting for hereditary and memory–driven growth dynamics.

Define the nonlinear mapping
$$F : \mathbb{I} \times \mathcal{B} \to \mathbb{Z}$$
by
$$F(t, \xi)(\nu) = \int_0^1 k(\nu, \eta)\, \phi\Big(\int_{-\infty}^{0} w(\theta)\, \xi(\theta)(\eta)\, d\theta\Big)\, d\eta.$$
Then problem (4.15) is of the form (1.1).
Verification of Hypothesis (H1). Assume that ϕ satisfies the linear growth condition
$$|\phi(s)| \le a|s| + b, \qquad s \in \mathbb{R},$$
for some constants $a, b > 0$.

Let $\xi \in \mathcal{B}$. Then
$$\begin{aligned}\|F(t, \xi)\|_{\mathbb{Z}} &= \sup_{\nu \in [0,1]} \left| \int_0^1 k(\nu, \eta)\, \phi\Big(\int_{-\infty}^{0} w(\theta)\, \xi(\theta)(\eta)\, d\theta\Big)\, d\eta \right| \\ &\le \|k\|_\infty \int_0^1 \left| \phi\Big(\int_{-\infty}^{0} w(\theta)\, \xi(\theta)(\eta)\, d\theta\Big) \right| d\eta.\end{aligned}$$

Using Jensen's inequality and the normalization of w, we obtain
$$\left| \int_{-\infty}^{0} w(\theta)\, \xi(\theta)(\eta)\, d\theta \right| \le \|\xi\|_{\mathcal{B}}.$$
Hence,
$$\|F(t, \xi)\|_{\mathbb{Z}} \le \|k\|_\infty \big(a\|\xi\|_{\mathcal{B}} + b\big).$$

Thus hypothesis (H1) holds with
$$\mu(t) \equiv \|k\|_\infty, \qquad \psi(r) = ar + b,$$
where ψ is continuous and nondecreasing on $\mathbb{R}_+$.

Verification of Hypothesis (H2). Assume additionally that ϕ is Lipschitz continuous with constant $L \in (0,1)$.

Let $\mathcal{K} \subset \mathcal{B}$ be nonempty, bounded, and convex. For $\xi_1, \xi_2 \in \mathcal{K}$, we have

$$\begin{aligned}\|F(t,\xi_1) - F(t,\xi_2)\|_{\mathbb{Z}} &\le \|k\|_\infty \int_0^1 \left|\phi\Big(\int_{-\infty}^0 w(\theta)\xi_1(\theta)(\eta)d\theta\Big) - \phi\Big(\int_{-\infty}^0 w(\theta)\xi_2(\theta)(\eta)d\theta\Big)\right| d\eta \\ &\le L\|k\|_\infty \sup_{\theta\in(-\infty,0]} \|\xi_1(\theta) - \xi_2(\theta)\|_{\mathbb{Z}}.\end{aligned}$$

Therefore,

$$\lambda(F(t,\mathcal{K})) \le \|k\|_\infty \sup_{\theta\in(-\infty,0]} h(\lambda(\mathcal{K}(\theta))),$$

where $h(r) = Lr \in \mathcal{F}$. Hence hypothesis (H2) is satisfied. Following the same procedure adopted in Example 4.1, the hypotheses (H3) and (H4) can be verified in a straightforward manner for the present model, and hence the corresponding conditions are satisfied.

Therefore, all assumptions of Theorem 3.5 are satisfied, and problem (4.15) admits at least one solution.

5. Conclusions

In this paper, we have established existence results for a class of $m\mathbb{ABC}$ fractional differential equations with infinite delay in real Banach spaces, addressing both semilinear and neutral systems. The analysis was carried out via the degree of nondensifiability (DND) and an associated fixed-point theorem, without imposing compactness or global Lipschitz conditions on the nonlinear terms.

The main theoretical contributions can be summarised as follows. For the semilinear system (1.1), Theorem 3.5 guarantees the existence of at least one mild solution under the growth condition (H1), the DND-contractive estimate (H2), a priori bound (H3), and the contraction coefficient condition (H4). The proof employs a decomposition $\omega = z + \beta$, reduces the problem to a fixed-point equation for the modified operator $\mathcal{T}_e$ on an invariant ball $\widehat{\mathcal{K}}$, and verifies the DND-contractive condition via the explicit constant

$$C_\Theta = \frac{1-\sigma}{B(\sigma)} + \frac{\sigma\,\Theta^\sigma}{B(\sigma)\Gamma(\sigma+1)}.$$

For the neutral system (1.3), Theorem 3.6 extends the result to include a neutral term $G(t,\omega_t)$, under the additional linear growth condition (G2) and DND estimate (G3), with the combined contraction coefficient $C_\Theta\|K\|_\infty + \|\widetilde{K}\|_\infty < 1$.

The key novelty is that both results apply without classical measures of noncompactness, replacing the Darbo-type contraction with the weaker DND-contractiveness condition $\lambda(\mathcal{T}_e(\mathcal{D})) \le \widetilde{h}(\lambda(\mathcal{D}))$ for some $\widetilde{h} \in \mathcal{F}$. This represents, to the best of our knowledge, the first systematic combination of the $m\mathbb{ABC}$ fractional derivative, infinite delay via the Hale–Kato phase space, and densifiability techniques in a general Banach space setting.

The applicability of the abstract theory was demonstrated through two concrete examples: an academic model in the infinite-dimensional space $L^2([0,1],\mathbb{R})$ with explicit verification of all hypotheses (H1)–(H4), and a trait-structured population dynamics model with long-term hereditary memory, confirming that the results are relevant to biological systems governed by nonlinear interactions and cumulative past effects.

Several directions for future research naturally emerge. These include:

- the investigation of uniqueness of solutions via stricter Lipschitz-type estimates on F;
- Ulam–Hyers stability of the fixed point, which would quantify the structural robustness of solutions against bounded perturbations;

- the incorporation of impulsive effects or state-dependent delays, requiring appropriate modifications of the phase space axioms;
- the extension to fractional stochastic systems with infinite delay, where the DND framework would need to be adapted via Itô-type estimates;
- the study of approximate controllability of the controlled $m\mathbb{ABC}$ system, using resolvent operator techniques in conjunction with DND-based fixed-point arguments.

These extensions present distinct mathematical challenges and will be pursued in subsequent work.

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