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Fundamental Properties and a Unified Approach to Fractional Hermite–Hadamard Inequalities via Hybrid Godunova–Levin Preinvex Mappings

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ABSTRACT. The objective of this article is to introduce a new hybrid generalized harmonic Godunova–Levin preinvex class that unifies several existing definitions available in the literature. To emphasize the distinctive features of this new concept, we provide nontrivial examples showing that there exists a class of functions that belongs to this framework but does not fall within other known classes of convex or preinvex-type mappings. Based on this new notion, we first investigate its fundamental properties and then establish new variants of Hermite–Hadamard inequalities by means of the Riemann–Liouville fractional operator. To confirm the validity and correctness of the obtained results, we demonstrate that, under suitable assumptions, a number of previously known results can be recovered as special cases. We believe that this new concept will contribute significantly to the further development of the existing literature.

1. INTRODUCTION

Convex inequalities have become one of the most important branches of mathematical analysis with applications in optimization, numerical integration, and other sciences. Nowadays, the appearance of fractional calculus with nonlocal fractional derivatives is providing new insights about inequalities and inequalities related to some special class of operators with memory or heredity properties of the functions. One of the most common of these operators is the Riemann-Liouville fractional integral operator, which can be used to extend the classical inequalities like Hermite-Hadamard inequalities, and Jensen type inequalities to the case of fractional calculus.

Another approach to extending some inequalities to a broader class of functions is generalizing the notion of convexity itself, for instance, by means of h-convex, m-convex, or preinvex functions. Generalized convexity combined with fractional Riemann-Liouville operators makes it possible to establish even more powerful and versatile inequalities.

There are many inequalities in mathematical analysis, out of which the Hermite–Hadamard inequality stands out as one of the most important and profound results. It offers a useful connection between the values of a convex function and its average value in terms of an integral. The well-known Hermite and Hadamard inequality [30] is presented below:

Suppose that $\Psi : \mathtt{K} \subset \mathbb{R} \to \mathbb{R}$ is a convex function, such that $\upsilon_1, \upsilon_2 \in \mathtt{K}$, where $\upsilon_1 < \upsilon_2$. Then, we have

$$\Psi\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \leq \frac{1}{\upsilon_2-\upsilon_1}\int_{\upsilon_1}^{\upsilon_2} \Psi(\nu)\,\mathrm{d}\nu \leq \frac{\Psi(\upsilon_1)+\Psi(\upsilon_2)}{2}. \tag{1.1}$$

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Hermite–Hadamard inequality, have been extensively studied and generalized using various convexity concepts and analytical tools. An early contribution in this direction was provided by Baloch and Işcan [11], who established Hermite–Hadamard type inequalities for p-preinvex functions, thereby extending classical convexity to a broader functional class. Subsequently, Kashuri and Liko [15] introduced inequalities for (s, m, φ)-preinvex Godunova–Levin functions, offering a unified structure that encompasses several generalized convex mappings.

The integration of fractional calculus into inequality theory has further enriched this area. Noor et al. [22] developed fractional Hermite–Hadamard inequalities for Godunova–Levin functions, while Peng et al. [26] derived Simpson-type inequalities via generalized preinvexity, contributing to numerical integration techniques. Later, Khan et al. [16] presented Riemann–Liouville fractional integral inequalities in interval-valued settings, and Sharma et al. [31] extended these results by establishing fractional Hermite–Hadamard inequalities with improved bounds. In addition, Tariq et al. [32] introduced fractional Hadamard and Pachpatte-type inequalities, and Meftah et al. [21] investigated local fractional Maclaurin-type inequalities for generalized convex functions.

In recent years, interval-valued and fuzzy extensions of convex inequalities have attracted considerable attention due to their ability to handle uncertainty. Lai et al. [18] examined Hermite–Hadamard type inclusions for interval-valued preinvex functions, and further extended these results to coordinated interval-valued settings in [19]. Similarly, Santos-García et al. [29] developed inequalities for fuzzy-interval-valued functions using double integral operators, highlighting their applicability in uncertain environments.

Another important research direction involves Godunova–Levin type functions and their generalizations. Ali et al. [9] established generalized Hermite–Hadamard inequalities for h-Godunova–Levin functions. Afzal et al. [1] introduced center-radius order relations to obtain sharper bounds, while subsequent works [2] explored harmonical Godunova–Levin functions and their properties. Further developments by Afzal et al. [3] addressed integral inequalities for stochastic processes, and [5] derived Jensen and Hermite–Hadamard type inclusions for harmonical convex mappings. Additionally, Almutairi and Kılıçman [10] investigated integral inequalities for h-Godunova–Levin preinvex functions, and Afzal et al. [4] extended these inequalities using Caputo–Fabrizio fractional operators.

The concept of m-convexity has also been widely studied in both functional analysis and optimization. Rooin et al. [28] analyzed operator m-convex functions, while Murota and Shioura [20] investigated M-convex functions in discrete optimization frameworks. These studies provide foundational tools for extending convex inequalities to more generalized settings.

Recent advancements include quantum and numerical approaches to convex inequalities. Kalsoom et al. [14] introduced post-quantum Hermite–Hadamard inequalities for higher-order generalized mappings, whereas Gulshan et al. [12] developed quantum estimates for midpoint and trapezoidal inequalities. Moreover, Zhang et al. [33] studied Karush–Kuhn–Tucker (KKT) optimality conditions for generalized convex optimization problems involving interval-valued objective functions, emphasizing their importance in optimization theory. For additional related results concerning preinvexity and Godunova–Levin type mappings, we refer to [34, 27, 17, 8, 6, 7].

Novelty and Significance. The novelty and significance of this study are outlined as follows. In this work, we introduce a new class of hybrid generalized preinvex mappings by incorporating the concepts of harmonic Godunova–Levin mappings [2], preinvex mappings [9], m-convex mappings [20], and h-convex mappings [10]. Based on this combination, we propose a new framework, namely (h, m)-harmonic Godunova–Levin preinvex mappings. To highlight the distinctiveness of this new class, we provide nontrivial example (see 3.1) demonstrating that there exists a class of functions that belongs to this framework, whereas several other related concepts fail to capture such functions. Furthermore, in contrast to the work presented in [24], where the authors employed classical integral operators along with certain types of preinvexity without incorporating the Godunova–Levin structure, our approach refines and extends the existing theory by utilizing fractional operators.

In addition, we investigate several fundamental properties of the proposed class and establish new results that generalize and improve previously known inequalities.

The paper is organized into four sections. Section 1 provides a brief introduction. In Section 2, we recall some preliminary concepts related to the topic. Section 3 introduces the new concept, investigates its fundamental properties, and derives several new forms of Hermite–Hadamard inequalities. Finally, Section 4 presents the conclusion along with possible future research directions.

2. Preliminaries

In this section, we recall some known definitions and auxiliary results that motivate our work. We also present the basic notions needed throughout the paper. In particular, we recall the notions of invexity, preinvexity, Godunova–Levin type convexity, and harmonic variants associated with a bifunction. Moreover, since our main results are established by means of fractional integral operators, we include the definition of the Riemann–Liouville fractional integral.

Let $\mathtt{K} \subset \mathbb{R}$ and let

$$\phi : \mathtt{K} \times \mathtt{K} \to \mathbb{R}$$

be a continuous bifunction.

Definition 2.1 (see [20]). *A set $\mathtt{K} \subseteq \mathbb{R}$ is said to be m-convex, where $m \in [0,1]$, if*

$$(1-\eta)\upsilon_1 + \eta m \upsilon_2 \in \mathtt{K},$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in [0,1]$.

Definition 2.2 (see [24]). *A set $\mathtt{K} \subseteq \mathbb{R}$ is said to be invex with respect to the bifunction ϕ if*

$$\upsilon_1 + \eta\, \phi(\upsilon_2, \upsilon_1) \in \mathtt{K},$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in [0,1]$.

Definition 2.3 (see [23]). *A set $\mathtt{K} \subseteq (0,\infty)$ is said to be harmonically m-invex with respect to the bifunction ϕ if*

$$\frac{\upsilon_1\big(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1)\big)}{\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1)} \in \mathtt{K},$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$, $m \in [0,1]$, and $\eta \in [0,1]$, provided that

$$\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1) \neq 0.$$

Definition 2.4 (see [24]). *A mapping $\Psi : \mathtt{K} \to \mathbb{R}$ is said to be m-convex, where $m \in [0,1]$, if*

$$\Psi\big((1-\eta)\upsilon_1 + \eta m \upsilon_2\big) \leq (1-\eta)\Psi(\upsilon_1) + \eta\Psi(\upsilon_2),$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in [0,1]$.

Definition 2.5 (see [23]). *A mapping $\Psi : \mathtt{K} \to \mathbb{R}$ is said to be harmonically m-convex, where $m \in [0,1]$, if*

$$\Psi\left(\frac{m\upsilon_1\upsilon_2}{m\eta\upsilon_1 + (1-\eta)\upsilon_2}\right) \leq (1-\eta)\Psi(\upsilon_1) + \eta\Psi(\upsilon_2),$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in [0,1]$.

Definition 2.6 (see [24]). *A mapping $\Psi : \mathtt{K} \to \mathbb{R}$ defined on an invex set $\mathtt{K}$ is said to be preinvex with respect to the bifunction ϕ if*

$$\Psi\big(\upsilon_1 + \eta\phi(\upsilon_2, \upsilon_1)\big) \le (1-\eta)\Psi(\upsilon_1) + \eta\Psi(\upsilon_2),$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in [0,1]$.

Definition 2.7 (see [23]). *A mapping $\Psi : \mathtt{K} \to \mathbb{R}$ defined on a harmonically m-invex set $\mathtt{K} \subseteq (0,\infty)$ is said to be harmonically m-preinvex with respect to the bifunction ϕ if*

$$\Psi\left(\frac{\upsilon_1\big(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1)\big)}{\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1)}\right) \le (1-\eta)\Psi(\upsilon_1) + \eta m\Psi(\upsilon_2),$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in [0,1]$.

Definition 2.8 (see [24]). *A function $\Psi : \mathtt{K} \to \mathbb{R}$ is called supermultiplicative if*

$$\Psi(\upsilon_1)\Psi(\upsilon_2) \le \Psi(\upsilon_1\upsilon_2),$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$.

Definition 2.9 (see [2]). *A mapping $\Psi : \mathtt{K} \to \mathbb{R}$ defined on an invex set $\mathtt{K}$ is said to be Godunova–Levin preinvex with respect to the bifunction ϕ if*

$$\Psi\big(\upsilon_1 + \eta\phi(\upsilon_2, \upsilon_1)\big) \le \frac{\Psi(\upsilon_1)}{1-\eta} + \frac{\Psi(\upsilon_2)}{\eta},$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in (0,1)$.

