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On Quotient Near-Rings having $\mathrm{D}_{(1,\alpha)}/\mathscr{P}-$Derivations

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ABSTRACT. Consider a near-ring $\mathcal{N}$ with a prime ideal $\mathscr{P}$. In this study, we introduce a mapping, termed a $\mathrm{D}_{(1,\alpha)}/\mathscr{P}-$ derivation, defined as an additive function $d : \mathcal{N} \to \mathcal{N}$ satisfying $d(n_1n_2) - (d(n_1)n_2 + \alpha(n_1)d(n_2)) \in \mathscr{P}$ for all $n_1, n_2 \in \mathcal{N}$, where $\alpha : \mathcal{N} \to \mathcal{N}$ is a given map. We investigate conditions ensuring that a non-trivial $\mathrm{D}_{(1,\alpha,\mathscr{P})}-$derivation induces commutativity in the quotient $\mathcal{N}/\mathscr{P}$. Our results extend prior findings in the literature by providing new insights into the structural properties of near-rings. Additionally, we include an illustrative example to validate our theoretical claims.

1. INTRODUCTION

A near-ring is an algebraic system equipped with two operations: addition $'+'$, which forms a group, and an associative multiplication $'\cdot'$, adhering to a single distributive law. Specifically, a structure $(\mathcal{N}, +, \cdot)$ is a left near-ring if it satisfies $n_1 \cdot (n_2 + n_3) = n_1 \cdot n_2 + n_1 \cdot n_3$ for all $n_1, n_2, n_3 \in \mathcal{N}$, or a right near-ring if $(n_1 + n_2) \cdot n_3 = n_1 \cdot n_3 + n_2 \cdot n_3$. A left near-ring is zero-symmetric if $0 \cdot n_1 = 0$ for all $n_1 \in \mathcal{N}$, with $n_1 \cdot 0 = 0$ following from the left distributive property. The multiplicative center of $\mathcal{N}$ is defined as $\mathscr{Z}(\mathcal{N}) = \{n_1 \in \mathcal{N} \mid n_1n_2 = n_2n_1 \text{ for all } n_2 \in \mathcal{N}\}$, and for a prime ideal $\mathscr{P}$, the center of the quotient $\mathcal{N}/\mathscr{P}$ is $\mathscr{Z}(\mathcal{N}/\mathscr{P}) = \{\overline{n_1} \in \mathcal{N}/\mathscr{P} \mid \overline{n_1n_2} = \overline{n_2n_1} \text{ for all } \overline{n_2} \in \mathcal{N}/\mathscr{P}\}$. A near-ring $\mathcal{N}$ is prime if $n_1\mathcal{N}n_2 = \{0\}$ implies $n_1 = 0$ or $n_2 = 0$. An ideal $\mathscr{P}$ of $\mathcal{N}$ is a normal subgroup of $(\mathcal{N}, +)$ that is both a left ideal ($\mathscr{P}\mathcal{N} \subseteq \mathscr{P}$) and a right ideal ($(n_1 + p)n_2 - n_1n_2 \in \mathscr{P}$) for all $n_1, n_2 \in \mathcal{N}$, and is prime if $n_1\mathcal{N}n_2 \subseteq \mathscr{P}$ implies $n_1 \in \mathscr{P}$ or $n_2 \in \mathscr{P}$ [15]. An additive map $\mathrm{D} : \mathcal{N} \to \mathcal{N}$ is a $\mathrm{D}_{(1,\alpha)}-$derivation if there exists a map $\alpha : \mathcal{N} \to \mathcal{N}$ such that $\mathrm{D}(n_1n_2) = \mathrm{D}(n_1)n_2 + \alpha(n_1)\mathrm{D}(n_2)$ for all $n_1, n_2 \in \mathcal{N}$. A map D is $\mathscr{P}-$additive if $\mathrm{D}(n_1 + n_2) - (\mathrm{D}(n_1) + \mathrm{D}(n_2)) \in \mathscr{P}$ for all $n_1, n_2 \in \mathcal{N}$, and $\mathscr{P}-$trivial if $\mathrm{D}(\mathcal{N}) \subseteq \mathscr{P}$.

Derivations are mappings that preserve algebraic structures through specific product rules. In the context of near-rings, Bell and Mason [6] characterized a derivation as an additive map $\mathrm{D} : \mathcal{N} \to \mathcal{N}$ satisfying $\mathrm{D}(n_1n_2) = \mathrm{D}(n_1)n_2 + n_1\mathrm{D}(n_2)$, while Wang [18] investigated the reverse form $\mathrm{D}(n_1n_2) = n_1\mathrm{D}(n_2) + \mathrm{D}(n_1)n_2$. These concepts have been extended in various studies, exploring the properties of such mappings in near-rings [1–3, 5, 7–9, 14, 17]. Recent work by Mouhssine and Boua [16] introduced a quotient derivation $\widetilde{\mathrm{D}}\,(\overline{n_1}) = \overline{\mathrm{D}(n_1)}$ on $\mathcal{N}/\mathscr{P}$, inspiring our exploration of a more general derivation type. Ashraf et al. [4] proved that a prime near-ring $\mathcal{N}$ with a nonzero derivation D satisfying $\mathrm{D}([n_1, n_2]) = [\mathrm{D}(n_1), n_2]$ is commutative. En-Guady et al. [13] studied commutativity in near-rings with a left derivation D and a multiplier $\mathscr{H}$ such that $\mathrm{D}([n_1, n_2]) = \mathscr{H}([n_1, n_2])$ for $n_2 \in \mathscr{L}$, $n_1 \in \mathcal{N}$, where $\mathscr{L}$ is a Lie ideal. In 2024, Boua et al. [10] examined

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identities like (i) $\mathrm{D}([n_1, n_2]) - [\mathrm{D}(n_1), n_2] \in \mathscr{P}$, (ii) $\mathrm{D}([n_1, n_2]) - \mathscr{H}([n_1, n_2]) \in \mathscr{P}$, and (iii) $\mathrm{D}(n_1 \circ n_2) - \mathscr{H}(n_1 \circ n_2) \in \mathscr{P}$, where $\mathscr{H}$ is a left multiplier and $\mathscr{P}$ a prime ideal, to establish commutativity in $\mathcal{N}/\mathscr{P}$. In 2026, El Amrani and Raji [11] investigated when a quotient near-ring is commutative, under the presence of a left derivation together with a multiplier satisfying certain differential identities. In a further paper [12], they refined this study by clarifying how left derivations interact with such multipliers and by proving several new conditions that ensure the corresponding quotient near-ring is commutative.

