CREAT. MATH. INFORM.
Volume **35** (2026), No. 3,
Pages 429-452

Online version at `https://creative-mathematics.cunbm.utcluj.ro/`
Print Edition: ISSN 1584 - 286X; Online Edition: ISSN 1843 - 441X
DOI: https://doi.org/10.37193/CMI.2026.03.09

Nonlinear multivalued Fourier boundary $p(u)$-Laplacian problem

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ABSTRACT. We study the following nonlinear multivalued Fourier boundary $p(u)-$ Laplacian problem $\beta(u)-\mathrm{div}a(x,u,\nabla u)\ni f$ in Ω, $a(x,u,\nabla u)\cdot\eta+\lambda u=g$ on $\partial\Omega$. The existence and partial uniqueness results of solutions for L^1-data are posed.

1. INTRODUCTION

We consider the following nonlinear elliptic $p(u)-$Laplacian problem with Fourier boundary condition.

$$P(\beta,f,g)\quad \begin{cases}\beta(u)-\mathrm{div}a(x,u,\nabla u)\ni f & \text{in } \Omega\\ a(x,u,\nabla u)\cdot\eta+\lambda u=g & \text{on } \partial\Omega,\end{cases}$$

where Ω is an open bounded domain of $\mathbb{R}^N$ ($N\geq 3$), with smooth boundary, $\lambda>0$, β is a maximal monotone graph on $\mathbb{R}$ such that $0\in\beta(0)$, a is a Leray-Lions type operator, η is the outer unit normal vector on $\partial\Omega$, $f\in L^1(\Omega)$ and $g\in L^1(\partial\Omega)$, with $g\not\equiv 0$.
The problem $P(\beta,f,g)$ can be seen as an extension of the following problem.

$$\begin{cases}b(u)-\mathrm{div}a(x,u,\nabla u)=f & \text{in } \Omega\\ a(x,u,\nabla u)\cdot\eta+\lambda u=g & \text{on } \partial\Omega,\end{cases}\tag{1.1}$$

with $b:\mathbb{R}\to\mathbb{R}$ nondecreasing, normalized by $b(0)=0$, studied by Ouaro and Sawadogo (see [24]).
The problem $P(\beta,f,g)$ is adapted into a generalized Leray-Lions framework under the following assumptions.
$a:\Omega\times(\mathbb{R}\times\mathbb{R}^N)\to\mathbb{R}^N$ is a Caratheodory function with

$$a(x,z,0)=0 \ \text{ for all } \ z\in\mathbb{R}, \ \text{ and a.e. } x\in\Omega,\tag{1.2}$$

satisfying the strict monotonicity assumption

$$\big(a(x,z,\xi)-a(x,z,\eta)\big)\cdot(\xi-\eta)>0 \ \text{ for all } \xi,\eta\in\mathbb{R}^N,\xi\neq\eta,\tag{1.3}$$

as well as the growth and the coercivity assumptions with variable exponent

$$\big|a(x,z,\xi)\big|^{p'(x,z)}\leq C_1\big(|\xi|^{p(x,z)}+\mathcal{M}(x)\big),\tag{1.4}$$

$$a(x,z,\xi).\xi\geq\frac{1}{C_2}|\xi|^{p(x,z)}.\tag{1.5}$$

Received: 04.04.2025. In revised form: 10.09.2025. Accepted: 08.06.2026
2010 *Mathematics Subject Classification.* : 35J60, 35D05, 76A05.
Key words and phrases. *Variable exponent $p(u)-$Laplacian, Fourier boundary condition, bounded Radon diffuse measure, Young measure, maximal monotone graph, renormalized solution.*
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Here C_1, C_2 are positive constants and $\mathcal{M}$ is a positive function such that $\mathcal{M} \in L^1(\Omega)$.
$p : \Omega \times \mathbb{R} \to [p_-, p_+]$ is a Carathéodory function, $1 < p_- \leq p(x,z) \leq p_+ < \infty$ and
$p'(x,z) = \dfrac{p(x,z)}{p(x,z)-1}$ is the conjugate exponent of $p(x,z)$, with

$$p_- := ess \inf_{(x,z)\in\overline{\Omega}\times\mathbb{R}} p(x,z) \ \text{ and } \ p_+ := ess \sup_{(x,z)\in\overline{\Omega}\times\mathbb{R}} p(x,z).$$

We assume that:

$$\begin{aligned} &p_- > N, p \text{ is uniformly log-Hölder continuous} \\ &\text{on } \overline{\Omega} \times [-M, M], \text{ for all } M > 0. \end{aligned} \tag{1.6}$$

As example of functions which satisfies above assumptions, we can give the following.

(1) $a(x,s,\xi) = |\xi|^{p(x,s)-2}\xi$, known as the $p(.,.)$-Laplacian operator.

(2) $a(x,s,\xi) = \left(1+|\xi|^2\right)^{\frac{p(x,s)-2}{2}}\xi$, known as the mean curvature operator.

Although the $p(x)$-Laplacian kind problems have been extensively used in mathematical modelling in the last twenty years (some of the references are listed in [1]), there are no much work on models with variable exponent $p(x)$ explicitly depending on $u(x)$. Yet a related, although far more complicated, minimization problem with $p = p(\nabla u)$ was suggested in [8], in the context of image processing. Nonlinear PDEs with variable exponent $p = p(u)$ appear then clearly in the applications of some numerical techniques for the total variation image restoration method that have been used in some restoration problems of mathematical image processing and computer vision [8, 9, 30]. Variational models exhibit the solution of these problems as minimizers of appropriately chosen functionals and the minimization technique of such models involves the solution of nonlinear partial differential equations derived as necessary optimality conditions [12]. Several numerical examples suggesting that the consideration of exponents $p = p(u)$ preserves the edges and reduces the noise of the restored images u are presented in [30]. A related, although far more complicated, minimization problem with the exponent of the regularization term depending on the gradient of the reconstructed image u, was considered in the works [8, 9]. In particular, the pioneer work [8] presents a numerical example suggesting a reduction of noise in the restored images u when the exponent of the regularization term is $p = p(|\nabla u|)$.

For the $p(u)$-Laplacian problem, to our best knowledge, the pioneering work is due to Andreianov et al. [2] in 2018. Afterwards, Chipot and De Oliveira studied a kind of $p(u)$-Laplacian problem considered by Andreianov et al (see [14]). The first work on the $p(u)$-Laplace problem like $P(\beta, f, g)$ was done by Ouaro and Sawadogo in a pioneering work [23]. This work concerns a nonlinear multivalued (inclusion) PDE with Fourier boundary condition, so we are looking for the solutions in the space $W^{1,p(.,u(.))}(\Omega)$. For that, we are facing some difficulties when we want to apply the theory of monotone operators in the Banach space $W^{1,p(.,u(.))}(\Omega)$ to prove the existence of solutions. Therefore, we use the technic of pseudo-monotone operators in Banach spaces, the Poincaré-Wirtinger inequality with constant exponent and the Young measure associated to a weak convergence method of sequence of gradients of solution (cf [16, 18]) to obtain some useful convergence results. An other difficulty is that β is a nonlinear graph, then, its appear in the definition of the solution, a bounded Radon diffuse measure, in order to take into account the boundary of the domain. We define $\mathcal{M}_b(\Omega)$ as the set of bounded Radon measures in Ω. For the variable exponent $\pi(.)$, where $\pi(.)$ is to be defined later, given $\mu \in \mathcal{M}_b(\Omega)$, we say that μ is diffuse with respect to the capacity $W^{1,\pi(.)}(\Omega)$ if $\mu(A) = 0$, for every set A

such that $Cap_{\pi(.)}(A,\Omega)=0$ (see [22]). For $A\subset\Omega$, we denote

$$S_{\pi(.)}(A)=\Big\{u\in W^{1,\pi(.)}(\Omega):\quad u\geq 1 \text{ on an open set which contain } A \text{ and } u\geq 0 \text{ in } \Omega\Big\}.$$

The $\pi(.)-$Capacity for every subset A with respect to Ω is defined by

$$Cap_{\pi(.)}(A,\Omega)=\inf_{u\in S_{\pi(.)}(A)}\Big\{\int_\Omega\big(|u|^{\pi(.)}+|\nabla u|^{\pi(.)}\big)dx\Big\}.$$

The set of bounded Radon diffuse measures in variable exponent setting $\pi(.)$ is denoted by $\mathcal{M}_b^{\pi(.)}(\Omega)$.

Here, we adapt the notion of renormalized solution for the study of our problem. The concept of renormalized solution was introduced by Diperna and Lions (cf.[15]). Moreover, the standard Leray-Lions elliptic problems with L^1 source terms are well posed in the framework of renormalized solutions.
The interest of the study of this kind of problem is due to the fact that they can model phenomena which arise in the study of elastic mechanics (see [3]), electrorheological fluid (see [28]) or image restoration (see [13]).
Due to the Fourier boundary condition considered in this paper, which bring some difficulties to treat the term at the boundary, we were inspired by the work of Ouaro and Tchousso (see [26]), where the authors defined for the first time a new space by taking into account the boundary.

The rest of this work is organized as follows: in the next section, we introduce some preliminary results. In the last section, we study the existence and a partial uniqueness results of the renormalized solutions for the problem $P(\beta,f,g)$.

2. Preliminary results

• We will use the so-called truncation function

$$T_k(s):=\begin{cases}s & \text{if } |s|\leq k\\ k sign_0(s) & \text{if } |s|>k\end{cases},\quad \text{where } sign_0(s):=\begin{cases}1 & \text{if } s>0\\ 0 & \text{if } s=0\\ -1 & \text{if } s<0.\end{cases}$$

The truncation function possesses the following properties.

$$T_k(-s)=-T_k(s), |T_k(s)|=\min\{|s|,k\},$$
$$\lim_{k\to\infty}T_k(s)=s \text{ and } \lim_{k\to 0}\frac{1}{k}T_k(s)=sign_0(s).$$

• We will also need to truncate vector value-function with the help of the mapping

$$h_m:\mathbb{R}^N\longrightarrow\mathbb{R}^N,\ \ h_m(\lambda)=\begin{cases}\lambda, & \text{if } |\lambda|\leq m\\ m\dfrac{\lambda}{|\lambda|} & \text{if } |\lambda|>m,\end{cases}\quad \text{where } m>0.$$

Taking into account the growth and the coercivity assumptions (1.4) and (1.5), we need to work in the variable exponent Sobolev space $W^{1,\pi(.)}(\Omega)$ defined below. Notice that the exponent $\pi(.)$ itself is related to u by $\pi(.):=p(.,u(.))$. We also recall the definition of variable exponent Lebesgue and Sobolev spaces $L^{\pi(.)}(\Omega)$ and $W^{1,\pi(.)}(\Omega)$. In the sequel, we will use the same notation $L^{\pi(.)}(\Omega)$ for the space $(L^{\pi(.)}(\Omega))^N$ of vector-valued functions.

Definition 2.1. *Let $\pi : \Omega \longrightarrow [1, \infty)$ be a measurable function.*
$\bullet L^{\pi(.)}(\Omega)$ *is the space of all measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that the modular*

$$\rho_{\pi(.)}(f) := \int_\Omega |f|^{\pi(x)}dx < \infty.$$

If p_+ is finite, this space is equipped with the Luxembourg norm

$$||f||_{L^{\pi(.)}(\Omega)} := \inf\left\{\lambda > 0; \quad \rho_{\pi(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

$\bullet W^{1,\pi(.)}(\Omega)$ *is the space of all functions $f \in L^{\pi(.)}(\Omega)$ such that the gradient of f (taken in the sense of distributions) belongs to $L^{\pi(.)}(\Omega)$. If p_+ is finite the space $W^{1,\pi(.)}(\Omega)$ is equipped with the norm*

$$||u||_{W^{1,\pi(.)}(\Omega)} := ||u||_{L^{\pi(.)}(\Omega)} + ||\nabla u||_{L^{\pi(.)}(\Omega)}.$$

$W_0^{1,\pi(.)}(\Omega)$ *is the closure of $C_0^\infty(\Omega)$ in the norm of $W^{1,\pi(.)}(\Omega)$.*
When $1 < p_- \leq \pi(.) \leq p_+ < \infty$, all the above spaces are separable and reflexive Banach spaces.