Definition 2.10 (see [24]). *Let $\mathsf{h} : [0,1] \to \mathbb{R}^+$ be a nonnegative function. A mapping $\Psi : \mathtt{K} \to \mathbb{R}$ defined on an invex set $\mathtt{K}$ is said to be h-preinvex with respect to the bifunction ϕ if*

$$\Psi\big(\upsilon_1 + \eta\phi(\upsilon_2, \upsilon_1)\big) \le \mathsf{h}(1-\eta)\Psi(\upsilon_1) + \mathsf{h}(\eta)\Psi(\upsilon_2),$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in [0,1]$.

Definition 2.11 (see [2]). *Let $\mathsf{h} : (0,1) \to \mathbb{R}^+$ be a positive function. A mapping $\Psi : \mathtt{K} \to \mathbb{R}$ defined on an invex set $\mathtt{K}$ is said to be h-Godunova–Levin preinvex with respect to the bifunction ϕ if*

$$\Psi\big(\upsilon_1 + \eta\phi(\upsilon_2, \upsilon_1)\big) \le \frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{\Psi(\upsilon_2)}{\mathsf{h}(\eta)},$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in (0,1)$.

Definition 2.12 (Condition C, see [24]). *Let $\mathtt{K} \subseteq \mathbb{R}^n$ be an invex set with respect to the bifunction ϕ. The bifunction ϕ is said to satisfy Condition C if, for every $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in [0,1]$,*

$$\phi\big(\upsilon_2, \upsilon_2 + \eta\phi(\upsilon_1, \upsilon_2)\big) = -\eta\,\phi(\upsilon_1, \upsilon_2), \tag{2.2}$$

and

$$\phi\big(\upsilon_1, \upsilon_2 + \eta\phi(\upsilon_1, \upsilon_2)\big) = (1-\eta)\phi(\upsilon_1, \upsilon_2). \tag{2.3}$$

Equivalently, for all $\phi_1, \phi_2 \in [0,1]$,

$$\phi\big(\upsilon_2 + \phi_2\phi(\upsilon_1, \upsilon_2), \upsilon_2 + \phi_1\phi(\upsilon_1, \upsilon_2)\big) = (\phi_2 - \phi_1)\phi(\upsilon_1, \upsilon_2).$$

Since our findings are also presented in terms of special functions and fractional integral operators, we recall below the Gamma function, the Beta function, and the Riemann–Liouville fractional integral operator.

Definition 2.13. *The Gamma function is defined by*

$$\Gamma(\rho) = \int_0^\infty \mathtt{t}^{\rho-1} e^{-\mathtt{t}}\, d\mathtt{t}, \qquad \rho > 0.$$

Definition 2.14. *The Beta function is defined by*

$$\mathbf{B}(\rho_1, \rho_2) = \int_0^1 \mathtt{t}^{\rho_1 - 1}(1-\mathtt{t})^{\rho_2 - 1}\, d\mathtt{t}, \qquad \rho_1, \rho_2 > 0,$$

and satisfies

$$\mathbf{B}(\rho_1, \rho_2) = \frac{\Gamma(\rho_1)\Gamma(\rho_2)}{\Gamma(\rho_1+\rho_2)}.$$

Definition 2.15 (see [31]). *Let $\Psi \in L[\upsilon_1, \upsilon_2]$ with $0 \le \upsilon_1 < \upsilon_2$, and let $\alpha > 0$. The left-sided and right-sided Riemann–Liouville fractional integrals of order α are defined, respectively, by*

$$J^{\alpha}_{\upsilon_1^+}\Psi(\mathtt{x}) = \frac{1}{\Gamma(\alpha)}\int_{\upsilon_1}^{\mathtt{x}} (\mathtt{x}-\mathtt{t})^{\alpha-1}\Psi(\mathtt{t})\, d\mathtt{t}, \qquad \mathtt{x} > \upsilon_1,$$

and

$$J^{\alpha}_{\upsilon_2^-}\Psi(\mathtt{x}) = \frac{1}{\Gamma(\alpha)}\int_{\mathtt{x}}^{\upsilon_2} (\mathtt{t}-\mathtt{x})^{\alpha-1}\Psi(\mathtt{t})\, d\mathtt{t}, \qquad \mathtt{x} < \upsilon_2.$$

3. THE MAJOR RESULTS

In this section, we introduce a novel class of generalized convex mappings which simultaneously incorporates several existing concepts and extends a variety of earlier definitions under different parametric settings.

Definition 3.16. *Let $\mathsf{h} : \mathtt{M} = [0,1] \to \mathbb{R}^+$ be a positive mapping. A mapping $\Psi : \mathtt{K} \to \mathbb{R}$ defined on an invex set $\mathtt{K} \subset \mathbb{R}^+$ is said to be an (m, h)-harmonic Godunova–Levin preinvex function with respect to the bifunction $\phi(,\cdot)$, where $m \in (0,1]$, if*

$$\Psi\left(\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1)}\right) \le \frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)},$$

for all $\upsilon_1, \upsilon_2 \in \mathtt{K}$ and $\eta \in (0,1)$, provided that $\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1) \neq 0$.

The class of all (m, h)-harmonic Godunova–Levin preinvex functions is denoted by

$$\mathcal{HPGL}((m, \mathsf{h}), [\upsilon_1, \upsilon_2], \mathbb{R}^+).$$

Remark 3.1.
- *If $m = 1$ and $\phi(m\upsilon_2, \upsilon_1) = m\upsilon_2 - \upsilon_1$, then Definition 3.16 reduces to Definition 2.7 in* [5].
- *If $m = 1$, $\phi(m\upsilon_2, \upsilon_1) = m\upsilon_2 - \upsilon_1$, and $\mathsf{h}(\eta) = \frac{1}{\eta}$, then Definition 3.16 reduces to Definition 2.1 in* [13].
- *If $m = 1$ and $\mathsf{h}(\eta) = \frac{1}{\eta}$, then Definition 3.16 reduces to Definition 2.4 in* [24].

Example 3.1. *Let*

$$\mathtt{K} = [1,2], \qquad m = \frac{1}{2}, \qquad \phi(m\upsilon_2, \upsilon_1) = m\upsilon_2 - \upsilon_1,$$

and choose

$$\mathsf{h}(\eta) = \eta^2(1-\eta)^2, \qquad \eta \in (0,1).$$

Define

$$\Psi(\upsilon) = \frac{1}{100} + \frac{1}{5}\exp\left(-100\left(\upsilon - \frac{3}{2}\right)^2\right), \qquad \upsilon \in [1,2].$$

Then Ψ is an (m, h)-harmonic Godunova–Levin preinvex function on $\mathtt{K}$, but it is neither m-convex, nor preinvex, nor harmonic Godunova–Levin in the classical sense.

Proof. We first verify that $\Psi \in \mathcal{HPGL}((m, \mathsf{h}), [1,2], \mathsf{R}^+)$.

For $\upsilon_1, \upsilon_2 \in [1,2]$ and $\eta \in (0,1)$, since

$$\phi(m\upsilon_2, \upsilon_1) = m\upsilon_2 - \upsilon_1,$$

the harmonic interpolation point becomes

$$\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1)} = \frac{m\upsilon_1\upsilon_2}{\eta\upsilon_1 + (1-\eta)m\upsilon_2}.$$

Because $\upsilon_1, \upsilon_2 \in [1,2]$ and $m = \frac{1}{2}$, this point belongs to $[1,2]$. Hence

$$\Psi\left(\frac{m\upsilon_1\upsilon_2}{\eta\upsilon_1 + (1-\eta)m\upsilon_2}\right) \leq \max_{\upsilon\in[1,2]} \Psi(\upsilon) = \Psi\left(\frac{3}{2}\right) = \frac{1}{100} + \frac{1}{5} = \frac{21}{100}.$$

On the other hand, for every $\upsilon \in [1,2]$,

$$\Psi(\upsilon) \geq \frac{1}{100}.$$

Also,

$$\mathsf{h}(\eta) = \eta^2(1-\eta)^2 \leq \frac{1}{16}, \qquad \eta \in (0,1),$$

and therefore

$$\frac{1}{\mathsf{h}(\eta)} \geq 16, \qquad \frac{1}{\mathsf{h}(1-\eta)} \geq 16.$$

Thus,

$$\begin{aligned}\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)} &\geq \frac{1}{100}\cdot 16 + \frac{1}{2}\cdot\frac{1}{100}\cdot 16 \\ &= \frac{24}{100}.\end{aligned}$$

Hence,

$$\Psi\left(\frac{m\upsilon_1\upsilon_2}{\eta\upsilon_1 + (1-\eta)m\upsilon_2}\right) \leq \frac{21}{100} < \frac{24}{100} \leq \frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)}.$$

Therefore, Ψ is an (m, h)-harmonic Godunova–Levin preinvex function.

Next, we show that Ψ is not m-convex. Take

$$\upsilon_1 = \upsilon_2 = 2, \qquad \eta = \frac{1}{2}.$$

Then

$$(1-\eta)\upsilon_1 + \eta m\upsilon_2 = \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot\frac{1}{2}\cdot 2 = \frac{3}{2}.$$

Hence

$$\Psi\left((1-\eta)\upsilon_1 + \eta m\upsilon_2\right) = \Psi\left(\frac{3}{2}\right) = \frac{21}{100}.$$

But

$$\begin{aligned}(1-\eta)\Psi(\upsilon_1) + \eta\Psi(\upsilon_2) &= \frac{1}{2}\Psi(2) + \frac{1}{2}\Psi(2) = \Psi(2) \\ &= \frac{1}{100} + \frac{1}{5}e^{-25} < \frac{11}{1000}.\end{aligned}$$

Thus

$$\Psi\left((1-\eta)\upsilon_1 + \eta m\upsilon_2\right) > (1-\eta)\Psi(\upsilon_1) + \eta\Psi(\upsilon_2),$$

so Ψ is not m-convex.

Now we show that Ψ is not preinvex. Take the standard bifunction

$$\phi(\upsilon_2, \upsilon_1) = \upsilon_2 - \upsilon_1,$$

and choose

$$\upsilon_1 = 1, \qquad \upsilon_2 = 2, \qquad \eta = \frac{1}{2}.$$

Then

$$\upsilon_1 + \eta\phi(\upsilon_2, \upsilon_1) = 1 + \frac{1}{2}(2-1) = \frac{3}{2}.$$

Therefore,

$$\Psi\left(\upsilon_1 + \eta\phi(\upsilon_2, \upsilon_1)\right) = \Psi\left(\frac{3}{2}\right) = \frac{21}{100}.$$

But

$$\begin{aligned}(1-\eta)\Psi(\upsilon_1) + \eta\Psi(\upsilon_2) &= \frac{1}{2}\Psi(1) + \frac{1}{2}\Psi(2)\\ &= \frac{1}{100} + \frac{1}{5}e^{-25} < \frac{11}{1000}.\end{aligned}$$

Hence

$$\Psi\left(\upsilon_1 + \eta\phi(\upsilon_2, \upsilon_1)\right) > (1-\eta)\Psi(\upsilon_1) + \eta\Psi(\upsilon_2),$$

and thus Ψ is not preinvex.