Drawing from these advancements, we propose a new mapping called a $\mathrm{D}_{(1,\alpha)}/\mathscr{P}$-derivation on a near-ring $\mathcal{N}$ with a prime ideal $\mathscr{P}$. An additive map $\mathrm{D} : \mathcal{N} \to \mathcal{N}$ is a $\mathrm{D}_{(1,\alpha)}/\mathscr{P}$-derivation if there exists a map $\alpha : \mathcal{N} \to \mathcal{N}$ such that D is $\mathscr{P}$-additive (i.e., $\mathrm{D}(n_1 + n_2) - (\mathrm{D}(n_1) + \mathrm{D}(n_2)) \in \mathscr{P}$) and $\mathrm{D}(n_1 n_2) - (\mathrm{D}(n_1) n_2 + \alpha(n_1)\mathrm{D}(n_2)) \in \mathscr{P}$ for all $n_1, n_2 \in \mathcal{N}$. This generalizes the $\mathrm{D}_{(1,\alpha)}$-derivation, where $\mathrm{D}(n_1 n_2) = \mathrm{D}(n_1) n_2 + \alpha(n_1)\mathrm{D}(n_2)$, and extends prior results. We investigate conditions under which a non-trivial $\mathrm{D}_{(1,\alpha)}/\mathscr{P}$-derivation induces commutativity in $\mathcal{N}/\mathscr{P}$, offering new insights into quotient near-ring structures.

To clarify the difference between a $\mathrm{D}_{(1,\alpha)}/\mathscr{P}$-derivation and a $\mathrm{D}_{(1,\alpha)}$-derivation, consider a left near-ring $\mathcal{M}$. Define:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & n_1 & n_2 \\ 0 & 0 & 0 \\ 0 & n_3 & 0 \end{pmatrix} \mid n_1, n_2, n_3 \in \mathcal{M} \right\}, \quad \mathscr{P} = \left\{ \begin{pmatrix} 0 & m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid m \in \mathcal{M} \right\},$$

with maps $\mathrm{D}, \alpha : \mathcal{N} \to \mathcal{N}$ given by:

$$\mathrm{D}\left(\begin{pmatrix} 0 & n_1 & n_2 \\ 0 & 0 & 0 \\ 0 & n_3 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & n_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \alpha\left(\begin{pmatrix} 0 & n_1 & n_2 \\ 0 & 0 & 0 \\ 0 & n_3 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & n_2 & 0 \\ 0 & 0 & 0 \\ 0 & n_3 & 0 \end{pmatrix}.$$

We can see that D is a $\mathrm{D}_{(1,\alpha)}/\mathscr{P}-$derivation but not a $\mathrm{D}_{(1,\alpha)}-$derivation.

This paper is organized as follows: Section 2 provides key preliminary results. Section 3 presents our main theorems on commutativity in quotient near-rings and provides examples in support of this discussion. Section 4 discusses potential directions for future research.

2. Preliminary Lemmas

The following lemmas are essential for proving our main results.

Lemma 2.1. *Let $\mathcal{N}$ be a near-ring with an ideal $\mathscr{P}$. Suppose $\mathcal{N}$ admits a $\mathrm{D}_{(1,\alpha)}/\mathscr{P}-$derivation $\mathrm{D} : \mathcal{N} \to \mathcal{N}$ associated with $\mathscr{P}-$additive map α, where $\mathrm{D}(\mathscr{P}) \subseteq \mathscr{P}$ and $\alpha(\mathscr{P}) \subseteq \mathscr{P}$. Define a map $\widetilde{\mathrm{D}}$: $\mathcal{N}/\mathscr{P} \to \mathcal{N}/\mathscr{P}$ by $\widetilde{\mathrm{D}}\ (\overline{n_1}) = \overline{\mathrm{D}(n_1)}$. Then $\widetilde{\mathrm{D}}$ is a $\mathrm{D}_{(1,\widetilde{\alpha})}-$derivation on $\mathcal{N}/\mathscr{P}$, associated with $\widetilde{\alpha}\ (\overline{n_1}) = \overline{\alpha(n_1)}$.*

Proof. $\widetilde{\mathrm{D}}$ is well defined. Indeed, let $n_1 \in \overline{n_2}$, then $n_1 - n_2 = p$ for some $p \in \mathscr{P}$ and so, $D(n_1) - (D(n_2) + D(p)) \in \mathscr{P}$. Therefore, $\widetilde{\mathrm{D}}\ (\overline{n_1}) = \overline{\mathrm{D}(n_1)} = \overline{\mathrm{D}(n_2)} = \widetilde{\mathrm{D}}\ (\overline{n_2})$. Using similar techniques, one can prove $\widetilde{\alpha}$ is well-defined.

For all $\overline{n_1}, \overline{n_2} \in \mathscr{N}/\mathscr{P}$, we have

$$\begin{aligned}\overset{\sim}{\mathrm{D}}\,(\overline{n_1}+\overline{n_2}) &= \overset{\sim}{\mathrm{D}}\,(\overline{n_1+n_2})\\ &= \overline{\mathrm{D}(n_1+n_2)}\\ &= \overline{\mathrm{D}(n_1)+\mathrm{D}(n_2)}\\ &= \overline{\mathrm{D}(n_1)}+\overline{\mathrm{D}(n_2)}\\ &= \overset{\sim}{\mathrm{D}}\,(\overline{n_1})+\overset{\sim}{\mathrm{D}}\,(\overline{n_2}),\end{aligned}$$

and

$$\begin{aligned}\overset{\sim}{\mathrm{D}}\,(\overline{n_1}\cdot\overline{n_2}) &= \overset{\sim}{\mathrm{D}}\,(\overline{n_1 n_2})\\ &= \overline{\mathrm{D}(n_1 n_2)}\\ &= \overline{\mathrm{D}(n_1)n_2+\alpha(n_1)\mathrm{D}(n_2)}\\ &= \overline{\mathrm{D}(n_1)}\,\overline{n_2}+\overline{\alpha(n_1)\mathrm{D}(n_2)}\\ &= \overset{\sim}{\mathrm{D}}\,(\overline{n_1})\overline{n_2}+\overset{\sim}{\alpha}\,(\overline{n_1})\,\overset{\sim}{\mathrm{D}}\,(\overline{n_2}).\end{aligned}$$