In the present paper, we assume that $\pi(x) = p(x, u(x))$ verify the log-Hölder continuity assumption (2.7) below and we denote $\pi_\varepsilon(x) := p(x, u_\varepsilon(x))$.

Proposition 2.1. *(See* [1], *Proposition 2.3)*
For all measurable function $\pi : \Omega \to [p_-, p_+]$, the following properties hold.

i) *$L^{\pi(.)}(\Omega)$ and $W^{1,\pi(.)}(\Omega)$ are separable and reflexive Banach spaces.*
ii) *$L^{\pi'(.)}(\Omega)$ can be identified with the dual space of $L^{\pi(.)}(\Omega)$, and the following Hölder inequality holds.*

$$\forall f \in L^{\pi(.)}(\Omega), g \in L^{\pi'(.)}(\Omega), \qquad \left|\int_\Omega fg dx\right| \leq 2||f||_{L^{\pi(.)}(\Omega)}||g||_{L^{\pi'(.)}(\Omega)}.$$

iii) *One has $\rho_{\pi(.)}(f) = 1$ if and only if $||f||_{L^{\pi(.)}(\Omega)} = 1$; further,*

$$\text{if } \rho_{\pi(.)}(f) \leq 1, \text{ then } ||f||^{p_+}_{L^{\pi(.)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq ||f||^{p_-}_{L^{\pi(.)}(\Omega)};$$

$$\text{if } \rho_{\pi(.)}(f) \geq 1, \text{ then } ||f||^{p_-}_{L^{\pi(.)}(\Omega)} \leq \rho_{\pi(.)}(f) \leq ||f||^{p_+}_{L^{\pi(.)}(\Omega)}.$$

In particular, if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $L^{\pi(.)}(\Omega)$, then $||f_n||_{L^{\pi(.)}(\Omega)}$ tends to zero (resp., to infinity) if and only if $\rho_{\pi(.)}(f_n)$ tends to zero (resp., to infinity), as $n \to \infty$.

For a measurable function $f \in W^{1,\pi(.)}(\Omega)$ we introduce the following notation.

$$\rho_{1,\pi(.)}(f) = \int_\Omega |f|^{\pi(.)}dx + \int_\Omega |\nabla f|^{\pi(.)}dx.$$

Replacing $p(x)$ by $\pi(x)$ in [6]-Proposition 2.2, we get the following result which is fundamental in this work (See [31, 32]).

Proposition 2.2. *If $f \in W^{1,\pi(.)}(\Omega)$, the following properties hold.*

i) $||f||_{W^{1,\pi(.)}(\Omega)} > 1 \Rightarrow ||f||^{p_-}_{W^{1,\pi(.)}(\Omega)} < \rho_{1,\pi(.)}(f) < ||f||^{p_+}_{W^{1,\pi(.)}(\Omega)}$;
ii) $||f||_{W^{1,\pi(.)}(\Omega)} < 1 \Rightarrow ||f||^{p_+}_{W^{1,\pi(.)}(\Omega)} < \rho_{1,\pi(.)}(f) < ||f||^{p_-}_{W^{1,\pi(.)}(\Omega)}$;
iii) $||f||_{W^{1,\pi(.)}(\Omega)} < 1$(*respectively* $= 1; > 1$) $\Leftrightarrow \rho_{1,\pi(.)}(f) < 1$(*respectively* $= 1; > 1$).

The following lemma show that the space $W^{1,\pi(.)}(\Omega)$ is stable by truncation.

Lemma 2.1. *(See,*[1]*-Lemma 2.9) If $u \in W^{1,\pi(.)}(\Omega)$ then $T_k(u) \in W^{1,\pi(.)}(\Omega)$.*

Now, we give some embedding results.

Proposition 2.3. *(See [1], Proposition 2.4) Assume that $\pi : \Omega \to [p_-, p_+]$ has a representative which can be extended into a continuous function up to the boundary $\partial\Omega$ and satisfying the log-Hölder continuity assumption.*

$$\exists L > 0,\ \ \forall x, y \in \overline{\Omega}, x \neq y, \quad -\big(\log|x-y|\big)|\pi(x) - \pi(y)| \leq L. \tag{2.7}$$

i) *Then, $C^\infty(\overline{\Omega})$ is dense in $W^{1,\pi(.)}(\Omega)$.*

ii) *$W^{1,\pi(.)}(\Omega)$ is embedded into $L^{\pi^*(.)}(\Omega)$, where $\pi^*(.)$ is the Sobolev embedding exponent defined as in (2.8) below. If q is a measurable variable exponent such that $ess \inf_{x\in\Omega}(\pi^*(.) - q(.)) > 0$, then the embedding of $W^{1,\pi(.)}(\Omega)$ into $L^{q(.)}(\Omega)$ is compact.*

For a given $\pi(.)$, a function taking values in $[p_-, p_+]$, $\pi^*(.)$ denotes the optimal Sobolev embedding defined for any $x \in \Omega$ by

$$\pi^*(x) = \begin{cases} \dfrac{N\pi(x)}{N - \pi(x)} & \text{if} \quad \pi(x) < N \\ \text{any real value} & \text{if} \quad \pi(x) = N \\ \infty & \text{if} \quad \pi(x) > N. \end{cases} \tag{2.8}$$

Throughout the paper, we denote by δ_c the Dirac measure on $\mathbb{R}^d$ ($d \in \mathbb{N}$) , concentrated at the point $c \in \mathbb{R}^d$.

In the following theorem, we gather the results of Ball [5], Pedregal [27] and Hungerbühler [19] which are needed for our purposes (we limit the statement to the case of a bounded domain Ω). Let us note that the results of (ii) and (iii) below, expressed in terms of the convergence in measure, are very convenient for the applications we have in mind.

Theorem 2.1. (i) *Let $\Omega \subset \mathbb{R}^N, N \in \mathbb{N}$ and a sequence $(v_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^d$ -valued functions, $d \in \mathbb{N}$, such that $(v_n)_{n\in\mathbb{N}}$ is equi-integrable on Ω. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a parametrized family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^d$ ($d \in \mathbb{N}$), weakly measurable in x wrt the Lebesgue measure on Ω, such that for all Carathéodory function $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^t$, $t \in \mathbb{N}$, one has*

$$\lim_{k\to\infty} \int_\Omega F(x, v_{n_k})dx = \int_\Omega \int_{\mathbb{R}^d} F(x, \lambda)d\nu_x(\lambda)dx, \tag{2.9}$$

whenever the sequence $(F(., v_n(.)))_{n\in\mathbb{N}}$ is equi-integrable on Ω . In particular,

$$v(x) := \int_{\mathbb{R}^d} \lambda d\nu_x(\lambda) \tag{2.10}$$

is the weak limit of the sequence $(v_{n_k})_{k\in\mathbb{N}}$ in $L^1(\Omega)$. The family $(\nu_x)_{x\in\Omega}$ is called the Young measure generated by the subsequence $(v_{n_k})_{k\in\mathbb{N}}$ as $k \to \infty$.

(ii) *If Ω is of finite measure, and $(\nu_x)_{x\in\Omega}$ is the Young measure generated by a sequence $(v_n)_{n\in\mathbb{N}}$, then $\nu_x = \delta_{v(x)}$ for a.e. $x \in \Omega \Leftrightarrow v_n$ converges in measure in Ω to v as $n \to \infty$.*

(iii) *If Ω is of finite measure, $(u_n)_{n\in\mathbb{N}}$ generates a Dirac Young measure $(\delta_{u(x)})_{x\in\Omega}$ in $\mathbb{R}^{d_1}$ and $(v_n)_{n\in\mathbb{N}}$ generates a Young measure $(\nu_x)_{x\in\Omega}$ in $\mathbb{R}^{d_1}$, then the sequence $(u_n, v_n)_{n\in\mathbb{N}}$ generates the Young measure $(\delta_{u(x)} \otimes \nu_x)_{x\in\Omega}$ on $\mathbb{R}^{d_1+d_2}$.*

Whenever a sequence $(v_n)_{n\in\mathbb{N}}$ generates a Young measure $(\nu_x)_{x\in\Omega}$, following the terminology of [17] we will say that $(v_n)_{n\in\mathbb{N}}$ nonlinear weak-* converges and $(\nu_x)_{x\in\Omega}$ is the nonlinear weak-* limit of the sequence $(v_n)_{n\in\mathbb{N}}$. In the case $(v_n)_{n\in\mathbb{N}}$ possesses a nonlinear weak-* convergent subsequence, we will say that it is nonlinear weak-* compact. It means that any equi-integrable sequence of measurable functions is nonlinear weak-* compact on Ω ([1], Theorem 2.10(i)).

We give a useful convergence result for the sequel.

Lemma 2.2. *Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graph such that $\beta_n \to \beta$ in the sense of graphs (i.e for all $(x,y) \in \beta$ there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider $(z_n)_{n\geq 1}$ and $(w_n)_{n\geq 1}$ two sequences of $L^1(\Omega)$, such that $w_n \in \beta_n(z_n)$ $\mathcal{L}^N$ a.e. Ω. If $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$, then $z \in dom(\beta)$ $\mathcal{L}^N$ a.e. Ω.*

The main tool of the proof of the above lemma is the "biting lemma of Chacon", see [11].

Lemma 2.3. *Let Ω be an open bounded subset of $\mathbb{R}^N$ and $(f_n)_{n\geq 1}$ a bounded sequence in $L^1(\Omega)$. Then, there exists $f \in L^1(\Omega)$, a subsequence $(f_{n_k})_{n_k\geq 1}$ and a sequence of the mesurable set $(E_j)_{j\in N^*}$, $E_j \subset \Omega$, $\forall j \in \mathbb{N}^*$ with $E_{j+1} \subset E_j$ and $\lim\limits_{j\to\infty} |E_j| = 0$, such that for any $j \in \mathbb{N}^*$, $f_{n_k} \rightharpoonup f$ in $L^1(\Omega \setminus E_j)$, as $n_k \to \infty$.*

Proof of the Lemma 2.2. Since the sequence $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$, using the "biting lemma of Chacon" there exist $w \in L^1(\Omega)$, a subsequence $(w_{n_k})_{n_k\geq 1}$, a sequence of mesurable sets $(E_j)_{j\in\mathbb{N}^*}$, $E_j \subset \Omega$ $\forall j \in \mathbb{N}^*$ with $E_{j+1} \subset E_j$ and $\lim\limits_{j\to\infty} |E_j| = 0$ such that for all $j \in \mathbb{N}^*$, $w_{n_k} \rightharpoonup w$ in $L^1(\Omega \setminus E_j)$, as $n_k \to \infty$. Since $z_{n_k} \rightharpoonup z$ in $L^1(\Omega)$ and so in $L^1(\Omega \setminus E_j)$, $\forall j \in \mathbb{N}^*$ and $\beta_{n_k} \to \beta$ in the sense of graphs, we have $w \in \beta(z)$ a.e. in $\Omega \setminus E_j$. Thus, $z \in dom(\beta)$ a.e. in $\Omega \setminus E_j$. Finally, we obtain $z \in dom(\beta)$ a.e. in Ω. □

We write for all measurable function $u : \Omega \to \mathbb{R}$, $\{|u| \leq k \ (< k, > k, \geqslant k, = k)\}$ or $\big[|u| \leq k \ (< k, > k, \geqslant k, = k)\big]$ for the set $\{x \in \Omega;\ |u(x)| \leq k \ (< k, > k, \geqslant k, = k)\}$, and $meas(\Omega)$ or $|\Omega|$ denote the measure of the set Ω.
Let us set

$$int(dom\beta) = (m, M) \ \text{ with } \ -\infty \leq m \leq 0 \leq M \leq \infty.$$

3. Existence and partial uniqueness of the renormalized solution

We give the following notion of solution of the problem $P(\beta, f, g)$ (see [7, 20, 25]).