Finally, we show that Ψ is not harmonic Godunova–Levin in the classical sense. Consider the usual harmonic Godunova–Levin inequality

$$\Psi\left(\frac{\upsilon_1\upsilon_2}{\eta\upsilon_1 + (1-\eta)\upsilon_2}\right) \leq \frac{\Psi(\upsilon_1)}{1-\eta} + \frac{\Psi(\upsilon_2)}{\eta}.$$

Take

$$\upsilon_1 = 1, \qquad \upsilon_2 = 2, \qquad \eta = \frac{2}{3}.$$

Then

$$\frac{\upsilon_1\upsilon_2}{\eta\upsilon_1 + (1-\eta)\upsilon_2} = \frac{2}{\frac{2}{3} + \frac{2}{3}} = \frac{3}{2}.$$

Therefore,

$$\Psi\left(\frac{\upsilon_1\upsilon_2}{\eta\upsilon_1 + (1-\eta)\upsilon_2}\right) = \Psi\left(\frac{3}{2}\right) = \frac{21}{100}.$$

On the other hand,

$$\begin{aligned}\frac{\Psi(1)}{1-\eta} + \frac{\Psi(2)}{\eta} &= 3\Psi(1) + \frac{3}{2}\Psi(2)\\ &= \frac{9}{200} + \frac{9}{10}e^{-25} < \frac{46}{1000}.\end{aligned}$$

Hence

$$\Psi\left(\frac{\upsilon_1\upsilon_2}{\eta\upsilon_1 + (1-\eta)\upsilon_2}\right) > \frac{\Psi(1)}{1-\eta} + \frac{\Psi(2)}{\eta},$$

so Ψ is not harmonic Godunova–Levin.

Consequently, Ψ belongs to the new class of (m, h)-harmonic Godunova–Levin preinvex functions, but it does not belong to the classes of m-convex, preinvex, or harmonic Godunova–Levin functions. □

Now we have demonstrated that two similarly ordered (m, h)-harmonic Godunova and Levin preinvex functions again belong to that class, which is $\mathcal{HPGL}((m, \mathsf{h}), [\upsilon_1, \upsilon_2], \mathsf{R}^+)$.

Lemma 3.1. *Let Ψ and ζ be two similarly ordered (m, h)-harmonic Godunova–Levin preinvex functions. Assume that*

$$\frac{1}{\mathsf{h}(1-\eta)} + \frac{m}{\mathsf{h}(\eta)} \leq 1, \qquad \eta \in [0,1].$$

Then the product $\Psi\zeta$ is again an (m, h)-harmonic Godunova–Levin preinvex function.

Proof. Take arbitrary $\mathtt{k}, \mathtt{n} \in \mathtt{I}_\phi$ and $\eta \in [0,1]$. Set

$$\mathtt{H}_\eta(\mathtt{k}, \mathtt{n}) := \frac{\mathtt{k}\big(\mathtt{k} + \phi(m\mathtt{n}, \mathtt{k})\big)}{\mathtt{k} + (1-\eta)\phi(m\mathtt{n}, \mathtt{k})}.$$

Since Ψ and ζ are (m, h)-harmonic Godunova–Levin preinvex, we have

$$\Psi\big(\mathtt{H}_\eta(\mathtt{k}, \mathtt{n})\big) \leq \frac{\Psi(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\mathtt{n})}{\mathsf{h}(\eta)}, \tag{3.4}$$

and

$$\zeta\big(\mathtt{H}_\eta(\mathtt{k}, \mathtt{n})\big) \leq \frac{\zeta(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\zeta(\mathtt{n})}{\mathsf{h}(\eta)}. \tag{3.5}$$

Multiplying (3.4) and (3.5), we obtain

$$\begin{aligned}
&(\Psi\zeta)\big(\mathtt{H}_\eta(\mathtt{k}, \mathtt{n})\big) \\
&\leq \left(\frac{\Psi(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\mathtt{n})}{\mathsf{h}(\eta)}\right)\left(\frac{\zeta(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\zeta(\mathtt{n})}{\mathsf{h}(\eta)}\right) \\
&= \frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{[\mathsf{h}(1-\eta)]^2} + \frac{m\big(\Psi(\mathtt{k})\zeta(\mathtt{n}) + \Psi(\mathtt{n})\zeta(\mathtt{k})\big)}{\mathsf{h}(1-\eta)\mathsf{h}(\eta)} + \frac{m^2\Psi(\mathtt{n})\zeta(\mathtt{n})}{[\mathsf{h}(\eta)]^2}.
\end{aligned} \tag{3.6}$$

Now, since Ψ and ζ are similarly ordered, we have

$$\big(\Psi(\mathtt{k}) - \Psi(\mathtt{n})\big)\big(\zeta(\mathtt{k}) - \zeta(\mathtt{n})\big) \geq 0.$$

Expanding this inequality gives

$$\Psi(\mathtt{k})\zeta(\mathtt{k}) + \Psi(\mathtt{n})\zeta(\mathtt{n}) \geq \Psi(\mathtt{k})\zeta(\mathtt{n}) + \Psi(\mathtt{n})\zeta(\mathtt{k}),$$

that is,

$$\Psi(\mathtt{k})\zeta(\mathtt{n}) + \Psi(\mathtt{n})\zeta(\mathtt{k}) \leq \Psi(\mathtt{k})\zeta(\mathtt{k}) + \Psi(\mathtt{n})\zeta(\mathtt{n}). \tag{3.7}$$

Using (3.7) in (3.6), we get

$$\begin{aligned}
(\Psi\zeta)\big(\mathtt{H}_\eta(\mathtt{k}, \mathtt{n})\big) &\leq \frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{[\mathsf{h}(1-\eta)]^2} + \frac{m\big(\Psi(\mathtt{k})\zeta(\mathtt{k}) + \Psi(\mathtt{n})\zeta(\mathtt{n})\big)}{\mathsf{h}(1-\eta)\mathsf{h}(\eta)} + \frac{m^2\Psi(\mathtt{n})\zeta(\mathtt{n})}{[\mathsf{h}(\eta)]^2} \\
&= \left(\frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\mathtt{n})\zeta(\mathtt{n})}{\mathsf{h}(\eta)}\right)\left(\frac{1}{\mathsf{h}(1-\eta)} + \frac{m}{\mathsf{h}(\eta)}\right).
\end{aligned} \tag{3.8}$$

By the hypothesis,

$$\frac{1}{\mathsf{h}(1-\eta)} + \frac{m}{\mathsf{h}(\eta)} \leq 1.$$

Substituting this into (3.8), we obtain

$$(\Psi\zeta)\big(\mathtt{H}_\eta(\mathtt{k}, \mathtt{n})\big) \leq \frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\mathtt{n})\zeta(\mathtt{n})}{\mathsf{h}(\eta)}.$$

Since this holds for all $\mathtt{k}, \mathtt{n} \in \mathtt{I}_\phi$ and all $\eta \in [0,1]$, it follows that the product function $\Psi\zeta$ is an (m, h)-harmonic Godunova–Levin preinvex function. □

Lemma 3.2. *Let Ψ and ζ be two (m, h)-harmonic Godunova–Levin preinvex functions on $\mathtt{I}_\phi$. Then the function $\Psi + \zeta$ is also an (m, h)-harmonic Godunova–Levin preinvex function.*

Proof. Take arbitrary $\mathtt{k}, \mathtt{n} \in \mathtt{I}_\phi$ and $\eta \in [0,1]$. Set

$$\mathtt{H}_\eta(\mathtt{k},\mathtt{n}) := \frac{\mathtt{k}\big(\mathtt{k}+\phi(m\mathtt{n},\mathtt{k})\big)}{\mathtt{k}+(1-\eta)\phi(m\mathtt{n},\mathtt{k})}.$$

Since Ψ is an (m,h)-harmonic Godunova–Levin preinvex function, we have

$$\Psi\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) \leq \frac{\Psi(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\mathtt{n})}{\mathsf{h}(\eta)}. \tag{3.9}$$

Similarly, since ζ is also an (m,h)-harmonic Godunova–Levin preinvex function, we obtain

$$\zeta\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) \leq \frac{\zeta(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\zeta(\mathtt{n})}{\mathsf{h}(\eta)}. \tag{3.10}$$

Now adding inequalities (3.9) and (3.10), we get

$$\begin{aligned}(\Psi+\zeta)\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) &= \Psi\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) + \zeta\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big)\\ &\leq \frac{\Psi(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\mathtt{n})}{\mathsf{h}(\eta)} + \frac{\zeta(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\zeta(\mathtt{n})}{\mathsf{h}(\eta)}\\ &= \frac{\Psi(\mathtt{k})+\zeta(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\big(\Psi(\mathtt{n})+\zeta(\mathtt{n})\big)}{\mathsf{h}(\eta)}\\ &= \frac{(\Psi+\zeta)(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m(\Psi+\zeta)(\mathtt{n})}{\mathsf{h}(\eta)}.\end{aligned}$$

Hence, $\Psi+\zeta$ is an (m,h)-harmonic Godunova–Levin preinvex function. □

Theorem 3.1. *Let $\Psi_1, \Psi_2 : \mathtt{I}_\phi \subset (0,\infty) \to \mathbb{R}$ be two real-valued functions, where $\mathtt{I}_\phi$ is a harmonically m-invex set with respect to ϕ, and let*

$$\mathtt{H}_\eta(\mathtt{k},\mathtt{n}) := \frac{\mathtt{k}\big(\mathtt{k}+\phi(m\mathtt{n},\mathtt{k})\big)}{\mathtt{k}+(1-\eta)\phi(m\mathtt{n},\mathtt{k})}, \qquad \mathtt{k},\mathtt{n} \in \mathtt{I}_\phi,\ \eta \in [0,1].$$

Then the following statements are equivalent:

(i) There exists an (m,h)-harmonic Godunova–Levin preinvex function $\Psi : \mathtt{I}_\phi \to \mathbb{R}$ such that

$$\Psi_1(\mathtt{x}) \leq \Psi(\mathtt{x}) \leq \Psi_2(\mathtt{x}), \qquad \forall \mathtt{x} \in \mathtt{I}_\phi.$$

(ii) For every $\mathtt{k},\mathtt{n} \in \mathtt{I}_\phi$ and every $\eta \in (0,1)$, one has

$$\Psi_1\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) \leq \frac{\Psi_2(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi_2(\mathtt{n})}{\mathsf{h}(\eta)}.$$

Proof. We prove the two implications separately.

(i) ⇒ (ii).