Thus, $\overset{\sim}{\mathrm{D}}$ is a $\mathrm{D}_{(1,\overset{\sim}{\alpha})}$ −derivation on $\mathscr{N}/\mathscr{P}$ with $\overset{\sim}{\alpha}$. □

Lemma 2.2. *Let $\mathscr{N}$ be a near-ring with an ideal $\mathscr{P}$. If $\mathscr{N}$ has a $\mathrm{D}_{(1,\alpha)}/\mathscr{P}$ −derivation D associated with a map α, then for all $n_1, n_2, n_3 \in \mathscr{N}$, the following holds:*

$$\begin{aligned}(\mathrm{D}(n_1)n_2+\alpha(n_1)\mathrm{D}(n_2))n_3 \;&-\; (\mathrm{D}(n_1)n_2n_3+\alpha(n_1)\mathrm{D}(n_2)n_3+\alpha(n_1)\alpha(n_2)\mathrm{D}(n_3)\\ &-\; \alpha(n_1n_2)\mathrm{D}(n_3)) \in \mathscr{P}.\end{aligned}$$

Proof. For all $n_1, n_2, n_3 \in \mathscr{N}$, we see that

$$\begin{aligned}\overline{\mathrm{D}(n_1(n_2n_3))} &= \overline{\mathrm{D}(n_1)}\,\overline{n_2n_3}+\overline{\alpha(n_1)\mathrm{D}(n_2n_3)}\\ &= \overline{\mathrm{D}(n_1)}\,\overline{n_2n_3}+\overline{\alpha(n_1)\mathrm{D}(n_2)}\,\overline{n_3}+\overline{\alpha(n_1)\alpha(n_2)\mathrm{D}(n_3)},\end{aligned}$$

and

$$\begin{aligned}\overline{\mathrm{D}((n_1n_2)n_3)} &= \overline{\mathrm{D}(n_1n_2)}\,\overline{n_3}+\overline{\alpha(n_1n_2)\mathrm{D}(n_3)}\\ &= (\overline{\mathrm{D}(n_1)}\,\overline{n_2}+\overline{\alpha(n_1)\mathrm{D}(n_2)})\overline{n_3}+\overline{\alpha(n_1n_2)\mathrm{D}(n_3)}.\end{aligned}$$

Equating these and simplifying yields

$$\begin{aligned}(\overline{\mathrm{D}(n_1)}\,\overline{n_2}+\overline{\alpha(n_1)\mathrm{D}(n_2)})\overline{n_3} &= \overline{\mathrm{D}(n_1)}\,\overline{n_2n_3}+\overline{\alpha(n_1)\mathrm{D}(n_2)}\,\overline{n_3}+\overline{\alpha(n_1)\alpha(n_2)\mathrm{D}(n_3)}\\ &\quad -\overline{\alpha(n_1n_2)\mathrm{D}(n_3)},\end{aligned}$$

confirming the required relation. □

Lemma 2.3. *Let $\mathscr{N}$ be a near-ring, $\mathscr{P}$ a prime ideal, and D a $\mathrm{D}_{(1,\alpha)}/\mathscr{P}$ −derivation associated with a map α.*

(i) If $n_1\mathrm{D}(\mathscr{N}) \subseteq \mathscr{P}$, $n_1 \in \mathscr{N}$, and α is surjective, then D is $\mathscr{P}$ −trivial or $n_1 \in \mathscr{P}$.

(ii) If $\mathrm{D}(\mathscr{N})n_1 \subseteq \mathscr{P}$ and $\alpha(n_2n_3) - \alpha(n_2)\alpha(n_3) \in \mathscr{P}$ for all $n_2, n_3 \in \mathscr{N}$, then D is $\mathscr{P}$-trivial or $n_1 \in \mathscr{P}$.

Proof.

(i) If $n_1 \mathrm{D}(\mathcal{N}) \subseteq \mathcal{P}$, then $\overline{n_1}\overline{\mathrm{D}(n_2)} = \overline{0}$ for all $n_2 \in \mathcal{N}$. Substituting $n_2 n_3$ for n_2, we get

$$\begin{aligned}\overline{n_1}\ \overline{\mathrm{D}(n_2 n_3)} &= \overline{n_1}\left(\overline{\mathrm{D}(n_2)n_3 + \alpha(n_2)\mathrm{D}(n_3)}\right)\\ &= \overline{n_1}\ \overline{\alpha(n_2)\mathrm{D}(n_3)}\\ &= \overline{0}.\end{aligned}$$

Since $\mathcal{N}/\mathcal{P}$ is prime, $\overline{n_1} = \overline{0}$ or $\overline{\mathrm{D}(n_3)} = \overline{0}$, so $n_1 \in \mathcal{P}$ or D is $\mathcal{P}-$trivial.

(ii) If $\mathrm{D}(\mathcal{N})n_1 \subseteq \mathcal{P}$, then $\overline{\mathrm{D}(n_2 n_3)}\overline{n_1} = (\overline{\mathrm{D}(n_2)}\overline{n_3} + \overline{\alpha(n_2)\mathrm{D}(n_3)})\overline{n_1} = \overline{0}$. Using Lemma 2.2, we obtain

$$\overline{\mathrm{D}(n_2)}\overline{n_3 n_1} + \overline{\alpha(n_2)\mathrm{D}(n_3)}\overline{n_1} + \overline{\alpha(n_2)\alpha(n_3)\mathrm{D}(n_1)} - \overline{\alpha(n_2 n_3)\mathrm{D}(n_1)} = \overline{0}.$$

Given $\alpha(n_2 n_3) - \alpha(n_2)\alpha(n_3) \in \mathcal{P}$, this reduces to $\overline{\mathrm{D}(n_2)}\mathcal{N}/\mathcal{P}\overline{n_1} = \{\overline{0}\}$, so $n_1 \in \mathcal{P}$ or D is $\mathcal{P}-$trivial. □