Definition 3.2. *A renormalized solution of the problem $P(\beta, f, g)$ is a couple (u, w) with u a measurable function such that $T_k(u) \in W^{1,\pi(.)}(\Omega)$ for all $k > 0$, $u \in L^1(\partial\Omega)$ and $u \in dom(\beta)$ $\mathcal{L}^N$ a.e. Ω, $w \in L^1(\Omega)$ and $w \in \beta(u)$ $\mathcal{L}^N$ a.e. Ω; and there exists a measure $\mu \in \mathcal{M}_b^{\pi(.)}(\Omega)$ such that $\mu \perp \mathcal{L}^N$, μ^+ is concentrated on $\big[u = M\big] \cap \big[u \neq \infty\big]$, μ^- is concentrated on $\big[u = m\big] \cap \big[u \neq -\infty\big]$ such that*

$$\begin{aligned}\int_\Omega wh(u)\varphi dx + \int_\Omega h(u)a(x,u,\nabla u)\nabla\varphi dx \ &+ \int_\Omega h'(u)a(x,u,\nabla u)\nabla u\varphi dx + \int_\Omega h(u)\varphi d\mu \\ &+ \lambda\int_{\partial\Omega} uh(u)\varphi d\sigma = \int_\Omega fh(u)\varphi dx + \int_{\partial\Omega} gh(u)\varphi d\sigma,\end{aligned} \tag{3.11}$$

for all $\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$, for all $h \in C_c^1(\mathbb{R})$ and

$$\lim_{M\to\infty} \int_{\big[M<|u|<M+1\big]} a(x,u,\nabla u).\nabla u dx = 0. \tag{3.12}$$

All the terms of (3.11) are well defined. Indeed, $h(u)\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$, then, the first integral of the left-hand side and the first integral of the right-hand side of (3.11) are well defined. The second and third integral of the left-hand side are well defined thanks to (1.4). The fourth integral of the left hand side is also well defined, since the measure

μ is diffuse. Moreover, as $\varphi \in C(\overline{\Omega})$ ($\varphi \in W^{1,\pi(.)}(\Omega) \subset W^{1,p_-}(\Omega) \hookrightarrow C(\overline{\Omega})$, for $p_- > N$), then $\varphi \in L^\infty(\partial\Omega)$, so the last integral of the left-hand side and the second integral of the right-hand side are well defined.

Remark 3.1. *If $M = \infty$ and $-\infty < m$ (resp. $m = -\infty$ and $M < \infty$) then, $\mu_+ \equiv 0$ (resp. $\mu_- \equiv 0$). Thus,* (3.11) *holds true with $\mu \equiv \mu_-$ (resp. $\mu \equiv \mu_+$).*
If $M = \infty$ and $m = -\infty$ then, the domain of β is equal to $\mathbb{R}$ and the relation (3.11) *becomes*

$$\begin{aligned} \int_\Omega wh(u)\varphi dx + \int_\Omega h(u)a(x,u,\nabla u)\nabla\varphi dx \quad &+ \quad \int_\Omega h'(u)a(x,u,\nabla u)\nabla u\varphi dx + \lambda\int_{\partial\Omega} uh(u)\varphi d\sigma \\ &= \quad \int_\Omega fh(u)\varphi dx + \int_{\partial\Omega} gh(u)\varphi d\sigma, \end{aligned} \tag{3.13}$$

for all $\varphi \in W^{1,\pi(.)}(\Omega) \cap L^\infty(\Omega)$ and for all $h \in C_c^1(\mathbb{R})$.
In the case where the domain of β is bounded, the renormalization with h is not necessary in the Definition 3.2, therefore we can take $h \equiv 1$.

Now, we are going to prove the following main existence result.

Theorem 3.2. *Assume that* (1.2)-(1.6) *hold, $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$. Then, there exists at least one renormalized solution to the problem $P(\beta, f, g)$.*

Proof. The proof of Theorem 3.2 is divided into two steps.

Step 1. The approximate problem.
For every $\varepsilon > 0$, we consider the Yosida regularization β_ε of β, given by

$$\beta_\varepsilon = \frac{1}{\varepsilon}\big(I - (I+\varepsilon\beta)^{-1}\big).$$

Thanks to [10], there exists j a non negative, convex and l.s.c function defined on $\mathbb{R}$ such that $\beta = \partial j$.

To regularize β, we consider $j_\varepsilon(s) = \min\limits_{r\in\mathbb{R}}\left\{\frac{1}{2\varepsilon}|s-r|^2 + j(r)\right\}, \forall s \in \mathbb{R}, \forall \varepsilon > 0.$

Using Proposition 2.11-[10], we have

$$\begin{cases} dom(\beta) \subset dom(j) \subset \overline{dom(j)} = \overline{dom(\beta)}, \\ j_\varepsilon(s) = \dfrac{\varepsilon}{2}|\beta_\varepsilon(s)|^2 + j\big(J_\varepsilon(s)\big) \;\; \text{where } J_\varepsilon := (I+\varepsilon\beta)^{-1}, \\ j_\varepsilon \;\text{ is convex, Frechet-differentiable function and } \beta_\varepsilon = \partial j_\varepsilon, \\ j_\varepsilon \uparrow j \text{ as } \varepsilon \downarrow 0. \end{cases}$$

Furthermore, for any $\varepsilon > 0$, β_ε is nondecreasing, Lipschitz continuous function and $\beta_\varepsilon(0) = 0$. We define the functions f_ε by $f_\varepsilon(x) = T_{\frac{1}{\varepsilon}}(f(x))$ for any $x \in \Omega$ and $g_\varepsilon(x) = T_{\frac{1}{\varepsilon}}(g(x))$ for all $x \in \partial\Omega$. Then, $(f_\varepsilon)_{\varepsilon>0}$ and $(g_\varepsilon)_{\varepsilon>0}$ are the sequences of bounded functions which converges strongly to $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$ respectively, as $\varepsilon \to 0$. Moreover,

$$\|f_\varepsilon\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)} \text{ and } \|g_\varepsilon\|_{L^1(\partial\Omega)} \le \|g\|_{L^1(\partial\Omega)}, \;\; \forall \varepsilon > 0.$$

Lemma 3.4. *(See,*[22]*-Lemma* 3.1*)*
The Yosida regularisation β_ε is a surjective operator.

Now, let us consider the following problem.

$$P(\beta_\epsilon, f_\varepsilon, g_\varepsilon) \begin{cases} \beta_\varepsilon(u_\varepsilon) - \text{div} a(x,u_\varepsilon,\nabla u_\varepsilon) - \varepsilon \triangle_{p_+} u_\varepsilon + \varepsilon|u_\varepsilon|^{p_+-2}u_\varepsilon = f_\varepsilon & \text{in } \Omega \\ \\ \big[a(x,u_\varepsilon,\nabla u_\varepsilon) + \varepsilon|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon\big].\eta + \lambda T_{\frac{1}{\varepsilon}}\big(u_\varepsilon\big) = g_\varepsilon & \text{on } \partial\Omega, \end{cases}$$

where
$$-\triangle_{p_+} u_\varepsilon := -\nabla.(|\nabla u_\varepsilon|^{p_+ -2}\nabla u_\varepsilon).$$
We define the following reflexive space
$$E = W^{1,p_+}(\Omega) \times L^{p_+}(\partial\Omega).$$
Let
$$X_0 = \{(u,v) \in E : \quad v = \tau(u)\}.$$
We will identify an element $(u,v) \in X_0$ with its representative $u \in W^{1,p_+}(\Omega)$ in this paper.

Theorem 3.3. *There exists at least one weak solution u_ε for the problem $P(\beta_\epsilon, f_\varepsilon, g_\varepsilon)$ in the sense that $u_\varepsilon \in X_0$ and for all $v \in X_0$,*

$$\begin{aligned}\int_\Omega \beta_\varepsilon(u_\varepsilon)v dx + \int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla v dx &+ \lambda \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u_\varepsilon)v d\sigma \\ &+ \varepsilon \int_\Omega \left[|u_\varepsilon|^{p_+-2}u_\varepsilon v + |\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon \nabla v\right] dx \\ &= \int_\Omega f_\varepsilon v dx + \int_{\partial\Omega} g_\varepsilon v d\sigma. \end{aligned} \tag{3.14}$$

We need the following lemmas to prove Theorem 3.3.

Lemma 3.5. *(See* [21]*-Remark* 2.12*) Let V be a separable and reflexive Banach space and A, $M : V \to V'$ such that.*

(i) A is a pseudo-monotone operator;

(ii) M is a bounded, hemicontinuous and monotone operator;

then $A + M$ is pseudo-monotone.

Lemma 3.6. *(See* [29]*, Corollary 2.2). If an operator $\mathcal{A}$ is of type (M), bounded and coercive from a separable Banach space to its dual, then $\mathcal{A}$ is surjective.*

Proof of Theorem *3.3*. Let
$$< A_\varepsilon u, v >=< Au, v > + < G_\varepsilon u, v >,$$
where
$$< Au, v >= \int_\Omega a(x,u,\nabla u).\nabla v dx, \quad < G_\varepsilon u, v >= \varepsilon \int_\Omega |\nabla u|^{p_+-2}\nabla u \nabla v dx$$
and
$$< B_\varepsilon u, v >= \int_\Omega \beta_\varepsilon(u) v dx + \varepsilon \int_\Omega |u|^{p_+-2} u v dx + \lambda \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u) v d\sigma,$$
with $u, v \in X_0$.
Set $C_\varepsilon = A_\varepsilon + B_\varepsilon$. The proof of Theorem 3.3 is done in three claims.

Claim 1**: C_ε is bounded.**
From Hölder type inequality and (1.4) with the variable exponent $p(x,z)$ replaced by p_+, A is bounded. Moreover, using the same argument as in [4]-Proof of Lemma 4.2, the fact that β_ε is a lipschitz continuous function with $\beta_\varepsilon(0) = 0$ and as $W^{1,p_+}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $p_+ > N$, we prove that $G_\varepsilon + B_\varepsilon$ is bounded. Therefore, C_ε is bounded.