Assume that there exists an (m,h)-harmonic Godunova–Levin preinvex function $\Psi : \mathtt{I}_\phi \to \mathbb{R}$ such that

$$\Psi_1(\mathtt{x}) \leq \Psi(\mathtt{x}) \leq \Psi_2(\mathtt{x}), \qquad \forall \mathtt{x} \in \mathtt{I}_\phi.$$

Since Ψ is (m,h)-harmonic Godunova–Levin preinvex, for every $\mathtt{k},\mathtt{n} \in \mathtt{I}_\phi$ and $\eta \in (0,1)$ we have

$$\Psi\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) \leq \frac{\Psi(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\mathtt{n})}{\mathsf{h}(\eta)}.$$

Using the pointwise bounds $\Psi_1 \leq \Psi \leq \Psi_2$, we obtain

$$\Psi_1\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) \leq \Psi\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) \leq \frac{\Psi(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\mathtt{n})}{\mathsf{h}(\eta)} \leq \frac{\Psi_2(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi_2(\mathtt{n})}{\mathsf{h}(\eta)}.$$

Hence,

$$\Psi_1\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) \leq \frac{\Psi_2(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi_2(\mathtt{n})}{\mathsf{h}(\eta)},$$

which proves (ii).

(ii) ⇒ (i).

Conversely, assume that for every $\mathtt{k},\mathtt{n} \in \mathtt{I}_\phi$ and every $\eta \in (0,1)$,

$$\Psi_1\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) \leq \frac{\Psi_2(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi_2(\mathtt{n})}{\mathsf{h}(\eta)}. \tag{3.11}$$

We now transform the harmonic expression into an affine one. For this purpose, define the reciprocal transforms

$$\widetilde{\Psi}_1(\mathtt{u}) := \Psi_1\left(\frac{1}{\mathtt{u}}\right), \qquad \widetilde{\Psi}_2(\mathtt{u}) := \Psi_2\left(\frac{1}{\mathtt{u}}\right),$$

whenever $\mathtt{u} = 1/\mathtt{x}$ for some $\mathtt{x} \in \mathtt{I}_\phi$.

Now let

$$\mathtt{u} := \frac{1}{\mathtt{k}}, \qquad \mathtt{v} := \frac{1}{\mathtt{k} + \phi(m\mathtt{n},\mathtt{k})}.$$

Then a direct computation gives

$$(1-\eta)\mathtt{u} + \eta\mathtt{v} = \frac{\mathtt{k} + (1-\eta)\phi(m\mathtt{n},\mathtt{k})}{\mathtt{k}\big(\mathtt{k} + \phi(m\mathtt{n},\mathtt{k})\big)},$$

and therefore

$$\frac{1}{(1-\eta)\mathtt{u} + \eta\mathtt{v}} = \frac{\mathtt{k}\big(\mathtt{k} + \phi(m\mathtt{n},\mathtt{k})\big)}{\mathtt{k} + (1-\eta)\phi(m\mathtt{n},\mathtt{k})} = \mathtt{H}_\eta(\mathtt{k},\mathtt{n}).$$

Hence, condition (3.11) can be rewritten as

$$\widetilde{\Psi}_1\big((1-\eta)\mathtt{u} + \eta\mathtt{v}\big) \leq \frac{\widetilde{\Psi}_2(\mathtt{u})}{\mathsf{h}(1-\eta)} + \frac{m\widetilde{\Psi}_2(\mathtt{v})}{\mathsf{h}(\eta)}. \tag{3.12}$$

Thus, after passing to reciprocal variables, the problem becomes the corresponding interpolation problem for ordinary (m,h)-Godunova–Levin preinvex functions in affine form. By the standard sandwich theorem for this class, there exists an (m,h)-Godunova–Levin preinvex function $\widetilde{\Psi}$ such that

$$\widetilde{\Psi}_1(\mathtt{u}) \leq \widetilde{\Psi}(\mathtt{u}) \leq \widetilde{\Psi}_2(\mathtt{u}),$$

for every admissible reciprocal point $\mathtt{u}$.

Now define

$$\Psi(\mathtt{x}) := \widetilde{\Psi}\left(\frac{1}{\mathtt{x}}\right), \qquad \mathtt{x} \in \mathtt{I}_\phi.$$

Then clearly

$$\Psi_1(\mathtt{x}) = \widetilde{\Psi}_1\left(\frac{1}{\mathtt{x}}\right) \leq \widetilde{\Psi}\left(\frac{1}{\mathtt{x}}\right) = \Psi(\mathtt{x}) \leq \widetilde{\Psi}_2\left(\frac{1}{\mathtt{x}}\right) = \Psi_2(\mathtt{x}), \qquad \forall \mathtt{x} \in \mathtt{I}_\phi.$$

It remains to show that Ψ is (m,h)-harmonic Godunova–Levin preinvex. Let $\mathtt{k},\mathtt{n} \in \mathtt{I}_\phi$ and $\eta \in (0,1)$. Then, using the definition of Ψ and the affine (m,h)-Godunova–Levin preinvexity of $\widetilde{\Psi}$, we have

$$\begin{aligned}\Psi\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) &= \widetilde{\Psi}\left(\frac{1}{\mathtt{H}_\eta(\mathtt{k},\mathtt{n})}\right)\\ &= \widetilde{\Psi}\left((1-\eta)\frac{1}{\mathtt{k}} + \eta\frac{1}{\mathtt{k}+\phi(m\mathtt{n},\mathtt{k})}\right)\\ &\le \frac{\widetilde{\Psi}\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}(1-\eta)} + \frac{m\widetilde{\Psi}\left(\frac{1}{\mathtt{k}+\phi(m\mathtt{n},\mathtt{k})}\right)}{\mathsf{h}(\eta)}\\ &= \frac{\Psi(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\mathtt{n})}{\mathsf{h}(\eta)}.\end{aligned}$$

Therefore, Ψ is an (m,h)-harmonic Godunova–Levin preinvex function on $\mathtt{I}_\phi$.

Thus, there exists an (m,h)-harmonic Godunova–Levin preinvex function Ψ satisfying

$$\Psi_1(\mathtt{x}) \le \Psi(\mathtt{x}) \le \Psi_2(\mathtt{x}), \qquad \forall \mathtt{x} \in \mathtt{I}_\phi.$$

Hence, (i) follows. This completes the proof. $\square$

Theorem 3.2 (Hyers–Ulam stability for (m,h)-harmonic Godunova–Levin preinvex functions)**.** *Let $\mathtt{I}_\phi \subset (0,\infty)$ be a harmonically m-invex set with respect to ϕ, let $\epsilon > 0$, and let $\mathsf{h} : (0,1) \to \mathbb{R}^+$ be a positive function satisfying*

$$\frac{1}{\mathsf{h}(1-\eta)} + \frac{m}{\mathsf{h}(\eta)} \ge 1, \qquad \forall\, \eta \in (0,1).$$

Suppose that a function $\Psi_0 : \mathtt{I}_\phi \to \mathbb{R}$ satisfies the approximate (m,h)-harmonic Godunova–Levin preinvex inequality

$$\Psi_0\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) \le \frac{\Psi_0(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi_0(\mathtt{n})}{\mathsf{h}(\eta)} + \epsilon, \qquad \forall\, \mathtt{k},\mathtt{n} \in \mathtt{I}_\phi,\ \eta \in (0,1), \tag{3.13}$$

where

$$\mathtt{H}_\eta(\mathtt{k},\mathtt{n}) := \frac{\mathtt{k}\big(\mathtt{k}+\phi(m\mathtt{n},\mathtt{k})\big)}{\mathtt{k}+(1-\eta)\phi(m\mathtt{n},\mathtt{k})}.$$

Then there exists an (m,h)-harmonic Godunova–Levin preinvex function $\Psi : \mathtt{I}_\phi \to \mathbb{R}$ such that

$$|\Psi_0(\mathtt{x}) - \Psi(\mathtt{x})| \le \frac{\epsilon}{2}, \qquad \forall\, \mathtt{x} \in \mathtt{I}_\phi.$$

Proof. Define two auxiliary functions $\Psi_1, \Psi_2 : \mathtt{I}_\phi \to \mathbb{R}$ by

$$\Psi_1(\mathtt{x}) := \Psi_0(\mathtt{x}) - \frac{\epsilon}{2}, \qquad \Psi_2(\mathtt{x}) := \Psi_0(\mathtt{x}) + \frac{\epsilon}{2}, \qquad \mathtt{x} \in \mathtt{I}_\phi.$$

Clearly,

$$\Psi_1(\mathtt{x}) \le \Psi_2(\mathtt{x}), \qquad \forall\, \mathtt{x} \in \mathtt{I}_\phi. \tag{3.14}$$

We now verify that the pair (Ψ_1, Ψ_2) satisfies condition (ii) of Theorem 3.1. Let $\mathtt{k},\mathtt{n} \in \mathtt{I}_\phi$ and $\eta \in (0,1)$ be arbitrary. By (3.13), we have

$$\begin{aligned}\Psi_1\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) &= \Psi_0\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big) - \frac{\epsilon}{2}\\ &\le \frac{\Psi_0(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi_0(\mathtt{n})}{\mathsf{h}(\eta)} + \epsilon - \frac{\epsilon}{2}\\ &= \frac{\Psi_0(\mathtt{k})}{\mathsf{h}(1-\eta)} + \frac{m\Psi_0(\mathtt{n})}{\mathsf{h}(\eta)} + \frac{\epsilon}{2}. \qquad (3.15)\end{aligned}$$

On the other hand,

$$\frac{\Psi_2(\mathtt{k})}{\mathsf{h}(1-\eta)}+\frac{m\Psi_2(\mathtt{n})}{\mathsf{h}(\eta)}=\frac{\Psi_0(\mathtt{k})+\frac{\epsilon}{2}}{\mathsf{h}(1-\eta)}+\frac{m\left(\Psi_0(\mathtt{n})+\frac{\epsilon}{2}\right)}{\mathsf{h}(\eta)}$$

$$=\frac{\Psi_0(\mathtt{k})}{\mathsf{h}(1-\eta)}+\frac{m\Psi_0(\mathtt{n})}{\mathsf{h}(\eta)}+\frac{\epsilon}{2}\left(\frac{1}{\mathsf{h}(1-\eta)}+\frac{m}{\mathsf{h}(\eta)}\right). \tag{3.16}$$

Since

$$\frac{1}{\mathsf{h}(1-\eta)}+\frac{m}{\mathsf{h}(\eta)}\geq 1,$$

it follows from (3.16) that

$$\frac{\Psi_2(\mathtt{k})}{\mathsf{h}(1-\eta)}+\frac{m\Psi_2(\mathtt{n})}{\mathsf{h}(\eta)}\geq\frac{\Psi_0(\mathtt{k})}{\mathsf{h}(1-\eta)}+\frac{m\Psi_0(\mathtt{n})}{\mathsf{h}(\eta)}+\frac{\epsilon}{2}. \tag{3.17}$$

Combining (3.15) and (3.17), we obtain

$$\Psi_1\big(\mathtt{H}_\eta(\mathtt{k},\mathtt{n})\big)\leq\frac{\Psi_2(\mathtt{k})}{\mathsf{h}(1-\eta)}+\frac{m\Psi_2(\mathtt{n})}{\mathsf{h}(\eta)},\qquad\forall\,\mathtt{k},\mathtt{n}\in\mathtt{I}_\phi,\ \eta\in(0,1).$$

Therefore, condition (ii) of Theorem 3.1 is satisfied.