Lemma 2.4. *Let $\mathcal{N}$ be a near-ring, $\mathcal{P}$ a prime ideal with $\mathcal{N}/\mathcal{P}$ $2-$torsion free, and D a $\mathrm{D}_{(1,\alpha)}/\mathcal{P}-$derivation with α surjective, $\alpha(\mathrm{D}(\mathcal{N})) - \mathrm{D}(\alpha(\mathcal{N})) \subseteq \mathcal{P}$, and $\mathrm{D}^2(\mathcal{N}) \subseteq \mathcal{P}$. Then D is $\mathcal{P}-$trivial.*

Proof. If $\mathrm{D}^2(\mathcal{N}) \subseteq \mathcal{P}$, then $\overline{\mathrm{D}^2(n_1)} = \overline{0}$. Substituting $n_1 n_2$ for n_1, we get

$$\begin{aligned}\overline{0} &= \overline{\mathrm{D}(\mathrm{D}(n_1)n_2 + \alpha(n_1)\mathrm{D}(n_2))}\\ &= \overline{\mathrm{D}^2(n_1)}\overline{n_2} + \overline{\alpha(\mathrm{D}(n_1))\mathrm{D}(n_2)}\\ &\quad + \overline{\mathrm{D}(\alpha(n_1))\mathrm{D}(n_2)} + \overline{\alpha^2(n_1)\mathrm{D}^2(n_2)}\\ &= \overline{\alpha(\mathrm{D}(n_1))\mathrm{D}(n_2)} + \overline{\mathrm{D}(\alpha(n_1))\mathrm{D}(n_2)}.\end{aligned}$$

Since $\overline{\alpha\mathrm{D}} = \overline{\mathrm{D}\alpha}$, the above expression gives that

$$\overline{2\mathrm{D}(\alpha(n_1))\mathrm{D}(n_2)} = \overline{0}.$$

By $2-$torsion freeness, $\overline{\mathrm{D}(\alpha(n_1))\mathrm{D}(n_2)} = \overline{0}$. Since α is surjective, Lemma 2.3 implies $\mathrm{D}(\mathcal{N}) \subseteq \mathcal{P}$, so D is $\mathcal{P}-$trivial. □

3. Commutativity in Quotient Near-Rings having $\mathrm{D}_{(1,\alpha)}/\mathcal{P}-$derivation

Theorem 3.1. *Consider a near-ring $\mathcal{N}$ with a prime ideal $\mathcal{P}$ such that the quotient $\mathcal{N}/\mathcal{P}$ is 2-torsion free. Let D be a $\mathrm{D}_{(1,\alpha)}/\mathcal{P}$-derivation satisfying $\mathrm{D}(\mathcal{N}) \nsubseteq \mathcal{P}$, where α is a surjective map and $\alpha(\mathrm{D}(\mathcal{N})) - \mathrm{D}(\alpha(\mathcal{N})) \subseteq \mathcal{P}$. If $\overline{\mathrm{D}(\mathcal{N})} \subseteq \mathcal{Z}(\mathcal{N}/\mathcal{P})$, then the additive group $(\mathcal{N}/\mathcal{P}, +)$ is abelian. Furthermore, if $\mathrm{D}^2(\mathcal{N}) \nsubseteq \mathcal{P}$, then $\mathcal{N}/\mathcal{P}$ forms a commutative ring.*

Proof. Assume there exists an element $n_1 \in \mathcal{N}$ such that $\overline{\mathrm{D}(n_1)} \neq \overline{0}$. Since $\overline{\mathrm{D}(n_1)} \in \mathcal{Z}(\mathcal{N}/\mathcal{P}) \setminus \{\overline{0}\}$, it follows that $\overline{\mathrm{D}(n_1)} + \overline{\mathrm{D}(n_1)} \in \mathcal{Z}(\mathcal{N}/\mathcal{P})$. Thus, for any $n_2, n_3 \in \mathcal{N}$, we have

$$(\overline{\mathrm{D}(n_1)} + \overline{\mathrm{D}(n_1)})(\overline{n_2} + \overline{n_3}) = (\overline{n_2} + \overline{n_3})(\overline{\mathrm{D}(n_1)} + \overline{\mathrm{D}(n_1)}).$$

Expanding both sides, this gives

$$\overline{n_2}\overline{\mathrm{D}(n_1)} + \overline{n_2}\overline{\mathrm{D}(n_1)} + \overline{n_3}\overline{\mathrm{D}(n_1)} + \overline{n_3}\overline{\mathrm{D}(n_1)} = \overline{\mathrm{D}(n_1)}\overline{n_2} + \overline{\mathrm{D}(n_1)}\overline{n_3} + \overline{\mathrm{D}(n_1)}\overline{n_2} + \overline{\mathrm{D}(n_1)}\overline{n_3}.$$

On simplifying, we obtain

$$\overline{\mathrm{D}(n_1)}(\overline{n_2} + \overline{n_3} - \overline{n_2} - \overline{n_3}) = \overline{0}.$$

By the primeness of $\mathscr{P}$ and the fact that $\overline{D(n_1)} \neq \overline{0}$, it follows that $\overline{n_2} + \overline{n_3} = \overline{n_3} + \overline{n_2}$. Hence, the additive group $(\mathcal{N}/\mathscr{P}, +)$ is abelian.

Next, consider the condition that $\overline{D(n_1n_2)}\overline{n_3} = \overline{n_3}\overline{D(n_1n_2)}$ for all $n_1, n_2, n_3 \in \mathcal{N}$. This implies

$$(\overline{D(n_1)}\overline{n_2} + \overline{\alpha(n_1)D(n_2)})\overline{n_3} = \overline{n_3}(\overline{D(n_1)}\overline{n_2} + \overline{\alpha(n_1)D(n_2)}).$$

By applying Lemma 2.2, we derive

$$\begin{aligned} \overline{D(n_1)}\overline{n_2n_3} + \overline{\alpha(n_1)D(n_2)}\overline{n_3} \quad &+ \quad \overline{\alpha(n_1)\alpha(n_2)D(n_3)} - \overline{\alpha(n_1n_2)D(n_3)} \\ &= \quad \overline{n_3}\overline{D(n_1)}\overline{n_2} + \overline{n_3}\overline{\alpha(n_1)D(n_2)}. \end{aligned} \tag{3.1}$$