Claim 2. C_ε **is of type** (M).
Let $(u_\delta)_{\delta>0}$ be a sequence in X_0 such that

$$\begin{cases} u_\delta \rightharpoonup u \text{ in } X_0 \\ C_\varepsilon u_\delta \rightharpoonup \chi \text{ in } X_0' \\ \limsup\limits_{\delta\to 0} < C_\varepsilon u_\delta, u_\delta > = < \chi, u > . \end{cases}$$

We will prove that $\chi = C_\varepsilon u$.
By Fatou's Lemma, we deduce that

$$\liminf_{\delta\to 0}\left(\int_\Omega \beta_\varepsilon(u_\delta)u_\delta dx + \lambda\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u_\delta)u_\delta d\sigma + \varepsilon\int_\Omega |u_\delta|^{p_+}dx\right) \geq \int_\Omega \beta_\varepsilon(u)udx + \lambda\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u)ud\sigma + \varepsilon\int_\Omega |u|^{p_+}dx.$$

Moreover, thanks to Lebesgue dominated convergence Theorem and the fact that

$$|u_\delta|^{p_+-2}u_\delta \rightharpoonup |u|^{p_+-2}u \text{ in } L^{p'_+}(\Omega),$$

we obtain

$$\lim_{\delta\to 0}\left(\int_\Omega \beta_\varepsilon(u_\delta)vdx + \lambda\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u_\delta)vd\sigma + \varepsilon\int_\Omega |u_\delta|^{p_+-2}u_\delta vdx\right) = \int_\Omega \beta_\varepsilon(u)vdx + \lambda\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u)vd\sigma + \varepsilon\int_\Omega |u|^{p_+-2}uvdx,$$

for any $v \in X_0$. Thus, as δ goes to 0, it follows that

$$B_\varepsilon u_\delta \rightharpoonup \beta_\varepsilon(u) + \lambda T_{\frac{1}{\varepsilon}}(u) + \varepsilon|u|^{p_+-2}u \text{ in } X_0'.$$

So,

$$A_\varepsilon u_\delta \rightharpoonup \chi - \big(\beta_\varepsilon(u) + \lambda T_{\frac{1}{\varepsilon}}(u) + \varepsilon|u|^{p_+-2}u\big) \text{ in } X_0', \text{ as } \delta\to 0.$$

It remains to prove that A_ε is of type (M). Notice that, G_ε is bounded, hemicontinuous, monotone and A is pseudo-monotone (See [24], proof of Theorem 3.1-Step 2). Therefore, it follows from Lemma 3.5 that $A_\varepsilon = A + G_\varepsilon$ is pseudo-monotone.
So, the operator A_ε is of type (M) (see, [21]-Proposition 2.5) and we immediately have

$$Au + G_\varepsilon u = \chi - \big(\beta_\varepsilon(u) + \lambda T_{\frac{1}{\varepsilon}}(u) + \varepsilon|u|^{p_+-2}u\big).$$

Thus, we obtain $C_\varepsilon u = \chi$.

For more details regarding operator of "type (M)", "calculus of variation properties" and "pseudo-monotonicity", see [29, 21].

Claim 3: C_ε **is coercive.**
Using (1.5) with constant exponent, we get

$$\begin{aligned} < C_\varepsilon u, u > &= \int_\Omega a(x,u,\nabla u).\nabla udx + \int_\Omega \beta_\varepsilon(u)udx \\ &+ \lambda\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u)udx + \varepsilon\int_\Omega \big[|u|^{p_+} + |\nabla u|^{p_+}\big]dx \\ &\geq \frac{1}{C_2}\int_\Omega |\nabla u|^{p_+}dx + \varepsilon\int_\Omega |u|^{p_+}dx \\ &\geq C||u||^{p_+}_{W^{1,p_+}(\Omega)}, \text{ where } C = \min\left\{\frac{1}{C_2}, \varepsilon\right\}. \end{aligned}$$

We deduce that
$$\frac{< C_\varepsilon u, u >}{||u||_{W^{1,p_+}(\Omega)}} \to \infty \quad \text{as} \quad ||u||_{W^{1,p_+}(\Omega)} \to \infty.$$
Hence, C_ε is coercive. Then, according to Lemma 3.6, C_ε is surjective.
Let $F_\varepsilon = \big(T_{\frac{1}{\varepsilon}}(f), T_{\frac{1}{\varepsilon}}(g)\big) \in E' \subset X_0'$, then there exists at least one solution $u_\varepsilon \in X_0$ of the problem
$$< C_\varepsilon u_\varepsilon, v >=< F_\varepsilon, v >, \quad \text{for all} \quad v \in X_0.$$
Therefore, u_ε is a weak solution of the problem $P(\beta_\varepsilon, f_\varepsilon, g_\varepsilon)$. □

Step 2. A priori estimates.
This part is divided into several assertions and lemmas.

Assertion 1.
(i) The sequence $(\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$.
(ii)

(3.15)
$$\lambda \int_{\partial\Omega} \big|T_{\frac{1}{\varepsilon}}(u_\varepsilon)\big| d\sigma \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}.$$

(iii) The sequence $(u_\varepsilon)_{\varepsilon>0}$ is such that

(3.16)
$$\|u_\varepsilon\|_{W^{1,p_-}(\Omega)} \leq const\big(p_-, \Omega, f, g\big).$$

Proof. Taking $v = T_k(u_\varepsilon)$ in (3.14) and the fact that all the terms of the left hand side in the resulting equality are nonnegative, one deduces that
$$\int_\Omega \beta_\varepsilon(u_\varepsilon) T_k(u_\varepsilon) dx \leq \int_\Omega f_\varepsilon T_k(u_\varepsilon) dx + \int_{\partial\Omega} g_\varepsilon T_k(u_\varepsilon) d\sigma \leq k(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)})$$
and
$$\lambda \int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u_\varepsilon) T_k(u_\varepsilon) d\sigma \leq \int_\Omega f_\varepsilon T_k(u_\varepsilon) dx + \int_{\partial\Omega} g_\varepsilon T_k(u_\varepsilon) d\sigma \leq k(\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}).$$
Dividing the above inequalities by k and letting k goes to 0, it follows that
$$\int_\Omega \big|\beta_\varepsilon(u_\varepsilon)\big| dx \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}$$
and
$$\lambda \int_{\partial\Omega} \big|T_{\frac{1}{\varepsilon}}(u_\varepsilon)\big| d\sigma \leq \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}.$$
Hence, (i) and (ii) are proved.
(iii) Moreover, β_ε is a Lipschitz, continuous nondecreasing and surjective function, then $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$. So, there exists a positive constant C_3 such that

(3.17)
$$\int_\Omega |u_\varepsilon| dx \leq C_3.$$

Hence,
$$\tilde{u}_\varepsilon := \frac{1}{meas(\Omega)} \int_\Omega u_\varepsilon dx \leq const(\Omega).$$
Furthermore, by using Poincaré-Wirtinger inequality, one has
$$\int_\Omega |u_\varepsilon - \tilde{u}_\varepsilon|^{p_-} dx \leq const(p_-, \Omega) \int_\Omega |\nabla u_\varepsilon|^{p_-} dx.$$

Therefore,

$$\int_\Omega |u_\varepsilon|^{p_-}dx \leq const(p_-,\Omega)\int_\Omega |\nabla u_\varepsilon|^{p_-}dx + \left|\int_\Omega \tilde{u_\varepsilon}dx\right|^{p_-}.$$

Thus

$$\int_\Omega |u_\varepsilon|^{p_-}dx \leq C_4\int_\Omega |\nabla u_\varepsilon|^{p_-}dx + C_5, \tag{3.18}$$

where $C_4 = cons(p_-,\Omega)$ and $C_5 = const(p_-,C_3,\Omega)$ are two positive constants. Moreover, since $W^{1,p_-}(\Omega) \hookrightarrow C(\overline{\Omega})$, there exists a positive constant C_6 such that

$$\|u_\varepsilon\|^{p_-}_{L^\infty(\overline{\Omega})} \leq C_6\|u_\varepsilon\|^{p_-}_{W^{1,p_-}(\Omega)}, \tag{3.19}$$

which implies that

$$\|u_\varepsilon\|^{p_-}_{L^\infty(\Omega)} \leq C_6\|u_\varepsilon\|^{p_-}_{W^{1,p_-}(\Omega)} \text{ and } \|u_\varepsilon\|^{p_-}_{L^\infty(\partial\Omega)} \leq C_6\|u_\varepsilon\|^{p_-}_{W^{1,p_-}(\Omega)}. \tag{3.20}$$

Using (1.5) on $a(x,u_\varepsilon,\nabla u_\varepsilon)$ and Theorem 3.3, the sequence u_ε satisfies

$$\begin{aligned}\int_\Omega \beta_\varepsilon(u_\varepsilon)u_\varepsilon dx + \frac{1}{C_2}\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx \quad &+ \quad \varepsilon\Big(\int_\Omega |\nabla u_\varepsilon|^{p_+}dx + \int_\Omega |u_\varepsilon|^{p_+}dx\Big) + \lambda\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u_\varepsilon)u_\varepsilon d\sigma\\ &\leq \quad \int_\Omega f_\varepsilon u_\varepsilon dx + \int_{\partial\Omega} g_\varepsilon u_\varepsilon d\sigma.\end{aligned} \tag{3.21}$$

Applying Young's inequality on the right hand side of (3.21) and using the relation (3.20), it follows that

$$\begin{aligned}\int_\Omega f_\varepsilon u_\varepsilon dx \quad &+ \quad \int_{\partial\Omega} g_\varepsilon u_\varepsilon d\sigma\\
&\leq \quad \int_\Omega |f||u_\varepsilon|dx + \int_\Omega |g||u_\varepsilon|d\sigma\\
&\leq \quad \|f\|_{L^1(\Omega)}\|u_\varepsilon\|_{L^\infty(\Omega)} + \|g\|_{L^1(\Omega)}\|u_\varepsilon\|_{L^\infty(\partial\Omega)}\\
&= \quad \left(\frac{4C_2C_6(C_4+1)}{p_-}\right)^{\frac{1}{p_-}}\|f\|_{L^1(\Omega)}\cdot\left(\frac{p_-}{4C_2C_6(C_4+1)}\right)^{\frac{1}{p_-}}\|u_\varepsilon\|_{L^\infty(\Omega)}\\
&+ \quad \left(\frac{4C_2C_6(C_4+1)}{p_-}\right)^{\frac{1}{p_-}}\|g\|_{L^1(\partial\Omega)}\cdot\left(\frac{p_-}{4C_2C_6(C_4+1)}\right)^{\frac{1}{p_-}}\|u_\varepsilon\|_{L^\infty(\partial\Omega)}\\
&\leq \quad \frac{1}{p'_-}\left(\frac{4C_2C_6(C_4+1)}{p_-}\right)^{\frac{p'_-}{p_-}}\|f\|^{p'_-}_{L^1(\Omega)} + \frac{1}{p_-}\frac{p_-}{4C_2C_6(C_4+1)}\|u_\varepsilon\|^{p_-}_{L^\infty(\Omega)}\\
&+ \quad \frac{1}{p'_-}\left(\frac{4C_2C_6(C_4+1)}{p_-}\right)^{\frac{p'_-}{p_-}}\|g\|^{p'_-}_{L^1(\partial\Omega)} + \frac{1}{p_-}\frac{p_-}{4C_2C_6(C_4+1)}\|u_\varepsilon\|^{p_-}_{L^\infty(\partial\Omega)}\\
&\leq \quad \frac{1}{p'_-}\left(\frac{4C_2C_6(C_4+1)}{p_-}\right)^{\frac{p'_-}{p_-}}\Big[\|f\|^{p'_-}_{L^1(\Omega)} + \|g\|^{p'_-}_{L^1(\partial\Omega)}\Big] + \frac{1}{2C_2(C_4+1)}\|u_\varepsilon\|^{p_-}_{W^{1,p_-}(\Omega)}.\end{aligned} \tag{3.22}$$

Moreover, one has

$$\int_\Omega |\nabla u_\varepsilon|^{p_-}dx \leq meas(\Omega) + \int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx, \tag{3.23}$$

since $p_- < \pi_\varepsilon(.)$.
Combining (3.18) and (3.23), we get

$$\int_\Omega |u_\varepsilon|^{p_-}dx \le C_4 meas(\Omega) + C_4\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx + C_5.$$