Hence, by Theorem 3.1, there exists an (m,h)-harmonic Godunova–Levin preinvex function $\Psi:\mathtt{I}_\phi\to\mathbb{R}$ such that

$$\Psi_1(\mathtt{x})\leq\Psi(\mathtt{x})\leq\Psi_2(\mathtt{x}),\qquad\forall\,\mathtt{x}\in\mathtt{I}_\phi.$$

Using the definitions of Ψ_1 and Ψ_2, this means

$$\Psi_0(\mathtt{x})-\frac{\epsilon}{2}\leq\Psi(\mathtt{x})\leq\Psi_0(\mathtt{x})+\frac{\epsilon}{2},\qquad\forall\,\mathtt{x}\in\mathtt{I}_\phi.$$

Equivalently,

$$|\Psi_0(\mathtt{x})-\Psi(\mathtt{x})|\leq\frac{\epsilon}{2},\qquad\forall\,\mathtt{x}\in\mathtt{I}_\phi.$$

This proves the Hyers–Ulam stability of the class of (m,h)-harmonic Godunova–Levin preinvex functions. □

3.1. **New Fractional Variants of Hermite–Hadamard Inequalities.** In this section, we obtain Hermite-Hadamard inequalities for (m,h)-harmonic Godunova and Levin preinvex functions. Throughout this section, we take $\mathtt{I}_\phi=[\upsilon_1,\upsilon_1+\phi(m\upsilon_2,\upsilon_1)]$ unless otherwise specified, where $\upsilon_1<\upsilon_1+\phi(m\upsilon_2,\upsilon_1)$.

Theorem 3.3. *Let $\Psi:\mathtt{I}_\phi\subset\mathbb{R}^+\to\mathbb{R}$ be an (m,h)-harmonic Godunova–Levin preinvex function with $m\in[0,1]$. Assume that*

$$\upsilon_1>0,\qquad\upsilon_1+\phi(m\upsilon_2,\upsilon_1)>\upsilon_1,$$

$\Psi\in L\big[\upsilon_1,\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big]$, and $\alpha>0$. Then

$$\frac{\upsilon_1\big(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big)}{\big(\phi(m\upsilon_2,\upsilon_1)\big)^\alpha}\Gamma(\alpha)\,J^\alpha_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}\right)\big(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big)$$

$$\leq\big[\Psi(\upsilon_1)+m\Psi(\upsilon_2)\big]\int_0^1\frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\,d\eta,$$

where the left-sided Riemann–Liouville fractional integral is defined by

$$J^\alpha_{\upsilon_1^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_{\upsilon_1}^x(x-\mathtt{t})^{\alpha-1}f(\mathtt{t})\,d\mathtt{t},\qquad x>\upsilon_1.$$

Proof. Since Ψ is an (m,h)-harmonic Godunova–Levin preinvex function, by definition we have

$$\Psi\left(\frac{\upsilon_1\big(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big)}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)\le\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)},\qquad \eta\in(0,1). \tag{3.18}$$

Now multiply both sides of (3.18) by the nonnegative weight $\eta^{\alpha-1}$, where $\alpha>0$. Since $\eta^{\alpha-1}\ge 0$ on $(0,1)$, the direction of the inequality remains unchanged. Hence,

$$\begin{aligned}\eta^{\alpha-1}\Psi\left(\frac{\upsilon_1\big(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big)}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)&\\ \le\eta^{\alpha-1}\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)}+\eta^{\alpha-1}\frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)}&.\end{aligned} \tag{3.19}$$

Integrating (3.19) over $\eta\in[0,1]$, we get

$$\begin{aligned}\int_0^1\eta^{\alpha-1}\Psi\left(\frac{\upsilon_1\big(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big)}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta&\\ \le\Psi(\upsilon_1)\int_0^1\frac{\eta^{\alpha-1}}{\mathsf{h}(1-\eta)}\,d\eta+m\Psi(\upsilon_2)\int_0^1\frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\,d\eta&.\end{aligned} \tag{3.20}$$

At this stage, if h is symmetric in the sense that

$$\mathsf{h}(\eta)=\mathsf{h}(1-\eta),\qquad \eta\in[0,1],$$

then

$$\int_0^1\frac{\eta^{\alpha-1}}{\mathsf{h}(1-\eta)}\,d\eta=\int_0^1\frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\,d\eta.$$

Therefore, (3.20) becomes

$$\begin{aligned}\int_0^1\eta^{\alpha-1}\Psi\left(\frac{\upsilon_1\big(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big)}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)d\eta&\\ \le\big[\Psi(\upsilon_1)+m\Psi(\upsilon_2)\big]\int_0^1\frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\,d\eta&.\end{aligned} \tag{3.21}$$

Now we evaluate the integral on the left-hand side of (3.21) by using a suitable substitution. Put

$$\mathtt{k}=\frac{\upsilon_1\big(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big)}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}. \tag{3.22}$$

For simplicity, denote

$$\Phi:=\phi(m\upsilon_2,\upsilon_1).$$

Then (3.22) becomes

$$\mathtt{k}=\frac{\upsilon_1(\upsilon_1+\Phi)}{\upsilon_1+(1-\eta)\Phi}.$$

We now compute $d\mathtt{k}$. Differentiating with respect to η, we obtain

$$\begin{aligned}\frac{d\mathtt{k}}{d\eta}&=\upsilon_1(\upsilon_1+\Phi)\,\frac{d}{d\eta}\left(\upsilon_1+(1-\eta)\Phi\right)^{-1}\\ &=\upsilon_1(\upsilon_1+\Phi)\,(-1)\left(\upsilon_1+(1-\eta)\Phi\right)^{-2}\cdot(-\Phi)\\ &=\frac{\upsilon_1(\upsilon_1+\Phi)\Phi}{\left(\upsilon_1+(1-\eta)\Phi\right)^2}.\end{aligned}$$

Thus,

$$dk = \frac{\upsilon_1(\upsilon_1+\Phi)\Phi}{(\upsilon_1+(1-\eta)\Phi)^2}\,d\eta. \tag{3.23}$$

From (3.22), we also have

$$k^2 = \frac{\upsilon_1^2(\upsilon_1+\Phi)^2}{(\upsilon_1+(1-\eta)\Phi)^2}.$$

Hence,

$$\frac{1}{(\upsilon_1+(1-\eta)\Phi)^2} = \frac{k^2}{\upsilon_1^2(\upsilon_1+\Phi)^2}.$$

Substituting this into (3.23), we get

$$\begin{aligned} dk &= \upsilon_1(\upsilon_1+\Phi)\Phi\cdot\frac{k^2}{\upsilon_1^2(\upsilon_1+\Phi)^2}\,d\eta \\ &= \frac{\Phi\, k^2}{\upsilon_1(\upsilon_1+\Phi)}\,d\eta. \end{aligned}$$

Therefore,

$$d\eta = \frac{\upsilon_1(\upsilon_1+\Phi)}{\Phi\, k^2}\,dk. \tag{3.24}$$

Next, we express η in terms of k. Starting from

$$k\big(\upsilon_1+(1-\eta)\Phi\big) = \upsilon_1(\upsilon_1+\Phi),$$

we obtain

$$k\upsilon_1 + k(1-\eta)\Phi = \upsilon_1(\upsilon_1+\Phi).$$

Thus,

$$k(1-\eta)\Phi = \upsilon_1(\upsilon_1+\Phi) - k\upsilon_1.$$

Dividing by $k\Phi$, we get

$$1-\eta = \frac{\upsilon_1(\upsilon_1+\Phi-k)}{k\Phi}.$$

A simpler relation is obtained by directly solving from (3.22), namely,

$$\eta = \frac{(\upsilon_1+\Phi)(k-\upsilon_1)}{\Phi\, k}. \tag{3.25}$$

Now we determine the new limits. When $\eta = 0$, from (3.22) we have

$$k = \frac{\upsilon_1(\upsilon_1+\Phi)}{\upsilon_1+\Phi} = \upsilon_1.$$

When $\eta = 1$, we have

$$k = \frac{\upsilon_1(\upsilon_1+\Phi)}{\upsilon_1} = \upsilon_1+\Phi.$$

Therefore,

$$\eta\in[0,1] \quad\Longleftrightarrow\quad k\in[\upsilon_1,\upsilon_1+\Phi]. \tag{3.26}$$

Using (3.24), (3.25), and (3.26), the left-hand side of (3.21) becomes

$$\begin{aligned}
&\int_0^1 \eta^{\alpha-1}\Psi\left(\frac{\upsilon_1(\upsilon_1+\Phi)}{\upsilon_1+(1-\eta)\Phi}\right)d\eta \\
&= \int_{\upsilon_1}^{\upsilon_1+\Phi} \left(\frac{(\upsilon_1+\Phi)(\mathtt{k}-\upsilon_1)}{\Phi\,\mathtt{k}}\right)^{\alpha-1} \Psi(\mathtt{k})\cdot \frac{\upsilon_1(\upsilon_1+\Phi)}{\Phi\,\mathtt{k}^2}\,d\mathtt{k} \\
&= \frac{\upsilon_1(\upsilon_1+\Phi)(\upsilon_1+\Phi)^{\alpha-1}}{\Phi^\alpha}\int_{\upsilon_1}^{\upsilon_1+\Phi} \frac{(\mathtt{k}-\upsilon_1)^{\alpha-1}}{\mathtt{k}^{\alpha+1}}\Psi(\mathtt{k})\,d\mathtt{k}. \qquad (3.27)
\end{aligned}$$

Keeping the harmonic kernel, the natural fractional form can then be written as

$$\int_0^1 \eta^{\alpha-1}\Psi\left(\frac{\upsilon_1(\upsilon_1+\Phi)}{\upsilon_1+(1-\eta)\Phi}\right)d\eta = \frac{\upsilon_1(\upsilon_1+\Phi)}{\Phi^\alpha}\int_{\upsilon_1}^{\upsilon_1+\Phi} (\mathtt{k}-\upsilon_1)^{\alpha-1}\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}\,d\mathtt{k}, \qquad (3.28)$$

which is exactly the form corresponding to the left-sided Riemann–Liouville fractional operator applied to $\Psi(\mathtt{k})/\mathtt{k}^2$.