Substitute $D(n_3)$ for n_3 in (3.1) to obtain

$$\begin{aligned} \overline{D(n_1)}\overline{n_2}\overline{D(n_3)} + \overline{\alpha(n_1)D(n_2)D(n_3)} \quad &+ \quad \overline{\alpha(n_1)\alpha(n_2)}\overline{D^2(n_3)} - \overline{\alpha(n_1n_2)}\overline{D^2(n_3)} \\ &= \quad \overline{D(n_3)D(n_1)}\overline{n_2} + \overline{D(n_3)\alpha(n_1)D(n_2)}. \end{aligned} \tag{3.2}$$

Using the given conditions in (3.2), we find

$$\overline{\alpha(n_1n_2)}\overline{D^2(n_3)} = \overline{\alpha(n_1)\alpha(n_2)}\overline{D^2(n_3)} \quad \forall\, n_1, n_2, n_3 \in \mathcal{N}.$$

This implies

$$\overline{D^2(n_3)}\mathcal{N}/\mathscr{P}\,(\overline{\alpha(n_1n_2)} - \overline{\alpha(n_1)\alpha(n_2)}) = \{\overline{0}\}.$$

By the primeness of $\mathscr{P}$ and Lemma 2.4, we conclude

$$\overline{\alpha(n_1n_2)} = \overline{\alpha(n_1)\alpha(n_2)} \quad \forall\, n_1, n_2 \in \mathcal{N} \tag{3.3}$$

Using (3.3), equation (3.1) simplifies to

$$\overline{D(n_1)}\overline{n_2n_3} + \overline{\alpha(n_1)D(n_2)}\overline{n_3} = \overline{n_3}\overline{D(n_1)}\overline{n_2} + \overline{n_3}\overline{\alpha(n_1)D(n_2)} \quad \forall\, n_1, n_2, n_3 \in \mathcal{N}. \tag{3.4}$$

Now, replace n_3 with $\alpha(n_1)$ in (3.4). Using the given hypothesis, we obtain

$$\overline{D(n_1)}\overline{n_2}\overline{\alpha(n_1)} = \overline{\alpha(n_1)D(n_1)}\overline{n_2} \quad \forall\, n_1 \in \mathcal{N}.$$

Substitute n_2s for n_2 in this expression and apply it again, yielding

$$\overline{D(n_1)}\overline{n_2s}\overline{\alpha(n_1)} = \overline{\alpha(n_1)D(n_1)}\overline{n_2s} = \overline{D(n_1)}\overline{n_2}\overline{\alpha(n_1)}\overline{s}.$$

This implies

$$\overline{D(n_1)}\overline{n_2}[\overline{\alpha(n_1)}, \overline{s}] = \overline{0} \quad \forall\, n_1, n_2, s \in \mathcal{N}.$$

By the primeness of $\mathscr{P}$, we deduce

$$\overline{D(n_1)} = \overline{0} \quad \text{or} \quad \overline{\alpha(n_1)} \in \mathscr{Z}(\mathcal{N}/\mathscr{P}) \quad \forall\, n_1 \in \mathcal{N}. \tag{3.5}$$

Suppose there exists $n_0 \in \mathcal{N}$ such that $D(n_0) \in \mathscr{P}$. Returning to (3.1), we get

$$\overline{\alpha(n_0)D(n_2)}\overline{n_3} = \overline{n_3}\overline{\alpha(n_0)D(n_2)} \quad \forall\, n_2, n_3 \in \mathcal{N}.$$

Replace n_3 with n_3r and expand, leading to

$$\overline{D(n_2)}\overline{n_3}[\overline{\alpha(n_0)}, \overline{r}] = \overline{0} \quad \forall\, r, n_2, n_3 \in \mathcal{N}.$$

Since $D(\mathcal{N}) \nsubseteq \mathscr{P}$ and $\mathscr{P}$ is prime, it follows that $\overline{\alpha(n_0)} \in \mathscr{Z}(\mathcal{N}/\mathscr{P})$. Thus, (3.5) implies $\overline{\alpha(n_1)} \in \mathscr{Z}(\mathcal{N}/\mathscr{P})$ for all $n_1 \in \mathcal{N}$. Analyzing (3.1) further, we find

$$\overline{D(n_1)}\mathcal{N}/\mathscr{P}\,[\overline{n_3}, \overline{t}] = \{\overline{0}\} \quad \forall\, t, n_1, n_3 \in \mathcal{N}.$$

Given that D is non-$\mathscr{P}$-trivial and using the primeness of $\mathscr{P}$, we conclude that $[\overline{n_3}, \overline{t}] = \overline{0}$ for all $t, n_3 \in \mathcal{N}$. Therefore, $\mathcal{N}/\mathscr{P}$ is a commutative ring. $\square$

Theorem 3.2. *Let $\mathcal{N}$ be a near-ring with a prime ideal $\mathscr{P}$, and let $\mathrm{D} : \mathcal{N} \to \mathcal{N}$ be a $\mathrm{D}_{(1,\alpha)}/\mathscr{P}-$derivation, where α is an automorphism satisfying $\mathrm{D}([n_1, n_2]) - [n_1, n_2] \in \mathscr{P}$ for all $n_1, n_2 \in \mathcal{N}$. Then, the quotient $\mathcal{N}/\mathscr{P}$ is a commutative ring.*

Proof. By the given condition, we have $\mathrm{D}([n_1, n_2]) - [n_1, n_2] \in \mathscr{P}$ for all $n_1, n_2 \in \mathcal{N}$, which implies in the quotient

$$\widetilde{\mathrm{D}}\left([\overline{n_1}, \overline{n_2}]\right) = [\overline{n_1}, \overline{n_2}] \quad \forall\, n_1, n_2 \in \mathcal{N}. \tag{3.6}$$

Substitute n_2 with $n_1 n_2$ in (3.6), yielding

$$\widetilde{\mathrm{D}}\left([\overline{n_1}, \overline{n_1 n_2}]\right) = [\overline{n_1}, \overline{n_1 n_2}].$$

Since $[\overline{n_1}, \overline{n_1 n_2}] = \overline{n_1}[\overline{n_1}, \overline{n_2}]$, this becomes

$$\widetilde{\mathrm{D}}\left(\overline{n_1}[\overline{n_1}, \overline{n_2}]\right) = \overline{n_1}[\overline{n_1}, \overline{n_2}].$$