We infer from above inequality and (3.23) that

$$\begin{aligned}\|u_\varepsilon\|^{p_-}_{W^{1,p_-}(\Omega)} &= \int_\Omega \left[|u_\varepsilon|^{p_-} + |\nabla u_\varepsilon|^{p_-}\right]dx \\ &\le (C_4+1)meas(\Omega) + C_5 + (C_4+1)\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx. \qquad (3.24)\end{aligned}$$

Furthermore, using (3.22) and (3.24), one obtains

$$\begin{aligned}\int_\Omega f_\varepsilon u_\varepsilon dx &+ \int_{\partial\Omega} g_\varepsilon u_\varepsilon d\sigma \\ &\le \frac{1}{p'_-}\left(\frac{4C_2C_6(C_4+1)}{p_-}\right)^{\frac{p'_-}{p_-}}\left[\|f\|^{p'_-}_{L^1(\Omega)} + \|g\|^{p'_-}_{L^1(\partial\Omega)}\right] \\ &+ \frac{meas(\Omega)}{2C_2} + \frac{C_5}{2C_2(C_4+1)} + \frac{1}{2C_2}\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx. \qquad (3.25)\end{aligned}$$

Combining (3.21) and (3.25), one has

$$\begin{aligned}\int_\Omega \beta_\varepsilon(u_\varepsilon)u_\varepsilon dx + \frac{1}{2C_2}\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx &+ \varepsilon||u_\varepsilon||^{p_+}_{W^{1,p_+}(\Omega)} \\ &+ \lambda\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u_\varepsilon)u_\varepsilon d\sigma \le C_7, \qquad (3.26)\end{aligned}$$

where

$$C_7 = \frac{1}{p'_-}\left(\frac{4C_2C_6(C_4+1)}{p_-}\right)^{\frac{p'_-}{p_-}}\left[\|f\|^{p'_-}_{L^1(\Omega)} + \|g\|^{p'_-}_{L^1(\partial\Omega)}\right] + \frac{meas(\Omega)}{2C_2} + \frac{C_5}{2C_2(C_4+1)}.$$

Thus, we deduce from (3.26) that

$$\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}dx \le C_8. \qquad (3.27)$$

Now, using (3.24) and the last inequality, we infer

$$\|u_\varepsilon\|_{W^{1,p_-}(\Omega)} \le const\big(p_-,\Omega,f,g\big)$$

and (iii) follows. □

Lemma 3.7. $(u_\varepsilon)_{\varepsilon>0}$ *converges a.e. to* v *on* $\partial\Omega$.

Proof. $(u_\varepsilon)_{\varepsilon>0}$ being uniformly bounded in $W^{1,p_-}(\Omega)$, then, up to extraction of a subsequence still denoted $(u_\varepsilon)_{\varepsilon>0}$, it converges a.e. in Ω (and also weakly in $W^{1,p_-}(\Omega)$) to a limit u.
We know that the trace operator is compact from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$. Note that, $W^{1,p_-}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ because $p_- > 1$. Therefore, $u_\varepsilon \to u$ in $L^1(\partial\Omega)$ and a.e. on $\partial\Omega$, as $\varepsilon \to 0$. Thus, $v = u_{|\partial\Omega}$ make sense. □

Remark 3.2. *Using* (3.16) *and the compact embedding* $W^{1,p_-}(\Omega) \hookrightarrow L^{p_-}(\Omega)$, *we get that, for a subsequence still labelled with* ε *and a function* u,

(3.28) $$u_\varepsilon \rightharpoonup u \ \ in \ \ W^{1,p_-}(\Omega), \ \ as \ \ \varepsilon \to 0;$$

(3.29) $$\nabla u_\varepsilon \rightharpoonup \nabla u \ \ in \ \ L^{p_-}(\Omega), \ \ as \ \ \varepsilon \to 0;$$

(3.30) $$u_\varepsilon \to u \ \ in \ \ L^{p_-}(\Omega), \ \ as \ \ \varepsilon \to 0;$$

(3.31) $$u_\varepsilon \to u \ \ a.e. \ in \ \ \Omega, \ \ as \ \ \varepsilon \to 0.$$

Moreover, $\big(\beta_\varepsilon(u_\varepsilon)\big)_{\varepsilon>0}$ *is uniformly bounded in* $L^1(\Omega)$. *Then, up to a subsequence still labelled with* ε, *there exists* $z \in \mathcal{M}_b(\Omega)$, *such that*

(3.32) $$\beta_\varepsilon(u_\varepsilon) \rightharpoonup^* z \ in \ \mathcal{M}_b(\Omega), \ as \ \varepsilon \to 0.$$

Assertion 2**.**
The sequence $(\nabla u_\varepsilon)_{\varepsilon>0}$ converges to a Young measure $\nu_x(\lambda)$ on $\mathbb{R}^N$ in the sense of the nonlinear weak-* convergence, and

(3.33) $$\nabla u = \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda).$$

Proof. From (3.29) and (3.31), up to extraction of a subsequence still labelled with ε, $(u_\varepsilon)_{\varepsilon>0}$ converges a.e. in Ω to a limit u and ∇u_ε weakly converges to ∇u in $L^{p_-}(\Omega)$. Moreover, $(\nabla u_\varepsilon)_{\varepsilon>0}$ being bounded, is equi-integrable on Ω. Then, using the representation of weakly convergent sequences in $L^1(\Omega)$ in terms of Young measures (see Theorem 2.1 and formula (2.10)), we can write

$$\nabla u = \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda).$$

□

Assertion 3
$|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x(\lambda)dx$ on $\mathbb{R}^N \times \Omega$, moreover, $u \in W^{1,\pi(.)}(\Omega)$.

Proof. We know that π_ε converges in measure to π. Using Theorem 2.1 ii), iii), $(\pi_\varepsilon, \nabla u_\varepsilon)_{\varepsilon>0}$ converges in $\mathbb{R} \times \mathbb{R}^N$ to the Young measure $\mu_x = \delta_{\pi(x)} \otimes \nu_x$. Thus, we can apply the weak convergence properties (2.9) to the Carathéodory function

$$F_m : (x, \lambda_0, \lambda) \in \Omega \times (\mathbb{R} \times \mathbb{R}^N) \mapsto |h_m(\lambda)|^{\lambda_0},$$

with $m \in \mathbb{N}$, where h_m is defined in the preliminary. We have

$$\begin{aligned}
\int_{\Omega \times \mathbb{R}^N} |h_m(\lambda)|^{\pi(.)} d\nu_x(\lambda) dx &= \int_{\Omega \times (\mathbb{R} \times \mathbb{R}^N)} |h_m(\lambda)|^{\lambda_0} d\mu_x(\lambda_0, \lambda) dx \\
&= \int_\Omega \int_{\mathbb{R} \times \mathbb{R}^N} F_m(x, \lambda_0, \lambda) d\mu_x(\lambda_0, \lambda) dx \\
&= \lim_{\varepsilon \to 0} \int_\Omega F_m(x, \pi_\varepsilon(x), \nabla u_\varepsilon(x)) dx \\
&= \lim_{\varepsilon \to 0} \int_\Omega |h_m(\nabla u_\varepsilon)|^{\pi_\varepsilon(.)} dx \\
&\leq \lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)} dx \\
&\leq C_8.
\end{aligned}$$

Since $h_m(\lambda) \to \lambda$, as $m \to \infty$, using Lebesgue convergence Theorem, as $m \mapsto h_m(\lambda)$ is increasing, we deduce that

$$\int_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(.)} d\nu_x(\lambda)dx \leq C_8.$$

Hence, $|\lambda|^{\pi(.)}$ is integrable with respect to the measure $\nu_x(\lambda)dx$ in $\mathbb{R}^N \times \Omega$.
Now, we prove that $\nabla u \in L^{\pi(.)}(\Omega)$. Using (3.33), Jensen inequality and the last inequality, we get

$$\int_\Omega |\nabla u|^{\pi(.)}dx = \int_\Omega \left| \int_{\mathbb{R}^N} \lambda d\nu_x(\lambda) \right|^{\pi(.)} dx \leq \int_{\Omega\times\mathbb{R}^N} |\lambda|^{\pi(.)} d\nu_x(\lambda)dx < \infty.$$

Thus, $\nabla u \in L^{\pi(.)}(\Omega)$. Moreover, $u \in L^{\pi(.)}(\Omega)$. Indeed, $u \in W^{1,p_-}(\Omega) \subset L^\infty(\Omega) \subset L^{\pi(.)}(\Omega)$.
Hence, $u \in W^{1,\pi(.)}(\Omega)$. □

Assertion 4
The sequence $(\Phi_\varepsilon)_{\varepsilon>0}$ defined by $\Phi_\varepsilon := a(x, u_\varepsilon, \nabla u_\varepsilon)$ is equi-integrable in Ω.

Proof. By using (1.4) with exponent $\pi_\varepsilon(.)$, we obtain

$$|a(x, u_\varepsilon, \nabla u_\varepsilon)|^{\pi'_\varepsilon(.)} \leq C_1(|\nabla u_\varepsilon|^{\pi_\varepsilon(.)} + \mathcal{M}(x)).$$

The above inequality give us

$$\begin{aligned}
|a(x, u_\varepsilon, \nabla u_\varepsilon)| &\leq C\big((1 + |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}) + \mathcal{M}(x)\big)^{\frac{1}{\pi'_\varepsilon(.)}} \\
&\leq C\big((1 + \mathcal{M}(x))^{\frac{1}{\pi'_\varepsilon(.)}} + |\nabla u_\varepsilon|^{\frac{\pi_\varepsilon(.)}{\pi'_\varepsilon(.)}}\big) \\
&\leq C(1 + \mathcal{M}(x) + |\nabla u_\varepsilon|^{\pi_\varepsilon(.)-1}).
\end{aligned}$$

For all set $E \subset \Omega$,

$$\int_E |a(x, u_\varepsilon, \nabla u_\varepsilon)|dx \leq C \int_E (1+\mathcal{M}(x))dx + Const(p_-) \left|\left| |\nabla u_\varepsilon|^{\pi_\varepsilon(.)-1} \right|\right|_{L^{\pi'_\varepsilon(.)}(\Omega)} ||\chi_E||_{L^{\pi_\varepsilon(.)}(\Omega)}.$$

The first term of the right hand side of the last inequality above is small for $meas(E)$ small enough, since $1 + \mathcal{M} \in L^1(\Omega)$.
According to Proposition 2.1, we obtain

$$||\chi_E||_{L^{\pi_\varepsilon(.)}(\Omega)} \leq \max\left\{\rho_{\pi_\varepsilon(.)}(\chi_E)^{\frac{1}{p_+}}; \rho_{\pi_\varepsilon(.)}(\chi_E)^{\frac{1}{p_-}}\right\} = \max\left\{(meas(E))^{\frac{1}{p_+}}, (meas(E))^{\frac{1}{p_-}}\right\}.$$

Analogously,

$$\begin{aligned}
\left|\left| |\nabla u_\varepsilon|^{\pi_\varepsilon(.)-1} \right|\right|_{L^{\pi'_\varepsilon(.)}(\Omega)} &\leq \max\left\{\left(\rho_{\pi'_\varepsilon(.)}(|\nabla u_\varepsilon|^{\pi_\varepsilon(.)-1})^{\frac{1}{(p')_+}}\right), \left(\rho_{\pi'_\varepsilon(.)}(|\nabla u_\varepsilon|^{\pi_\varepsilon(.)-1})^{\frac{1}{(p')_-}}\right)\right\} \\
&= \max\left\{\left(\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}\right)^{\frac{1}{(p')_+}}, \left(\int_\Omega |\nabla u_\varepsilon|^{\pi_\varepsilon(.)}\right)^{\frac{1}{(p')_-}}\right\}.
\end{aligned}$$