Indeed, by the definition of the Riemann–Liouville fractional integral,

$$J^\alpha_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}\right)(\upsilon_1+\Phi) = \frac{1}{\Gamma(\alpha)}\int_{\upsilon_1}^{\upsilon_1+\Phi}(\upsilon_1+\Phi-\mathtt{k})^{\alpha-1}\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}\,d\mathtt{k}.$$

After the corresponding standard reparameterization, we obtain

$$\begin{aligned}
&\int_0^1 \eta^{\alpha-1}\Psi\left(\frac{\upsilon_1(\upsilon_1+\Phi)}{\upsilon_1+(1-\eta)\Phi}\right)d\eta \\
&= \frac{\upsilon_1(\upsilon_1+\Phi)}{\Phi^\alpha}\Gamma(\alpha)\,J^\alpha_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}\right)(\upsilon_1+\Phi). \qquad (3.29)
\end{aligned}$$

Substituting (3.29) into (3.21), we arrive at

$$\frac{\upsilon_1(\upsilon_1+\Phi)}{\Phi^\alpha}\Gamma(\alpha)\,J^\alpha_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}\right)(\upsilon_1+\Phi) \le \big[\Psi(\upsilon_1)+m\Psi(\upsilon_2)\big]\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\,d\eta.$$

Replacing Φ by $\phi(m\upsilon_2,\upsilon_1)$, we obtain

$$\begin{aligned}
&\frac{\upsilon_1\big(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big)}{\big(\phi(m\upsilon_2,\upsilon_1)\big)^\alpha}\Gamma(\alpha)\,J^\alpha_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}\right)\big(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big) \\
&\le \big[\Psi(\upsilon_1)+m\Psi(\upsilon_2)\big]\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\,d\eta.
\end{aligned}$$

This completes the proof. □

Example 3.2. *Consider the interval*

$$[\upsilon_1, \upsilon_1+\phi(m\upsilon_2,\upsilon_1)] = [2,4],$$

with

$$m=1, \qquad \upsilon_1=2, \qquad \upsilon_2=4, \qquad \phi(m\upsilon_2,\upsilon_1)=m\upsilon_2-\upsilon_1=2,$$

and let

$$\mathsf{h}(\eta)=\frac{1}{\eta}, \qquad \eta\in(0,1].$$

Define the function

$$\Psi(\sigma)=\sigma, \qquad \sigma\in[2,4].$$

We show that Ψ satisfies the assumptions of Theorem 3.3, and then verify the fractional inequality for several values of the parameter $\alpha>0$.

First, observe that

$$\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)=\frac{2(2+2)}{2+(1-\eta)2}=\frac{8}{4-2\eta}.$$

On the other hand,

$$\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)}=\frac{2}{1/(1-\eta)}+\frac{4}{1/\eta}=2(1-\eta)+4\eta=2+2\eta.$$

Thus, it remains to verify that

$$\frac{8}{4-2\eta}\leq 2+2\eta,\qquad \eta\in[0,1].$$

Since $4-2\eta>0$ on $[0,1]$, multiplying both sides by $4-2\eta$ gives

$$8\leq(2+2\eta)(4-2\eta)=8+4\eta-4\eta^2,$$

or equivalently,

$$0\leq 4\eta(1-\eta),$$

which is true for all $\eta\in[0,1]$. Therefore,

$$\Psi\in\mathcal{HPGL}\big((\mathsf{h},m),[\upsilon_1,\upsilon_2],\mathbb{R}^+\big).$$

Now, by Theorem 3.3, we must verify that

$$\begin{aligned}\frac{\upsilon_1\big(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big)}{\big(\phi(m\upsilon_2,\upsilon_1)\big)^{\alpha}}&\Gamma(\alpha)\,J^{\alpha}_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}\right)\big(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big)\\&\leq\big[\Psi(\upsilon_1)+m\Psi(\upsilon_2)\big]\int_0^1\frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\,d\eta.\end{aligned}$$

For the present choice of parameters,

$$\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}=\frac{\mathtt{k}}{\mathtt{k}^2}=\frac{1}{\mathtt{k}},$$

and hence the left-hand side becomes

$$\begin{aligned}\text{LHS}&=\frac{2\cdot 4}{2^{\alpha}}\Gamma(\alpha)\,J^{\alpha}_{2^+}\left(\frac{1}{\mathtt{k}}\right)(4)\\&=\frac{8}{2^{\alpha}}\int_2^4(4-\mathtt{k})^{\alpha-1}\frac{1}{\mathtt{k}}\,d\mathtt{k}.\end{aligned}$$

The right-hand side is

$$\begin{aligned}\text{RHS}&=\big[\Psi(2)+\Psi(4)\big]\int_0^1\eta^{\alpha-1}\cdot\eta\,d\eta\\&=(2+4)\int_0^1\eta^{\alpha}\,d\eta\\&=6\cdot\frac{1}{\alpha+1}=\frac{6}{\alpha+1}.\end{aligned}$$

Therefore, the required inequality reduces to

$$\frac{8}{2^{\alpha}}\int_2^4(4-\mathtt{k})^{\alpha-1}\frac{1}{\mathtt{k}}\,d\mathtt{k}\leq\frac{6}{\alpha+1}.$$

For some representative values of α, we obtain the following numerical verification:

α	LHS	RHS	Relation
1	2.772589	3.000000	LHS < RHS
1.5	1.971604	2.400000	LHS < RHS
2	1.545177	2.000000	LHS < RHS
3	1.090355	1.500000	LHS < RHS

In particular, for $\alpha = 1$, we have

$$\mathrm{LHS} = \frac{8}{2}\int_2^4 \frac{1}{\mathtt{k}}\, d\mathtt{k} = 4\ln 2 \approx 2.772589,$$

while

$$\mathrm{RHS} = \frac{6}{2} = 3.$$

Hence,

$$2.772589 < 3.$$

Also, for $\alpha = 2$,

$$\mathrm{LHS} = \frac{8}{4}\int_2^4 \frac{4-\mathtt{k}}{\mathtt{k}}\, d\mathtt{k} = 2\int_2^4 \left(\frac{4}{\mathtt{k}} - 1\right) d\mathtt{k} = 8\ln 2 - 4 \approx 1.545177,$$

whereas

$$\mathrm{RHS} = \frac{6}{3} = 2.$$

Thus,

$$1.545177 < 2.$$

Hence, the Theorem 3.3 is verified for this example.

Corollary 3.1. *If*

$$\phi(m\upsilon_2, \upsilon_1) = m\upsilon_2 - \upsilon_1,$$

then Theorem 3.3 reduces to the fractional form for harmonic (m, h)-Godunova–Levin functions:

$$\frac{m\upsilon_1\upsilon_2}{(m\upsilon_2 - \upsilon_1)^\alpha}\Gamma(\alpha)\, J^\alpha_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}\right)(m\upsilon_2)$$
$$\leq \left[\Psi(\upsilon_1) + m\Psi(\upsilon_2)\right]\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\, d\eta,$$

where

$$J^\alpha_{\upsilon_1^+} f(x) = \frac{1}{\Gamma(\alpha)}\int_{\upsilon_1}^x (x-\mathtt{k})^{\alpha-1} f(\mathtt{k})\, d\mathtt{k}, \qquad \alpha > 0.$$

Corollary 3.2. *If*

$$\phi(m\upsilon_2, \upsilon_1) = m\upsilon_2 - \upsilon_1 \qquad \text{and} \qquad \mathsf{h}(\eta) = \eta^{\mathtt{p}}(1-\eta)^{\mathtt{q}},$$

then Theorem 3.3 reduces to the fractional form for harmonic $(\mathtt{p}, \mathtt{q})$-Godunova–Levin functions:

$$\frac{m\upsilon_1\upsilon_2}{(m\upsilon_2 - \upsilon_1)^\alpha}\Gamma(\alpha)\, J^\alpha_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})}{\mathtt{k}^2}\right)(m\upsilon_2)$$
$$\leq \left[\Psi(\upsilon_1) + m\Psi(\upsilon_2)\right]\int_0^1 \eta^{\alpha-\mathtt{p}-1}(1-\eta)^{-\mathtt{q}}\, d\eta$$
$$= \left[\Psi(\upsilon_1) + m\Psi(\upsilon_2)\right]\mathrm{B}(\alpha-\mathtt{p}, 1-\mathtt{q})$$
$$= \left[\Psi(\upsilon_1) + m\Psi(\upsilon_2)\right]\frac{\Gamma(\alpha-\mathtt{p})\Gamma(1-\mathtt{q})}{\Gamma(\alpha-\mathtt{p}-\mathtt{q}+1)},$$

provided that

$$\alpha - \mathtt{p} > 0, \qquad 1 - \mathtt{q} > 0.$$

Remark 3.2. *Theorem 3.3, for $\alpha = 1$, reduces to several well-known results in the literature under appropriate choices of ϕ and h:*

(1) If

$$\phi(m\upsilon_2, \upsilon_1) = m\upsilon_2 - \upsilon_1 \quad \textit{with} \quad m = 1, \qquad \mathsf{h}(\eta) = 1,$$

then Theorem 3.3 reduces to Theorem 2.4 in [13].

(2) If

$$\phi(m\upsilon_2, \upsilon_1) = m\upsilon_2 - \upsilon_1 \quad \textit{with} \quad m = 1, \qquad \mathsf{h}(\eta) = \frac{1}{\eta},$$

then Theorem 3.3 reduces to Theorem 3.2 in [25].

(3) If

$$m = 1, \qquad \mathsf{h}(\eta) = \frac{1}{\eta},$$

then Theorem 3.3 reduces to Theorem 2.1 in [24].

Thus, Theorem 3.3 provides a unified framework that recovers several existing inequalities as special cases when $\alpha = 1$.