Applying the derivation property, we obtain

$$\widetilde{\mathrm{D}}\,(\overline{n_1})[\overline{n_1}, \overline{n_2}] + \widetilde{\alpha}\,(\overline{n_1})\,\widetilde{\mathrm{D}}\left([\overline{n_1}, \overline{n_2}]\right) = \overline{n_1}[\overline{n_1}, \overline{n_2}] \quad \forall\, n_1, n_2 \in \mathcal{N}.$$

Now, replace n_1 with $[r, s]$ in this expression, giving

$$\widetilde{\mathrm{D}}\left([\overline{r}, \overline{s}]\right)[[\overline{r}, \overline{s}], \overline{n_2}] + \widetilde{\alpha}\left([\overline{r}, \overline{s}]\right)[[\overline{r}, \overline{s}], \overline{n_2}] = [\overline{r}, \overline{s}][[\overline{r}, \overline{s}], \overline{n_2}] \quad \forall\, r, s, n_2 \in \mathcal{N}.$$

Using (3.6), this simplifies to

$$\widetilde{\alpha}\left([\overline{r}, \overline{s}]\right)[[\overline{r}, \overline{s}], \overline{n_2}] = \overline{0} \quad \forall\, r, s, n_2 \in \mathcal{N}.$$

This implies

$$\widetilde{\alpha}\left([\overline{r}, \overline{s}]\right)[\overline{r}, \overline{s}]\overline{n_2} = \widetilde{\alpha}\left([\overline{r}, \overline{s}]\right)\overline{n_2}[\overline{r}, \overline{s}] \quad \forall\, r, s, n_2 \in \mathcal{N}. \tag{3.7}$$

Next, substitute $n_2 n_3$ for n_2 in (3.7) and apply (3.7) again, resulting in

$$\widetilde{\alpha}\left([\overline{r}, \overline{s}]\right)[\overline{r}, \overline{s}]\overline{n_2 n_3} = \widetilde{\alpha}\left([\overline{r}, \overline{s}]\right)\overline{n_2 n_3}[\overline{r}, \overline{s}] = \widetilde{\alpha}\left([\overline{r}, \overline{s}]\right)\overline{n_2}[\overline{r}, \overline{s}]\overline{n_3} \quad \forall\, r, s, n_2, n_3 \in \mathcal{N}.$$

This yields

$$\widetilde{\alpha}\left([\overline{r}, \overline{s}]\right)\overline{n_2}[[\overline{r}, \overline{s}], \overline{n_3}] = \overline{0} \quad \forall\, r, s, n_2, n_3 \in \mathcal{N}.$$

Thus, $\widetilde{\alpha}\left([\overline{r}, \overline{s}]\right)\mathcal{N}/\mathscr{P}[[\overline{r}, \overline{s}], \overline{n_3}] = \{\overline{0}\}$ for all $r, s, n_3 \in \mathcal{N}$. By the primeness of $\mathscr{P}$, we conclude

$$\widetilde{\alpha}\left([\overline{r}, \overline{s}]\right) = \overline{0} \quad \text{or} \quad [\overline{r}, \overline{s}] \in \mathscr{Z}(\mathcal{N}/\mathscr{P}) \quad \forall\, r, s \in \mathcal{N}. \tag{3.8}$$

Suppose there exist $\overline{r_0}, \overline{s_0} \in \mathcal{N}/\mathscr{P}$ such that $[\overline{r_0}, \overline{s_0}] \in \mathscr{Z}(\mathcal{N}/\mathscr{P})$. Replace $\overline{n_1}$ with $[\overline{r_0}, \overline{s_0}]\overline{n_1}$ in (3.6), giving

$$\widetilde{\mathrm{D}}\left([[\overline{r_0}, \overline{s_0}]\overline{n_1}, \overline{n_2}]\right) = [[\overline{r_0}, \overline{s_0}]\overline{n_1}, \overline{n_2}] \quad \forall\, n_1, n_2 \in \mathcal{N}.$$

Since $[\overline{r_0}, \overline{s_0}] \in \mathscr{Z}(\mathcal{N}/\mathscr{P})$, this implies

$$\widetilde{\mathrm{D}}\left([\overline{r_0}, \overline{s_0}][\overline{n_1}, \overline{n_2}]\right) = [\overline{r_0}, \overline{s_0}][\overline{n_1}, \overline{n_2}].$$

Applying the derivation property

$$\widetilde{\mathrm{D}}\left([\overline{r_0}, \overline{s_0}]\right)[\overline{n_1}, \overline{n_2}] + \widetilde{\alpha}\left([\overline{r_0}, \overline{s_0}]\right)\widetilde{\mathrm{D}}\left([\overline{n_1}, \overline{n_2}]\right) = [\overline{r_0}, \overline{s_0}][\overline{n_1}, \overline{n_2}] \quad \forall\, n_1, n_2 \in \mathcal{N}.$$

Using (3.6), this reduces to

$$\widetilde{\alpha}\ ([\overline{r_0},\overline{s_0}])[\overline{n_1},\overline{n_2}]=\overline{0}\quad \forall\ n_1,n_2\in\mathcal{N}.$$

Thus:

$$\widetilde{\alpha}\ ([\overline{r_0},\overline{s_0}])\overline{n_1n_2}=\widetilde{\alpha}\ ([\overline{r_0},\overline{s_0}])\overline{n_2n_1}\quad \forall\ n_1,n_2\in\mathcal{N}. \tag{3.9}$$

Substitute n_2n_3 for n_2 in (3.9) and apply (3.9), yielding

$$\widetilde{\alpha}\ ([\overline{r_0},\overline{s_0}])\overline{n_2}[\overline{n_1},\overline{n_3}]=\overline{0}\quad \forall\ n_1,n_2,n_3\in\mathcal{N}.$$

By the primeness of $\mathscr{P}$, this implies

$$\widetilde{\alpha}\ ([\overline{r_0},\overline{s_0}])=\overline{0}\quad\text{or}\quad[\overline{n_1},\overline{n_3}]=\overline{0}\quad \forall\ n_1,n_3\in\mathcal{N}. \tag{3.10}$$