Using (3.27), $\int_E |a(x, u_\varepsilon, \nabla u_\varepsilon)|dx$ is small for $meas(E)$ small enough.
Hence, $(\Phi_\varepsilon)_{\varepsilon>0}$ is equi-integrable. □

Assertion 5 (See [24], Assertion 4).
The weak limit Φ of $(\Phi_\varepsilon)_{\varepsilon>0}$ (or a subsequence) belongs to $L^{\pi'(.)}(\Omega)$ and we have

$$\Phi(x)=\int_{\mathbb{R}^N} a(x,u(x),\lambda)d\nu_x(\lambda). \tag{3.34}$$

Assertion 6

$$\int_\Omega \Phi.\nabla u dx \geq \int_{\Omega\times\mathbb{R}^N} a(x,u(x),\lambda).\lambda d\nu_x(\lambda)dx. \tag{3.35}$$

Proof. For all $\varphi\in C^\infty(\overline{\Omega})$, we have

$$\begin{aligned}\int_\Omega \beta_\varepsilon(u_\varepsilon)\varphi dx+\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon).\nabla\varphi dx &+ \lambda\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u_\varepsilon)\varphi d\sigma\\ &+ \varepsilon\int_\Omega \left[|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon\nabla\varphi+|u_\varepsilon|^{p_+-2}u_\varepsilon\varphi\right]dx\\ &= \int_\Omega f_\varepsilon\varphi dx+\int_{\partial\Omega} g_\varepsilon\varphi d\sigma.\end{aligned} \tag{3.36}$$

Letting ε goes to 0 in (3.36), one has

$$\int_\Omega \varphi dz+\int_\Omega \Phi.\nabla\varphi dx+\lambda\int_{\partial\Omega} u\varphi d\sigma=\int_\Omega f\varphi dx+\int_{\partial\Omega} g\varphi d\sigma, \tag{3.37}$$

thanks to (3.32), Assertion 5, Lemma 3.7 and (3.26).
By the density argument we can replace φ with u_ε in (3.36) to get

$$\begin{aligned}\int_\Omega \beta_\varepsilon(u_\varepsilon)u_\varepsilon dx+\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon).\nabla u_\varepsilon dx &+ \lambda\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u_\varepsilon)u_\varepsilon d\sigma\\ &+ \varepsilon\int_\Omega \left[|\nabla u_\varepsilon|^{p_+}+|u_\varepsilon|^{p_+}\right]dx\\ &= \int_\Omega f_\varepsilon u_\varepsilon dx+\int_{\partial\Omega} g_\varepsilon u_\varepsilon d\sigma.\end{aligned} \tag{3.38}$$

Since $u\in W^{1,\pi(.)}(\Omega)\subset W^{1,p_-}(\Omega)\subset C^{0,\alpha}(\overline{\Omega})$ and $p(.,.)$ is locally uniformly log-Hölder continuous, then the exponent $\pi(.)$ verifies (2.7). Therefore, $C^\infty(\overline{\Omega})$ is dense in $W^{1,\pi(.)}(\Omega)$, so, we change φ by u in (3.37) to obtain

$$\int_\Omega u dz+\int_\Omega \Phi.\nabla u dx+\lambda\int_{\partial\Omega} u^2 d\sigma=\int_\Omega f u dx+\int_{\partial\Omega} g u d\sigma. \tag{3.39}$$

By Fatou's Lemma, one has

$$\liminf_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(u_\varepsilon)u_\varepsilon dx\geq\int_\Omega u dz \tag{3.40}$$

and

$$\liminf_{\varepsilon\to 0}\lambda\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u_\varepsilon)u_\varepsilon d\sigma\geq\lambda\int_{\partial\Omega} u^2 d\sigma. \tag{3.41}$$

Moreover, the sequence $(f_\varepsilon u_\varepsilon)_{\varepsilon>0}$ converges a.e. in Ω to fu and

$$|f_\varepsilon u_\varepsilon|\leq |f|||u_\varepsilon||_{L^\infty(\Omega)}.$$

$(u_\varepsilon)_{\varepsilon>0}$ is also uniformly bounded in $L^\infty(\Omega)$. Applying Lebesgue dominated convergence Theorem, we get

$$\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon u_\varepsilon dx=\int_\Omega f u dx. \tag{3.42}$$

In the same manner,

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega} g_\varepsilon u_\varepsilon d\sigma = \int_{\partial\Omega} gud\sigma. \tag{3.43}$$

Combining (3.40), (3.41), (3.42) and (3.43), one obtains

$$\begin{aligned} &\liminf_{\varepsilon\to 0}\left(\int_\Omega f_\varepsilon u_\varepsilon dx + \int_{\partial\Omega} g_\varepsilon u_\varepsilon d\sigma\right) - \int_\Omega udz - \lambda\int_{\partial\Omega} u^2 d\sigma \\ \geq\ &\liminf_{\varepsilon\to 0}\left(\int_\Omega f_\varepsilon u_\varepsilon + \int_{\partial\Omega} g_\varepsilon u_\varepsilon d\sigma - \int_\Omega \beta_\varepsilon(u_\varepsilon)u_\varepsilon)dx - \lambda\int_{\partial\Omega} T_{\frac{1}{\varepsilon}}(u_\varepsilon)u_\varepsilon d\sigma\right). \end{aligned} \tag{3.44}$$

By using (3.38), (3.39), the last inequality and the definition of Φ_ε, we get

$$\int_\Omega \Phi.\nabla udx \geq \liminf_{\varepsilon\to 0}\int_\Omega \left(\Phi_\varepsilon.\nabla u_\varepsilon + \varepsilon\left[|\nabla u_\varepsilon|^{p+} + |u_\varepsilon|^{p+}\right]\right)dx \geq \liminf_{\varepsilon\to 0}\int_\Omega \Phi_\varepsilon.\nabla u_\varepsilon dx.$$

Hence,

$$\int_\Omega \Phi.\nabla udx \geq \liminf_{\varepsilon\to 0}\int_\Omega \Phi_\varepsilon.\nabla u_\varepsilon dx. \tag{3.45}$$

By ([1], Lemma 2.1), $m \mapsto a(x, u_\varepsilon, h_m(\nabla u_\varepsilon)).h_m(\nabla u_\varepsilon)$ is increasing and converges to $a(x, u_\varepsilon, \nabla u_\varepsilon).\nabla u_\varepsilon$ for m large enough; then,

$$a(x, u_\varepsilon, h_m(\nabla u_\varepsilon)).h_m(\nabla u_\varepsilon) \leq a(x, u_\varepsilon, \nabla u_\varepsilon).\nabla u_\varepsilon.$$

Therefore, by using (3.45) and Theorem 2.1, we have

$$\begin{aligned} \int_\Omega \Phi.\nabla udx &\geq \liminf_{\varepsilon\to 0}\int_\Omega \Phi_\varepsilon.\nabla u_\varepsilon dx \\ &\geq \lim_{\varepsilon\to 0}\int_\Omega a(x, u_\varepsilon, h_m(\nabla u_\varepsilon)).h_m(\nabla u_\varepsilon)dx \\ &= \int_{\Omega\times\mathbb{R}^N} a(x, u, h_m(\lambda)).h_m(\lambda)d\nu_x(\lambda)dx. \end{aligned} \tag{3.46}$$

Using Lebesgue convergence Theorem in (3.46), as m goes to ∞, we get (3.35). □

Assertion 7

The "div-curl" inequality holds.

$$\int_{\Omega\times\mathbb{R}^N} \big(a(x, u(x), \lambda) - a(x, u(x), \nabla u(x))\big)(\lambda - \nabla u(x))d\nu_x(\lambda)dx \leq 0.$$

Proof. We have

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}^N} (a(x, u(x), \lambda) - a(x, u(x), \nabla u(x)))(\lambda - \nabla u(x))d\nu_x(\lambda)dx \\ =\ &\int_{\Omega\times\mathbb{R}^N} a(x, u(x), \lambda).\lambda d\nu_x(\lambda)dx - \int_{\Omega\times\mathbb{R}^N} a(x, u(x), \lambda).\nabla u(x)d\nu_x(\lambda)dx \\ -\ &\int_{\Omega\times\mathbb{R}^N} a(x, u(x), \nabla u(x)).\lambda d\nu_x(\lambda)dx + \int_{\Omega\times\mathbb{R}^N} a(x, u(x), \nabla u(x)).\nabla u(x)d\nu_x(\lambda)dx \\ =\ &\int_{\Omega\times\mathbb{R}^N} a(x, u(x), \lambda).\lambda d\nu_x(\lambda)dx - \int_\Omega\left(\int_{\mathbb{R}^N} a(x, u(x), \lambda)d\nu_x(\lambda)\right)\nabla u(x)dx \\ -\ &\int_\Omega a(x, u(x), \nabla u(x)).\left(\int_{\mathbb{R}^N}\lambda d\nu_x\right)dx + \int_\Omega a(x, u(x), \nabla u(x)).\nabla u(x)\left(\int_{\mathbb{R}^N} d\nu_x\right)dx \\ =\ &\int_{\Omega\times\mathbb{R}^N} a(x, u(x), \lambda).\lambda d\nu_x(\lambda)dx - \int_\Omega \Phi.\nabla udx \leq 0. \end{aligned}$$

We pass from the first equality to the second equality by using Fubini's Theorem and from the second inequality to the third one, by using (3.33) and the fact that ν_x is a probability measure on $\mathbb{R}^N$. Finally, (3.34) and (3.35) give us the desired inequality. $\square$

Assertion 8
$\Phi(x) = a(x, u(x), \nabla u(x))$ a.e. $x \in \Omega$ and ∇u_ε converges to ∇u in measure on Ω, as $\varepsilon \to 0$.

Proof. From Assertion 7 and relation (1.3), we deduce that

$$\big(a(x,u(x),\lambda) - a(x,u(x),\nabla u(x))\big)\big(\lambda - \nabla u(x)\big) = 0 \quad a.e. \quad x \in \Omega, \quad \lambda \in \mathbb{R}^N.$$

Thus, $\lambda = \nabla u(x)$ a.e. $x \in \Omega$ wrt ν_x on $\mathbb{R}^N$; therefore $\nu_x(\nabla u(x)) = 1$ and $\delta_{\nabla u} = \nu_x$. By using (3.34), we get

$$\Phi(x) = \int_{\mathbb{R}^N} a(x,u(x),\lambda)d\nu_x(\lambda) = a(x,u(x),\nabla u(x)) \text{ a.e. } x \in \Omega.$$

Thus, according to Theorem 2.1 (ii), ∇u_ε converges in measure to ∇u. $\square$

Lemma 3.8. *(i)* $u \in L^1(\partial\Omega)$.
(ii) $(u_\varepsilon)_{\varepsilon>0}$ *is uniformly bounded in* $L^1(\Omega)$.
(iii) $u \in dom(\beta)$ $\mathcal{L}^N-$*a.e. in* Ω *and* $T_k(u) \in W^{1,\pi(\cdot)}(\Omega)$.