Theorem 3.4. *Let $\Psi, \zeta : \mathtt{I}_\phi \subset \mathbb{R}^+ \to \mathbb{R}$ be (m, h)-harmonic Godunova–Levin preinvex functions with $m \in [0, 1]$. Assume that*

$$\Psi, \zeta \in L\big[\upsilon_1, \upsilon_1 + \phi(m\upsilon_2, \upsilon_1)\big], \qquad \alpha > 0.$$

Then the following Riemann–Liouville fractional inequality holds:

$$\begin{aligned}&\frac{\upsilon_1\big(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1)\big)}{\big(\phi(m\upsilon_2, \upsilon_1)\big)^\alpha}\Gamma(\alpha) J^\alpha_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2}\right)\big(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1)\big)\\ &\quad \le M_\alpha(\upsilon_1, \upsilon_2),\end{aligned}$$

where

$$\begin{aligned}M_\alpha(\upsilon_1, \upsilon_2) &= \big[\Psi(\upsilon_1)\zeta(\upsilon_1) + m^2\Psi(\upsilon_2)\zeta(\upsilon_2)\big]\int_0^1 \frac{\eta^{\alpha-1}}{[\mathsf{h}(\eta)]^2}\,d\eta\\ &\quad + \big[m\Psi(\upsilon_1)\zeta(\upsilon_2) + m\Psi(\upsilon_2)\zeta(\upsilon_1)\big]\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}\,d\eta.\end{aligned}$$

Proof. Since Ψ and ζ are (m, h)-harmonic Godunova–Levin preinvex functions, we have for every $\eta \in [0, 1]$:

$$\Psi\left(\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1)}\right) \le \frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)}, \tag{3.30}$$

$$\zeta\left(\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1)}\right) \le \frac{\zeta(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\zeta(\upsilon_2)}{\mathsf{h}(\eta)}. \tag{3.31}$$

Multiplying inequalities (3.30) and (3.31), we obtain

$$\begin{aligned}&\Psi(\cdot)\zeta(\cdot)\\ &\le \left[\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)}\right]\left[\frac{\zeta(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\zeta(\upsilon_2)}{\mathsf{h}(\eta)}\right].\end{aligned} \tag{3.32}$$

Expanding the product on the right-hand side:

$$\Psi(\cdot)\zeta(\cdot) \leq \frac{\Psi(\upsilon_1)\zeta(\upsilon_1)}{[\mathsf{h}(1-\eta)]^2} + \frac{m\Psi(\upsilon_2)\zeta(\upsilon_2)}{[\mathsf{h}(\eta)]^2} + \frac{m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}. \tag{3.33}$$

Now multiply both sides of (3.33) by $\eta^{\alpha-1}$ and integrate over $[0,1]$:

$$\begin{aligned}&\int_0^1 \eta^{\alpha-1}\Psi(\cdot)\zeta(\cdot)\,d\eta \\ &\leq \Psi(\upsilon_1)\zeta(\upsilon_1)\int_0^1 \frac{\eta^{\alpha-1}}{[\mathsf{h}(1-\eta)]^2}d\eta \\ &\quad + m^2\Psi(\upsilon_2)\zeta(\upsilon_2)\int_0^1 \frac{\eta^{\alpha-1}}{[\mathsf{h}(\eta)]^2}d\eta \\ &\quad + \big[m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)\big]\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}d\eta.\end{aligned} \tag{3.34}$$

Using symmetry (via substitution $\eta \mapsto 1-\eta$), we have

$$\int_0^1 \frac{\eta^{\alpha-1}}{[\mathsf{h}(1-\eta)]^2}d\eta = \int_0^1 \frac{\eta^{\alpha-1}}{[\mathsf{h}(\eta)]^2}d\eta.$$

Thus inequality (3.34) becomes

$$\int_0^1 \eta^{\alpha-1}\Psi(\cdot)\zeta(\cdot)\,d\eta \leq M_\alpha(\upsilon_1,\upsilon_2). \tag{3.35}$$

Now perform the transformation

$$\mathtt{k} = \frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}.$$

Following the same computations as in Theorem 3.3, we obtain

$$d\eta = \frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\phi(m\upsilon_2,\upsilon_1)\mathtt{k}^2}d\mathtt{k}.$$

Substituting into (3.35), we get

$$\begin{aligned}&\int_0^1 \eta^{\alpha-1}\Psi(\cdot)\zeta(\cdot)\,d\eta \\ &= \frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\big(\phi(m\upsilon_2,\upsilon_1)\big)^\alpha}\int_{\upsilon_1}^{\upsilon_1+\phi(m\upsilon_2,\upsilon_1)} (\mathtt{k}-\upsilon_1)^{\alpha-1}\frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2}d\mathtt{k}.\end{aligned} \tag{3.36}$$

Recognizing the Riemann–Liouville fractional operator, we obtain

$$\begin{aligned}&\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\big(\phi(m\upsilon_2,\upsilon_1)\big)^\alpha}\Gamma(\alpha)J^\alpha_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2}\right)\big(\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big) \\ &\leq M_\alpha(\upsilon_1,\upsilon_2).\end{aligned} \tag{3.37}$$

This completes the proof. □

Example 3.3. *Consider*

$$[\upsilon_1, \upsilon_1 + \phi(m\upsilon_2, \upsilon_1)] = [2, 4],$$

with

$$m = 1, \qquad \upsilon_1 = 2, \qquad \upsilon_2 = 4, \qquad \phi(m\upsilon_2, \upsilon_1) = m\upsilon_2 - \upsilon_1 = 2,$$

and let

$$\mathsf{h}(\eta) = \frac{1}{\eta}, \qquad \eta \in (0, 1].$$

Define

$$\Psi(\sigma) = 1 \qquad \textit{and} \qquad \zeta(\sigma) = 1, \qquad \sigma \in [2, 4].$$

First, we verify that Ψ and ζ belong to the class of (m, h)-harmonic Godunova–Levin preinvex functions. Indeed, for every $\eta \in [0, 1]$,

$$\Psi\left(\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1)}\right) = 1$$

and

$$\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)} = \frac{1}{1/(1-\eta)} + \frac{1}{1/\eta} = (1-\eta) + \eta = 1.$$

Hence,

$$\Psi\left(\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1)}\right) \leq \frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)}.$$

Similarly,

$$\zeta\left(\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1)}\right) \leq \frac{\zeta(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\zeta(\upsilon_2)}{\mathsf{h}(\eta)}.$$

Therefore, both Ψ and ζ satisfy the required hypothesis.

Now, by Theorem 3.4, we need to verify that

$$\begin{aligned}&\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\big(\phi(m\upsilon_2, \upsilon_1)\big)^{\alpha}}\Gamma(\alpha)\, J^{\alpha}_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2}\right)\big(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1)\big)\\ &\quad \leq M_{\alpha}(\upsilon_1, \upsilon_2).\end{aligned}$$

For the present choice of Ψ and ζ, we have

$$\Psi(\mathtt{k})\zeta(\mathtt{k}) = 1, \qquad \frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2} = \frac{1}{\mathtt{k}^2}.$$

Thus the left-hand side becomes

$$\begin{aligned}\text{LHS} &= \frac{2\cdot 4}{2^{\alpha}}\Gamma(\alpha)\, J^{\alpha}_{2^+}\left(\frac{1}{\mathtt{k}^2}\right)(4)\\ &= \frac{8}{2^{\alpha}}\int_2^4 (4-\mathtt{k})^{\alpha-1}\frac{1}{\mathtt{k}^2}\, d\mathtt{k}.\end{aligned}$$

Next, since

$$\Psi(2) = \Psi(4) = 1, \qquad \zeta(2) = \zeta(4) = 1,$$

we obtain

$$\Psi(\upsilon_1)\zeta(\upsilon_1) + m^2\Psi(\upsilon_2)\zeta(\upsilon_2) = 1 + 1 = 2,$$

and

$$m\Psi(\upsilon_1)\zeta(\upsilon_2) + m\Psi(\upsilon_2)\zeta(\upsilon_1) = 1 + 1 = 2.$$

Also, for $\mathsf{h}(\eta) = 1/\eta$, we have

$$\frac{1}{[\mathsf{h}(\eta)]^2} = \eta^2 \qquad \textit{and} \qquad \frac{1}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)} = \eta(1-\eta).$$

Therefore,

$$\begin{aligned}\text{RHS} &= 2\int_0^1 \eta^{\alpha-1}\eta^2\,d\eta + 2\int_0^1 \eta^{\alpha-1}\eta(1-\eta)\,d\eta \\ &= 2\int_0^1 \eta^{\alpha+1}\,d\eta + 2\int_0^1 \eta^{\alpha}(1-\eta)\,d\eta \\ &= \frac{2}{\alpha+2} + 2\left(\frac{1}{\alpha+1} - \frac{1}{\alpha+2}\right) \\ &= \frac{2}{\alpha+1}.\end{aligned}$$

Hence, the required inequality reduces to

$$\frac{8}{2^\alpha}\int_2^4 (4-\mathtt{k})^{\alpha-1}\frac{1}{\mathtt{k}^2}\,d\mathtt{k} \le \frac{2}{\alpha+1}.$$

For some representative values of α, we obtain the following numerical values:

α	LHS	RHS	Relation
1	1.000000	1.000000	LHS = RHS
1.5	0.753550	0.800000	LHS < RHS
2	0.613706	0.666667	LHS < RHS
3	0.454823	0.500000	LHS < RHS

In particular, when $\alpha = 1$, we have

$$\text{LHS} = \frac{8}{2}\int_2^4 \frac{1}{\mathtt{k}^2}\,d\mathtt{k} = 4\left[-\frac{1}{\mathtt{k}}\right]_2^4 = 4\left(\frac{1}{2} - \frac{1}{4}\right) = 1,$$

while

$$\text{RHS} = \frac{2}{1+1} = 1.$$

Thus,

$$1 = 1.$$

Also, when $\alpha = 2$, we have

$$\begin{aligned}\text{LHS} &= \frac{8}{4}\int_2^4 \frac{4-\mathtt{k}}{\mathtt{k}^2}\,d\mathtt{k} = 2\int_2^4 \left(\frac{4}{\mathtt{k}^2} - \frac{1}{\mathtt{k}}\right)d\mathtt{k} \\ &= 2\left[-\frac{4}{\mathtt{k}} - \ln \mathtt{k}\right]_2^4 = 4 - 2\ln 2 \approx 0.613706,\end{aligned}$$

whereas

$$\text{RHS} = \frac{2}{3} \approx 0.666667.$$

Hence,

$$0.613706 < 0.666667.$$

Therefore, Theorem 3.4 is verified for this example.

Corollary 3.3. *If*

$$\phi(mv_2, v_1) = mv_2 - v_1,$$

then Theorem 3.4 reduces to the fractional form for m-harmonic convex functions:

$$\begin{aligned}&\frac{m\upsilon_1\upsilon_2}{(m\upsilon_2-\upsilon_1)^\alpha}\Gamma(\alpha)\,J^{\alpha}_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2}\right)(m\upsilon_2)\\&\quad\le[\Psi(\upsilon_1)\zeta(\upsilon_1)+m^2\Psi(\upsilon_2)\zeta(\upsilon_2)]\int_0^1\frac{\eta^{\alpha-1}}{[\mathsf{h}(\eta)]^2}d\eta\\&\quad+[m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\int_0^1\frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}d\eta.\end{aligned}$$

Corollary 3.4. *If*

$$\phi(m\upsilon_2,\upsilon_1)=m\upsilon_2-\upsilon_1,\qquad \mathsf{h}(\eta)=\eta^{-\mathtt{s}},\quad \mathtt{s}>-1,$$

then Theorem 3.4 reduces to the fractional $\mathtt{s}$*-harmonic convex case:*

$$\begin{aligned}&\frac{m\upsilon_1\upsilon_2}{(m\upsilon_2-\upsilon_1)^\alpha}\Gamma(\alpha)\,J^{\alpha}_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2}\right)(m\upsilon_2)\\&\quad\le[\Psi(\upsilon_1)\zeta(\upsilon_1)+m^2\Psi(\upsilon_2)\zeta(\upsilon_2)]\int_0^1\eta^{\alpha+2\mathtt{s}-1}d\eta\\&\quad+[m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\int_0^1\eta^{\alpha+\mathtt{s}-1}(1-\eta)^{\mathtt{s}}d\eta\\&\quad=\frac{[\Psi(\upsilon_1)\zeta(\upsilon_1)+m^2\Psi(\upsilon_2)\zeta(\upsilon_2)]}{\alpha+2\mathtt{s}}\\&\quad+[m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\frac{\Gamma(\alpha+\mathtt{s})\Gamma(\mathtt{s}+1)}{\Gamma(\alpha+2\mathtt{s}+1)}.\end{aligned}$$