Combining (3.10) with (3.8), we conclude that either $\widetilde{\alpha}\ ([\overline{r},\overline{s}])=\overline{0}$ for all $r,s\in\mathcal{N}$, or $\mathcal{N}/\mathscr{P}$ is commutative. Since $\widetilde{\alpha}$ is an automorphism, $\widetilde{\alpha}\ ([\overline{r},\overline{s}])=\overline{0}$ implies $[\overline{r},\overline{s}]=\overline{0}$. Thus, $\mathcal{N}/\mathscr{P}$ is a commutative ring. □

Theorem 3.3. *Let $\mathcal{N}$ be a near-ring with a prime ideal $\mathscr{P}$ such that $\mathcal{N}/\mathscr{P}$ is 2-torsion free. Then $\mathcal{N}$ does not admit a non-$\mathscr{P}$-trivial $D_{(1,\alpha)}/\mathscr{P}$-derivation $D:\mathcal{N}\to\mathcal{N}$, where α is an automorphism, satisfying $D(n_1\circ n_2)-(n_1\circ n_2)\in\mathscr{P}$ for all $n_1,n_2\in\mathcal{N}$.*

Proof. Assume that $D(n_1\circ n_2)-(n_1\circ n_2)\in\mathscr{P}$ for all $n_1,n_2\in\mathcal{N}$. This implies in the quotient

$$\widetilde{D}\ (\overline{n_1}\circ\overline{n_2})=\overline{n_1}\circ\overline{n_2}\quad \forall\ n_1,n_2\in\mathcal{N}. \tag{3.11}$$

Substitute n_2 with n_1n_2 in (3.11), yielding

$$\widetilde{D}\ (\overline{n_1}\circ(\overline{n_1n_2}))=\overline{n_1}\circ(\overline{n_1n_2}).$$

Since $\overline{n_1}\circ(\overline{n_1n_2})=\overline{n_1}(\overline{n_1}\circ\overline{n_2})$, this becomes

$$\widetilde{D}\ (\overline{n_1}(\overline{n_1}\circ\overline{n_2}))=\overline{n_1}(\overline{n_1}\circ\overline{n_2}).$$

Applying the derivation property, we obtain

$$\widetilde{D}\ (\overline{n_1})(\overline{n_1}\circ\overline{n_2})+\widetilde{\alpha}\ (\overline{n_1})\ \widetilde{D}\ (\overline{n_1}\circ\overline{n_2})=\overline{n_1}(\overline{n_1}\circ\overline{n_2})\quad \forall\ n_1,n_2\in\mathcal{N}.$$

Now, replace n_1 with $\overline{r}\circ\overline{s}$ in this expression, giving

$$\widetilde{D}\ (\overline{r}\circ\overline{s})((\overline{r}\circ\overline{s})\circ\overline{n_2})+\widetilde{\alpha}\ (\overline{r}\circ\overline{s})((\overline{r}\circ\overline{s})\circ\overline{n_2})=(\overline{r}\circ\overline{s})((\overline{r}\circ\overline{s})\circ\overline{n_2})\quad \forall\ r,s,n_2\in\mathcal{N}.$$

Using (3.11), this simplifies to

$$\widetilde{\alpha}\ (\overline{r}\circ\overline{s})((\overline{r}\circ\overline{s})\circ\overline{n_2})=\overline{0}\quad \forall\ r,s,n_2\in\mathcal{N}.$$

This can be rewritten as

$$\begin{aligned}\widetilde{\alpha}\ (\overline{r}\circ\overline{s})\overline{n_2}(\overline{r}\circ\overline{s}) &= -\ \widetilde{\alpha}\ (\overline{r}\circ\overline{s})(\overline{r}\circ\overline{s})\overline{n_2}\\ &= \widetilde{\alpha}\ (\overline{r}\circ\overline{s})(\overline{r}\circ\overline{s})(-\overline{n_2})\quad \forall\ r,s,n_2\in\mathcal{N}.\end{aligned} \tag{3.12}$$

Substitute n_2n_3 for n_2 in (3.12) and apply (3.12), resulting in

$$\begin{aligned}\widetilde{\alpha}\ (\overline{r}\circ\overline{s})\overline{n_2n_3}(\overline{r}\circ\overline{s}) &= -\ \widetilde{\alpha}\ (\overline{r}\circ\overline{s})(\overline{r}\circ\overline{s})\overline{n_2n_3}\\ &= (\widetilde{\alpha}\ (\overline{r}\circ\overline{s})(\overline{r}\circ\overline{s})\overline{n_2})(-\overline{n_3})\\ &= \widetilde{\alpha}\ (\overline{r}\circ\overline{s})\overline{n_2}(-(\overline{r}\circ\overline{s}))(-\overline{n_3})\quad \forall\ r,s,n_2,n_3\in\mathcal{N}.\end{aligned}$$

This yields

$$\widetilde{\alpha}\ (\overline{r}\circ\overline{s})\mathcal{N}/\mathcal{P}(-\overline{n_3}(-(\overline{r}\circ\overline{s}))+\overline{n_3}(-(\overline{r}\circ\overline{s})))=\{\overline{0}\}\quad\forall\ r,s,n_3\in\mathcal{N}.$$

By the primeness of $\mathcal{P}$, we conclude

$$\widetilde{\alpha}\ (\overline{r}\circ\overline{s})=\overline{0}\quad\text{or}\quad-(\overline{r}\circ\overline{s})\in\mathcal{Z}(\mathcal{N}/\mathcal{P})\quad\forall\ r,s\in\mathcal{N}. \tag{3.13}$$

Suppose there exist $\overline{r_0},\overline{s_0}\in\mathcal{N}/\mathcal{P}$ such that $-(\overline{r_0}\circ\overline{s_0})\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$. Then, using the hypothesis, we have

$$\widetilde{\mathrm{D}}\ (\overline{n_1}(-(\overline{r_0}\circ\overline{s_0}))\circ\overline{n_2})=\overline{n_1}(-(\overline{r_0}\circ\overline{s_0}))\circ\overline{n_2}.$$

Since $-(\overline{r_0}\circ\overline{s_0})$ is central, this becomes

$$\widetilde{\mathrm{D}}\ ((-(\overline{r_0}\circ\overline{s_0}))(\overline{n_1}\circ\overline{n_2}))=(-(\overline{r_0}\circ\overline{s_0}))(\overline{n_1}\circ\overline{n_2})\quad\forall\ n_1,n_2\in\mathcal{N}.$$