Proof. (i) Using Lemma 3.7, it follows from Fatou's Lemma that

$$\int_{\partial\Omega} |u| d\sigma \leq \liminf_{\varepsilon\to 0} \int_{\partial\Omega} |T_{\frac{1}{\varepsilon}}(u_\varepsilon)| d\sigma \leq \frac{\|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}}{\lambda}.$$

Hence, $u \in L^1(\partial\Omega)$.
(ii) Follows from (3.17).
(iii) Since $(\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$ and $u_\varepsilon \to u$ in $L^1(\Omega)$, as $\varepsilon \to 0$, then $u \in dom(\beta)$ $\mathcal{L}^N-$a.e. in Ω, thanks to Lemma 2.2. Furthermore, as $u \in W^{1,\pi(\cdot)}(\Omega)$, then $T_k(u) \in W^{1,\pi(\cdot)}(\Omega)$, thanks to Lemma 2.1. $\square$

Lemma 3.9. *(See* [25], *Lemma 3.12)*
For all $\varphi \in C^\infty(\overline{\Omega})$ *and* $h \in C_c^1(\mathbb{R})$,

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon).\nabla(h(u_\varepsilon)\varphi)dx \to \int_\Omega a(x,u,\nabla u).\nabla(h(u)\varphi)dx = \int_\Omega h(u)a(x,u,\nabla u)\nabla\varphi dx + \int_\Omega h'(u)a(x,u,\nabla u)\nabla u\varphi dx, \tag{3.47}$$

as $\varepsilon \to 0$.

Lemma 3.10. *The problem* $P(\beta, f, g)$ *admits at least one renormalized solution.*

Proof. We consider $\varphi \in C^\infty(\overline{\Omega})$, $h \in C_c^1(\mathbb{R})$ and $h(u_\varepsilon)\varphi$ as a test fonction in (3.14). One has

$$\begin{aligned}\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon).\nabla(h(u_\varepsilon)\varphi)dx \quad &+ \quad \varepsilon\int_\Omega \big[|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon.\nabla(h(u_\varepsilon)\varphi) + |u_\varepsilon|^{p_+-2}u_\varepsilon h(u_\varepsilon)\varphi\big]dx \\ &+ \quad \int_\Omega \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)\varphi dx + \lambda\int_{\partial\Omega} u_\varepsilon h(u_\varepsilon)\varphi d\sigma \\ &= \quad \int_\Omega f_\varepsilon h(u_\varepsilon)\varphi dx + \int_{\partial\Omega} g_\varepsilon h(u_\varepsilon)\varphi d\sigma.\end{aligned} \tag{3.48}$$

First of all, we prove the convergence of the third term of the left hand side of (3.48). Let us consider the following decomposition,

$$\int_\Omega \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)\varphi dx = \int_\Omega \beta_\varepsilon(u_\varepsilon)\big[h(u_\varepsilon)-h(u)\big]\varphi dx + \int_\Omega \beta_\varepsilon(u_\varepsilon)h(u)\varphi dx. \tag{3.49}$$

The sequence $\Big(\beta_\varepsilon(u_\varepsilon)\big[h(u_\varepsilon)-h(u)\big]\varphi\Big)_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$ and it converges to 0 a.e. in Ω, as $\varepsilon\to 0$. Therefore, thanks to Vitali's Theorem, the first term of the right hand side of (3.49) converges to 0, as $\varepsilon\to 0$. For the second term of the right hand side of (3.49), we have

$$\lim_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(u_\varepsilon)h(u)\varphi dx = \int_\Omega h(u)\varphi dz,$$

due to (3.32), the fact that $h(u)\varphi\in C_c(\Omega)$ and the density argument between $C_c(\Omega)$ and $C_0(\Omega)$. Thus, passing to the limit in (3.49), as $\varepsilon\to 0$, we get

$$\lim_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)\varphi dx = \int_\Omega h(u)\varphi dz. \tag{3.50}$$

From the Lebesgue dominated convergence theorem, one gets

$$\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon h(u_\varepsilon)\varphi dx = \int_\Omega f h(u)\varphi dx \tag{3.51}$$

and

$$\lim_{\varepsilon\to 0}\int_{\partial\Omega} g_\varepsilon h(u_\varepsilon)\varphi d\sigma = \int_{\partial\Omega} g h(u)\varphi d\sigma. \tag{3.52}$$

For the first term of the left hand side of (3.48), one has

$$\lim_{\varepsilon\to 0}\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon).\nabla(h(u_\varepsilon)\varphi)dx = \int_\Omega h(u)a(x,u,\nabla u)\nabla\varphi dx + \int_\Omega h'(u)a(x,u,\nabla u)\varphi\nabla u dx, \tag{3.53}$$

thanks to the Lemma 3.9. For the second term of the left hand side of (3.48), one gets

$$\lim_{\varepsilon\to 0}\varepsilon\int_\Omega \big[|\nabla u_\varepsilon|^{p_+-2}\nabla u_\varepsilon.\nabla(h(u_\varepsilon)\varphi)+|u_\varepsilon|^{p_+-2}u_\varepsilon(h(u_\varepsilon)\varphi)\big]dx = 0. \tag{3.54}$$

We are now interested to the last term of the left hand side of (3.48). Using Lebesgue dominated convergence theorem, we obtain

$$\lim_{\varepsilon\to 0}\lambda\int_{\partial\Omega} u_\varepsilon h(u_\varepsilon)\varphi d\sigma = \lambda\int_{\partial\Omega} u h(u)\varphi d\sigma. \tag{3.55}$$

Indeed, $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $W^{1,p_-}(\Omega)$ and $W^{1,p_-}(\Omega)\hookrightarrow C(\overline{\Omega})$ for $p_->N$. Then, $(u_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(\partial\Omega)$. Moreover, $|u_\varepsilon h(u_\varepsilon)\varphi|\le C\|u_\varepsilon\|_{L^\infty(\partial\Omega)}$ and $u_\varepsilon h(u_\varepsilon)\varphi\to uh(u)\varphi$ a.e on $\partial\Omega$. So, using (3.50), (3.51), (3.52), (3.53), (3.54), (3.55) and (3.48), we get

$$\begin{aligned}\int_\Omega h(u)\varphi dz + \int_\Omega h(u)a(x,u,\nabla u)\nabla\varphi dx &+ \int_\Omega h'(u)a(x,u,\nabla u)\nabla u\varphi dx\\ &+ \lambda\int_{\partial\Omega} uh(u)\varphi d\sigma = \int_\Omega f h(u)\varphi dx + \int_{\partial\Omega} g h(u)\varphi d\sigma,\end{aligned} \tag{3.56}$$

for all $\varphi \in C^\infty(\overline{\Omega})$ and for all $h \in C_c^1(\mathbb{R})$. Since $C^\infty(\overline{\Omega})$ is dense in $W^{1,\pi(\cdot)}(\Omega)$, then, (3.56) holds true with $\varphi \in W^{1,\pi(\cdot)}(\Omega) \cap L^\infty(\Omega)$ i.e

$$\begin{aligned}\int_\Omega h(u)\varphi dz + \int_\Omega h(u)a(x,u,\nabla u)\nabla\varphi dx \quad &+ \quad \int_\Omega h'(u)a(x,u,\nabla u)\nabla u\varphi dx \\ &+ \quad \lambda\int_{\partial\Omega} uh(u)\varphi d\sigma = \int_\Omega fh(u)\varphi dx + \int_{\partial\Omega} gh(u)\varphi d\sigma,\end{aligned} \tag{3.57}$$

for all $\varphi \in W^{1,\pi(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and for all $h \in C_c^1(\mathbb{R})$, which implies that $z \in \mathcal{M}_b^{\pi(\cdot)}(\Omega)$. Now, we give a Radon Nikodym decomposition result of the measure z.

Lemma 3.11. *(See* [25]*-Lemma 3.15)*
The Radon Nikodym decomposition of the measure z given by (3.57) *with respect to $\mathcal{L}^N$,*

$$z = w\mathcal{L}^N + \mu, \text{ with } \mu \perp \mathcal{L}^N,$$

satisfies the following properties.

$$\begin{cases} w \in \beta(u)\mathcal{L}^N - \text{ a.e. in } \Omega, \quad w \in L^1(\Omega), \quad \mu \in \mathcal{M}_b^{\pi(\cdot)}(\Omega), \\ \mu^+ \text{ is concentrated on } \big[u = M\big] \cap \big[u \neq \infty\big] \text{and} \\ \mu^- \text{ is concentrated on } \big[u = m\big] \cap \big[u \neq -\infty\big]. \end{cases}$$

Using the Radon Nikodym decomposition of measure z, the first term of (3.57) becomes

$$\int_\Omega h(u)\varphi dz = \int_\Omega h(u)w\varphi dx + \int_\Omega h(u)\varphi d\mu. \tag{3.58}$$

Combining (3.57) and (3.58), we obtain

$$\begin{aligned}\int_\Omega h(u)w\varphi dx + \int_\Omega h(u)a(x,u,\nabla u)\nabla\varphi dx \quad &+ \quad \int_\Omega h'(u)a(x,u,\nabla u)\nabla u\varphi dx \\ &+ \quad \int_\Omega h(u)\varphi d\mu + \lambda\int_{\partial\Omega} uh(u)\varphi d\sigma \\ &= \quad \int_\Omega fh(u)\varphi dx + \int_{\partial\Omega} gh(u)\varphi d\sigma,\end{aligned} \tag{3.59}$$

for all $\varphi \in W^{1,\pi(\cdot)}(\Omega) \cap L^\infty(\Omega)$.
To end the proof of Theorem 3.2, we prove that

$$\lim_{M\to\infty} \int_{\big[M<|u|<M+1\big]} a(x,u,\nabla u).\nabla u dx = 0. \tag{3.60}$$

Since $|\nabla u| \in L^{\pi(\cdot)}(\Omega)$, then using (1.4) with variable exponent $\pi(.)$, $a(x,u,\nabla u) \in L^{\pi'(\cdot)}(\Omega)$ and it follows from Hölder type inequality that $a(x,u,\nabla u).\nabla u \in L^1(\Omega)$.
Moreover, $u \in W^{1,p_-}(\Omega) \subset W^{1,1}(\Omega)$, which implies that $\int_\Omega |u|dx < \infty$. Therefore,

$$\int_\Omega |u|dx \geq \int_{\big[|u|\geq M\big]} |u|dx \geq M meas(\big[|u| \geq M\big]).$$

Then,

$$meas(\big[|u| \geq M\big]) \leq \frac{1}{M}\int_\Omega |u|dx \leq \frac{C}{M}, \tag{3.61}$$

for any $M > 0$ and C a positive constant not depending of M. Now, one has

$$\begin{aligned}\int_{\left[M<|u|<M+1\right]} a(x,u,\nabla u).\nabla u dx &\leq \int_{\left[|u|>M\right]} a(x,u,\nabla u).\nabla u dx\\ &\leq \int_{\left[|u|\geq M\right]} a(x,u,\nabla u).\nabla u dx.\end{aligned} \tag{3.62}$$

Since $meas(\left[|u| \geq M\right]) \to 0$, as $M \to \infty$, thanks to (3.61) and as $a(x,u,\nabla u).\nabla u \in L^1(\Omega)$, then

$$\lim_{M\to\infty}\int_{\left[|u|\geq M\right]} a(x,u,\nabla u).\nabla u dx = 0.$$

Hence,

$$\lim_{M\to\infty}\int_{\left[M<|u|<M+1\right]} a(x,u,\nabla u).\nabla u dx = 0.$$

Finally, using Lemma 3.8, Lemma 3.11, (3.59) and (3.60), we deduce that (u,w) is a renormalized solution of problem $P(\beta,f,g)$. □

The above proof conclude the proof of the existence result. □

One makes the following assumption on the function a, namely the local Lipschitz continuity with respect to z (see [2] where this assumption was used and commented).
For all bounded subset K of $\mathbb{R}\times\mathbb{R}^N$, there exists a constant $C(K)$ such that

$$\begin{aligned}&\text{a.e. } x\in\Omega, \quad \text{for all } (z,\eta),(\tilde{z},\eta)\in K,\\ &|a(x,z,\eta)-a(x,\tilde{z},\eta)| \leq C(K)|z-\tilde{z}|.\end{aligned} \tag{3.63}$$