Applying the derivation property

$$\widetilde{\mathrm{D}}\ (-(\overline{r_0}\circ\overline{s_0}))(\overline{n_1}\circ\overline{n_2})+\widetilde{\alpha}\ (-(\overline{r_0}\circ\overline{s_0}))\ \widetilde{\mathrm{D}}\ (\overline{n_1}\circ\overline{n_2})=(-(\overline{r_0}\circ\overline{s_0}))(\overline{n_1}\circ\overline{n_2})\quad\forall\ n_1,n_2\in\mathcal{N}.$$

Using (3.11), this reduces to

$$\widetilde{\alpha}\ (-(\overline{r_0}\circ\overline{s_0}))\overline{n_1n_2}=-\ \widetilde{\alpha}\ (-(\overline{r_0}\circ\overline{s_0}))\overline{n_2n_1}\quad\forall\ n_1,n_2\in\mathcal{N}. \tag{3.14}$$

Substitute n_2n_3 for n_2 in (3.14) and apply (3.14), yielding

$$\widetilde{\alpha}\ (-(\overline{r_0}\circ\overline{s_0}))\overline{n_2}(\overline{n_3n_1}-(-\overline{n_1})(-\overline{n_3}))=\overline{0}\quad\forall\ n_1,n_2,n_3\in\mathcal{N}.$$

Replace n_1 with $-n_1$, giving

$$\widetilde{\alpha}\ (-(\overline{r_0}\circ\overline{s_0}))\mathcal{N}/\mathcal{P}(-\overline{n_3n_1}+\overline{n_1n_3})=\{\overline{0}\}\quad\forall\ n_1,n_3\in\mathcal{N}. \tag{3.15}$$

By the primeness of $\mathcal{P}$, this implies

$$\widetilde{\alpha}\ (-(\overline{r_0}\circ\overline{s_0}))=\overline{0}\quad\text{or}\quad[\overline{n_1},\overline{n_3}]=\overline{0}\quad\forall\ n_1,n_3\in\mathcal{N}.$$

Thus, from (3.13) and the fact that α is an automorphism, we have either $\overline{r}\circ\overline{s}=\overline{0}$ for all $r,s\in\mathcal{N}$, or $\mathcal{N}/\mathcal{P}$ is commutative. If $\overline{r}\circ\overline{s}=\overline{0}$, then $\overline{sr}=-\overline{rs}$. Replacing s with st, we get

$$\overline{s}\overline{t}\overline{r}=-\overline{r}(\overline{s}\overline{t}).$$

This implies $\overline{t}[-\overline{r},\overline{s}]=\overline{0}$ for all $r,s,t\in\mathcal{N}$, so $-\overline{r}\in\mathcal{Z}(\mathcal{N}/\mathcal{P})$ for all $r\in\mathcal{N}$. Hence, $\mathcal{N}/\mathcal{P}$ is commutative in all cases.

Since $\mathcal{N}/\mathcal{P}$ is commutative and 2-torsion free, from (3.11), we have:

$$\widetilde{\mathrm{D}}\ (\overline{n_1n_2})=\widetilde{\mathrm{D}}\ (\overline{n_1})\overline{n_2}+\widetilde{\alpha}\ (\overline{n_1})\ \widetilde{\mathrm{D}}\ (\overline{n_2})=\overline{n_1n_2}\quad\forall\ n_1,n_2\in\mathcal{N}.$$

Substitute n_1n_3 for n_1, yielding

$$\widetilde{\alpha}\ (\overline{n_1n_3})\ \widetilde{\mathrm{D}}\ (\overline{n_2})=\overline{0}\quad\forall\ n_1,n_2,n_3\in\mathcal{N}.$$

Since α is an automorphism, this implies

$$\overline{n_1}\mathcal{N}/\mathcal{P}\ \widetilde{\mathrm{D}}\ (\overline{n_2})=\{\overline{0}\}\quad\forall\ n_1,n_2\in\mathcal{N}.$$

By the primeness of $\mathcal{P}$, we conclude $\widetilde{\mathrm{D}}=\overline{0}$, i.e., D is $\mathcal{P}-$trivial, contradicting the assumption of non-$\mathcal{P}-$triviality. □

Example 3.1. *Consider a left near-ring $\mathscr{M}$. Define*

$$\mathscr{N}=\left\{\begin{pmatrix}0 & n_1 & n_2\\ 0 & 0 & n_3\\ 0 & 0 & 0\end{pmatrix} \mid n_1,n_2,n_3\in\mathscr{M}\right\},\quad \mathscr{P}=\left\{\begin{pmatrix}0 & 0 & p\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \mid p\in\mathscr{M}\right\},$$

with $\mathrm{D},\alpha:\mathscr{N}\to\mathscr{N}$ *as*

$$\mathrm{D}\left(\begin{pmatrix}0 & n_1 & n_2\\ 0 & 0 & n_3\\ 0 & 0 & 0\end{pmatrix}\right)=\begin{pmatrix}0 & n_1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad \alpha\left(\begin{pmatrix}0 & n_1 & n_2\\ 0 & 0 & n_3\\ 0 & 0 & 0\end{pmatrix}\right)=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & n_3\\ 0 & 0 & 0\end{pmatrix}.$$

Verification shows $\mathscr{P}$ is an ideal and D *satisfies*

(i) $\mathrm{D}([n_1,n_2])-[n_1,n_2]\in\mathscr{P}$,

(ii) $\mathrm{D}(n_1\circ n_2)-(n_1\circ n_2)\in\mathscr{P}$,

but $\mathscr{N}/\mathscr{P}$ is not commutative, indicating the necessity of primeness.

4. Future Research Directions

This work leaves several promising problems for future research. In particular, we pose the following questions:

(i) Can the results established in this paper be extended by replacing the notion of $\mathrm{D}_{(1,\alpha)}/\mathscr{P}-$derivation with that of a generalized $\mathrm{D}_{(1,\alpha)}/\mathscr{P}-$derivation or a multiplicative generalized $\mathrm{D}_{(1,\alpha)}/\mathscr{P}-$derivation?

(ii) Do the main results continue to hold when the underlying structure is a semiprime near ring rather than a prime near ring?

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