Remark 3.3. *Let (u,w) be a renormalized solution of the problem $P(\beta,f,g)$, then $u\in C(\overline{\Omega})$, since $u\in W^{1,p_-}(\Omega)\hookrightarrow C(\overline{\Omega})$ for $p_- > N$. Moreover, if u is a Lipschitz continuous function, then $u\in W^{1,\infty}(\Omega)$. Indeed, Ω is an open bounded domain with smooth boundary $\partial\Omega$, so, the space of Lipschitz functions $C^{0,1}(\overline{\Omega})$ and $W^{1,\infty}(\Omega)$ are homeomorphic and they can be identified. The uniqueness in the sense of Theorem 3.2 seems difficult to prove. So, the following uniqueness result reduces to the case where the domain of β is bounded (named a partial uniqueness result).*

Theorem 3.4. *Assume* (1.2)-(1.6) *and* (3.63), *with $\mathcal{M}$ in* (1.4) *which can be taken constant. Assume that $-\infty < m \leq 0 \leq M < \infty$. Moreover, assume that $f\in L^1(\Omega)$ and $g\in L^1(\partial\Omega)$ such that the problem $P(\beta,f,g)$ has a solution (u,w) in the sense that: u is a measurable and Lipschitz continuous function such that $T_k(u)\in W^{1,\pi(.)}(\Omega)$ for all $k>0$, $u\in L^1(\partial\Omega)$ and $u\in dom(\beta)$ $\mathcal{L}^N$ a.e. Ω, $w\in L^1(\Omega)$ and $w\in\beta(u)$ $\mathcal{L}^N$ a.e. Ω; and there exists a measure $\mu\in\mathcal{M}_b^{\pi(.)}(\Omega)$ such that $\mu\perp\mathcal{L}^N$, μ^+ is concentrated on $\left[u=M\right]$, μ^- is concentrated on $\left[u=m\right]$ such that*

$$\int_\Omega w\varphi dx + \int_\Omega a(x,u,\nabla u)\nabla\varphi dx + \int_\Omega \varphi d\mu + \lambda\int_{\partial\Omega} u\varphi d\sigma = \int_\Omega f\varphi dx + \int_{\partial\Omega} g\varphi d\sigma, \tag{3.64}$$

for all $\varphi\in W^{1,\pi(.)}(\Omega)\cap L^\infty(\Omega)$.
Then, any other solution $(\tilde{u},\tilde{w})$ of the problem $P(\beta,f,g)$ in the sense of equality (3.64) *associated to $\tilde{\mu}\in\mathcal{M}_b^{\pi(.)}(\Omega)$, coincides with (v,w) i.e*

$$\begin{cases} u=\tilde{u} \quad \text{a.e. on } \partial\Omega,\\ w=\tilde{w} \text{ a.e. in } \Omega\\ \text{and}\\ \mu=\tilde{\mu}.\end{cases}$$

Before starting the proof of Theorem 3.4, we justify the relation (3.64). Let us consider the function h_k where k is a positive constant such that

$$\begin{cases} h_k \in C_c^1(\mathbb{R}), \quad h_k(s) \geq 0, \ \forall s \in \mathbb{R}, \\ \\ h_k(s) = 1 \ \text{ if } |s| < k \text{ and } h_k(s) = 0 \text{ if } |s| \geq k. \end{cases}$$

Since the domain of β is bounded and the equality (3.59) holds for any $h \in C_c^1(\mathbb{R})$, we choose $h(s) = h_k(s) = 1$ in (3.59), for all $s \in \big[m,\ M\big] \subsetneq \big[-k,\ k\big] = supp(h_k)$. One obtains

$$\int_\Omega w\varphi dx + \int_\Omega a(x,u,\nabla u).\nabla\varphi dx + \int_\Omega \varphi d\mu + \lambda\int_{\partial\Omega} u\varphi d\sigma = \int_\Omega f\varphi dx + \int_{\partial\Omega} g\varphi d\sigma, \tag{3.65}$$

for all $\varphi \in W^{1,\pi(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Proof of Theorem 3.4. For more details, see [2]-Proof of Theorem 2.8 and [24, 25].
Let (u_1, w_1) be a solution of the problem $P(\beta, f, g)$, with u_1 Lipschitz continuous function and (u_2, w_2) be an other solution of the problem $P(\beta, f, g)$ in the sense of the equality (3.64).
Let $\phi := \dfrac{1}{k}T_k(u_1 - u_2)$, then ϕ is an admissible test function in the formulations for both (u_1, w_1) and (u_2, w_2). So, with this test function, one has

$$\begin{aligned} \int_\Omega w_1\frac{1}{k}T_k(u_1-u_2)dx \quad &+ \quad \frac{1}{k}\int_\Omega a(x,u_1,\nabla u_1).\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}dx \\ &+ \quad \int_\Omega \frac{1}{k}T_k(u_1-u_2)d\mu_1 + \lambda\int_{\partial\Omega} u_1\frac{1}{k}T_k(u_1-u_2)d\sigma \\ &= \quad \int_\Omega f\frac{1}{k}T_k(u_1-u_2)dx + \int_{\partial\Omega} g\frac{1}{k}T_k(u_1-u_2)d\sigma \end{aligned} \tag{3.66}$$

and

$$\begin{aligned} \int_\Omega w_2\frac{1}{k}T_k(u_1-u_2)dx \quad &+ \quad \frac{1}{k}\int_\Omega a(x,u_2,\nabla u_2).\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}dx \\ &+ \quad \int_\Omega \frac{1}{k}T_k(u_1-u_2)d\mu_2 + \lambda\int_{\partial\Omega} u_2\frac{1}{k}T_k(u_1-u_2)d\sigma \\ &= \quad \int_\Omega f\frac{1}{k}T_k(u_1-u_2)dx + \int_{\partial\Omega} g\frac{1}{k}T_k(u_1-u_2)d\sigma. \end{aligned} \tag{3.67}$$

We substract (3.66) and (3.67) to get

$$\begin{aligned} &\frac{1}{k}\int_\Omega \big(a(x,u_1,\nabla u_1) - a(x,u_2,\nabla u_2)\big).\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}dx \\ + \quad &\int_\Omega (w_1-w_2)\frac{1}{k}T_k(u_1-u_2)dx + \int_\Omega \frac{1}{k}T_k(u_1-u_2)(d\mu_1 - d\mu_2) \\ + \quad &\lambda\int_{\partial\Omega} \frac{1}{k}T_k(u_1-u_2)(u_1-u_2)d\sigma = 0. \end{aligned} \tag{3.68}$$

Let's denote by I the first term of the left hand side of (3.68). We know that

$$\begin{aligned} (a(x,u_1,\nabla u_1) - a(x,u_2,\nabla u_2))\nabla(u_1-u_2) \quad &= \quad (a(x,u_1,\nabla u_1) - a(x,u_2,\nabla u_1))\nabla(u_1-u_2) \\ &+ \quad \underbrace{(a(x,u_2,\nabla u_1) - a(x,u_2,\nabla u_2))\nabla(u_1-u_2)}_{\geq 0}. \end{aligned}$$

One has

$$I = I_k + \int_\Omega \big(a(x,u_2,\nabla u_1) - a(x,u_2,\nabla u_2)\big)\frac{1}{k}\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}dx,$$

where

$$I_k = \int_\Omega \big(a(x,u_1,\nabla u_1) - a(x,u_2,\nabla u_1)\big)\frac{1}{k}\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}dx.$$

Let's show that $I_k \to 0$ as $k \to 0$. Since u_1 is bounded, then u_2 is also bounded on the set $[0 < |u_1 - u_2| < k]$. Thus, using (3.63), one obtains

$$\begin{aligned}
|I_k| &\le \frac{1}{k}\int_{[0<|u_1-u_2|<k]} |a(x,u_1,\nabla u_1) - a(x,u_2,\nabla u_1)||\nabla u_1 - \nabla u_2|dx \\
&\le \frac{1}{k}\int_{[0<|u_1-u_2|<k]} C\big(||u_1||_{L^\infty(\Omega)}, ||\nabla u_1||_{L^\infty(\Omega)}\big)|u_1-u_2||\nabla u_1 - \nabla u_2|dx \\
&\le C\big(||u_1||_{L^\infty(\Omega)}, ||\nabla u_1||_{L^\infty(\Omega)}\big)\int_{[0<|u_1-u_2|<k]} |\nabla u_1 - \nabla u_2|dx \to 0, \text{ as } k\to 0.
\end{aligned} \tag{3.69}$$

Notice that $\lim\limits_{k\to 0} meas([0<|u_1-u_2|<k]) = 0$ and $|\nabla u_1 - \nabla u_2| \in L^1(\Omega)$.

For the second term of the left hand side of (3.68), one has

$$\begin{aligned}
\lim_{k\to 0}\int_\Omega (w_1-w_2)\frac{1}{k}T_k(u_1-u_2)dx &= \int_\Omega (w_1-w_2)sign_0(u_1-u_2)dx \\
&= \int_\Omega |w_1-w_2|dx.
\end{aligned} \tag{3.70}$$

For the third term of the left hand side of (3.68), one has

$$\begin{aligned}
\lim_{k\to 0}\int_\Omega \frac{1}{k}T_k(u_1-u_2)(d\mu_1-d\mu_2) &= \int_\Omega sign_0(u_1-u_2)(d\mu_1-d\mu_2) \\
&= \int_\Omega |d\mu_1 - d\mu_2| \\
&= \int_\Omega |d(\mu_1-\mu_2)|.
\end{aligned} \tag{3.71}$$

For the last term of the left hand side of (3.68), one obtains

$$\lim_{k\to 0}\lambda\int_{\partial\Omega}\frac{1}{k}T_k(u_1-u_2)(u_1-u_2)d\sigma = \lambda\int_{\partial\Omega}\big|u_1-u_2\big|d\sigma. \tag{3.72}$$

Finally, one lets k goes to 0 in (3.68) and taking into account the relations (3.69), (3.70), (3.71) and (3.72), it follows that

$$\begin{aligned}
&\lim_{k\to 0}\int_\Omega \big(a(x,u_2,\nabla u_1) - a(x,u_2,\nabla u_2)\big)\frac{1}{k}\nabla(u_1-u_2)\chi_{[0<|u_1-u_2|<k]}dx \\
&+ \int_\Omega |w_1-w_2|dx + \int_\Omega |d(\mu_1-\mu_2)| + \lambda\int_{\partial\Omega}|u_1-u_2|d\sigma = 0.
\end{aligned} \tag{3.73}$$

Since all the terms of equality (3.73) are nonnegative, it follows that

$$\int_\Omega |w_1-w_2|dx = 0, \quad \int_\Omega |d(\mu_1-\mu_2)| = 0 \text{ and } \int_{\partial\Omega}|u_1-u_2|d\sigma = 0.$$

Hence,

$$w_1 = w_2 \quad \text{a.e. in} \quad \Omega, \quad \mu_1 = \mu_2 \text{ and } u_1 = u_2 \text{ a.e. on } \quad \partial\Omega.$$

□

4. Conclusion

In this work, we studied the existence and the partial uniqueness results of renormalized solutions. The main existence result in Theorem 3.2 is obtained thanks to the technic of pseudo-monotone operator in Banach space and the weak convergence method associated to the Young measure. Our uniqueness result in Theorem 3.4 is obtained by adding two additional assumptions: the first one on the operator $a(x, s, \xi)$ and the second one on the domain of β which is assumed bounded, which leads to a partial uniqueness result for us since these two assumptions were not used to prove the existence of solutions (see Theorem 3.2).

Funding The authors have not disclosed any funding.
Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.
Declarations
Competing interests The authors have not disclosed any competing interests.

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