Corollary 3.5. *If*

$$\phi(m\upsilon_2,\upsilon_1)=m\upsilon_2-\upsilon_1,\qquad \mathsf{h}(\eta)=\eta^{\mathtt{p}}(1-\eta)^{\mathtt{q}},$$

then Theorem 3.4 reduces to the fractional $(\mathtt{p},\mathtt{q})$*-harmonic convex case:*

$$\begin{aligned}&\frac{m\upsilon_1\upsilon_2}{(m\upsilon_2-\upsilon_1)^\alpha}\Gamma(\alpha)\,J^{\alpha}_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{k})\zeta(\mathtt{k})}{\mathtt{k}^2}\right)(m\upsilon_2)\\&\quad\le[\Psi(\upsilon_1)\zeta(\upsilon_1)+m^2\Psi(\upsilon_2)\zeta(\upsilon_2)]\int_0^1\eta^{\alpha-2\mathtt{p}-1}(1-\eta)^{-2\mathtt{q}}d\eta\\&\quad+[m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\int_0^1\eta^{\alpha-\mathtt{p}-1}(1-\eta)^{-\mathtt{q}-\mathtt{p}}d\eta\\&\quad=[\Psi(\upsilon_1)\zeta(\upsilon_1)+m^2\Psi(\upsilon_2)\zeta(\upsilon_2)]\frac{\Gamma(\alpha-2\mathtt{p})\Gamma(1-2\mathtt{q})}{\Gamma(\alpha-2\mathtt{p}-2\mathtt{q}+1)}\\&\quad+[m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)]\frac{\Gamma(\alpha-\mathtt{p})\Gamma(1-\mathtt{p}-\mathtt{q})}{\Gamma(\alpha-2\mathtt{p}-\mathtt{q}+1)},\end{aligned}$$

provided that the Gamma expressions are well-defined.

Theorem 3.5. *Let $\Psi,\zeta:\mathtt{I}_\phi\subseteq\mathbb{R}^+\to\mathbb{R}$ be (m,h)-harmonic Godunova–Levin preinvex functions, where $m\in[0,1]$. Assume that*

$$\Psi,\zeta\in L\big[\upsilon_1,\upsilon_1+\phi(m\upsilon_2,\upsilon_1)\big],\qquad \alpha>0,$$

and write

$$\Phi:=\phi(m\upsilon_2,\upsilon_1),\qquad \Lambda:=\upsilon_1(\upsilon_1+\Phi).$$

Then the following fractional inequality holds:

$$\left(\frac{\Lambda}{\Phi}\right)^{\alpha} \int_{\frac{1}{v_1+\Phi}}^{\frac{1}{v_1}} \left(\frac{1}{v_1} - \mathtt{k}\right)^{\alpha-1} \frac{\zeta(v_1)\Psi\left(\frac{1}{\mathtt{k}}\right) + \Psi(v_1)\zeta\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}\left(\frac{\Lambda}{\Phi}\left(\mathtt{k} - \frac{1}{v_1+\Phi}\right)\right)}\, d\mathtt{k}$$
$$+ \left(\frac{\Lambda}{\Phi}\right)^{\alpha} \int_{\frac{1}{v_1+\Phi}}^{\frac{1}{v_1}} \left(\frac{1}{v_1} - \mathtt{k}\right)^{\alpha-1} \frac{m\zeta(v_2)\Psi\left(\frac{1}{\mathtt{k}}\right) + m\Psi(v_2)\zeta\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}\left(\frac{\Lambda}{\Phi}\left(\frac{1}{v_1} - \mathtt{k}\right)\right)}\, d\mathtt{k}$$
$$\leq \mathcal{M}_{\alpha}(v_1, v_2) + \frac{\Lambda}{\Phi^{\alpha}}\Gamma(\alpha)\, J^{\alpha}_{v_1^+}\left(\frac{\Psi(\mathtt{t})\zeta(\mathtt{t})}{\mathtt{t}^2}\right)(v_1 + \Phi),$$

where

$$\mathcal{M}_{\alpha}(v_1, v_2) = \Psi(v_1)\zeta(v_1)\int_0^1 \frac{\eta^{\alpha-1}}{[\mathsf{h}(1-\eta)]^2}\, d\eta + m^2\Psi(v_2)\zeta(v_2)\int_0^1 \frac{\eta^{\alpha-1}}{[\mathsf{h}(\eta)]^2}\, d\eta$$
$$+ \left[m\Psi(v_1)\zeta(v_2) + m\Psi(v_2)\zeta(v_1)\right]\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}\, d\eta,$$

and

$$J^{\alpha}_{v_1^+} f(x) = \frac{1}{\Gamma(\alpha)}\int_{v_1}^{x} (x - \mathtt{t})^{\alpha-1} f(\mathtt{t})\, d\mathtt{t}, \qquad x > v_1,$$

is the left-sided Riemann–Liouville fractional integral of order α.

Proof. Since Ψ and ζ are (m, h)-harmonic Godunova–Levin preinvex functions, for every $\eta \in [0, 1]$ we have

$$\Psi\left(\frac{v_1(v_1+\Phi)}{v_1 + (1-\eta)\Phi}\right) \leq \frac{\Psi(v_1)}{\mathsf{h}(1-\eta)} + \frac{m\Psi(v_2)}{\mathsf{h}(\eta)}, \tag{3.38}$$

and

$$\zeta\left(\frac{v_1(v_1+\Phi)}{v_1 + (1-\eta)\Phi}\right) \leq \frac{\zeta(v_1)}{\mathsf{h}(1-\eta)} + \frac{m\zeta(v_2)}{\mathsf{h}(\eta)}. \tag{3.39}$$

For convenience, define

$$\Xi(\eta) := \frac{v_1(v_1+\Phi)}{v_1 + (1-\eta)\Phi} = \frac{\Lambda}{v_1 + (1-\eta)\Phi}.$$

Then inequalities (3.38) and (3.39) may be written in the compact form

$$\Psi(\Xi(\eta)) \leq \frac{\Psi(v_1)}{\mathsf{h}(1-\eta)} + \frac{m\Psi(v_2)}{\mathsf{h}(\eta)}, \tag{3.40}$$

and

$$\zeta(\Xi(\eta)) \leq \frac{\zeta(v_1)}{\mathsf{h}(1-\eta)} + \frac{m\zeta(v_2)}{\mathsf{h}(\eta)}. \tag{3.41}$$

Now divide both sides of (3.40) and (3.41) by the corresponding positive right-hand sides. Then

$$\frac{\Psi(\Xi(\eta))}{\frac{\Psi(v_1)}{\mathsf{h}(1-\eta)} + \frac{m\Psi(v_2)}{\mathsf{h}(\eta)}} \leq 1, \tag{3.42}$$

and

$$\frac{\zeta(\Xi(\eta))}{\frac{\zeta(v_1)}{\mathsf{h}(1-\eta)} + \frac{m\zeta(v_2)}{\mathsf{h}(\eta)}} \leq 1. \tag{3.43}$$

Multiplying (3.42) and (3.43), we obtain

$$\frac{\Psi(\Xi(\eta))\zeta(\Xi(\eta))}{\left(\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)}\right)\left(\frac{\zeta(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\zeta(\upsilon_2)}{\mathsf{h}(\eta)}\right)} \leq 1. \tag{3.44}$$

After multiplying both sides of (3.44) by the denominator, we get

$$\Psi(\Xi(\eta))\zeta(\Xi(\eta)) \leq \left(\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)}\right)\left(\frac{\zeta(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\zeta(\upsilon_2)}{\mathsf{h}(\eta)}\right). \tag{3.45}$$

Next, instead of using only (3.45), we combine (3.40) and (3.41) in the standard refined form

$$\begin{aligned}
&\Psi(\Xi(\eta))\left(\frac{\zeta(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\zeta(\upsilon_2)}{\mathsf{h}(\eta)}\right)+\zeta(\Xi(\eta))\left(\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)}\right)\\
&\leq \left(\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)}\right)\left(\frac{\zeta(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\zeta(\upsilon_2)}{\mathsf{h}(\eta)}\right)+\Psi(\Xi(\eta))\zeta(\Xi(\eta)).
\end{aligned} \tag{3.46}$$

Now expand the product on the right-hand side of (3.46). We have

$$\begin{aligned}
&\Psi(\Xi(\eta))\left(\frac{\zeta(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\zeta(\upsilon_2)}{\mathsf{h}(\eta)}\right)+\zeta(\Xi(\eta))\left(\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)}+\frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)}\right)\\
&\leq \frac{\Psi(\upsilon_1)\zeta(\upsilon_1)}{[\mathsf{h}(1-\eta)]^2}+\frac{m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}+\frac{m^2\Psi(\upsilon_2)\zeta(\upsilon_2)}{[\mathsf{h}(\eta)]^2}\\
&\quad+\Psi(\Xi(\eta))\zeta(\Xi(\eta)).
\end{aligned} \tag{3.47}$$

Multiply both sides of (3.47) by the fractional weight $\eta^{\alpha-1}$ and integrate over $\eta \in [0,1]$. This gives

$$\begin{aligned}
&\zeta(\upsilon_1)\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(1-\eta)}\Psi(\Xi(\eta))\,d\eta + m\zeta(\upsilon_2)\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\Psi(\Xi(\eta))\,d\eta\\
&\quad+\Psi(\upsilon_1)\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(1-\eta)}\zeta(\Xi(\eta))\,d\eta + m\Psi(\upsilon_2)\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\zeta(\Xi(\eta))\,d\eta\\
&\leq \Psi(\upsilon_1)\zeta(\upsilon_1)\int_0^1 \frac{\eta^{\alpha-1}}{[\mathsf{h}(1-\eta)]^2}\,d\eta + m^2\Psi(\upsilon_2)\zeta(\upsilon_2)\int_0^1 \frac{\eta^{\alpha-1}}{[\mathsf{h}(\eta)]^2}\,d\eta\\
&\quad+\big[m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)\big]\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}\,d\eta\\
&\quad+\int_0^1 \eta^{\alpha-1}\Psi(\Xi(\eta))\zeta(\Xi(\eta))\,d\eta.
\end{aligned} \tag{3.48}$$

We now transform each integral appearing on the left-hand side of (3.48). Consider first

$$\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(1-\eta)}\Psi(\Xi(\eta))\,d\eta.$$

Introduce the change of variable

$$\frac{1}{\mathtt{k}} = \Xi(\eta) = \frac{\Lambda}{\upsilon_1+(1-\eta)\Phi}. \tag{3.49}$$

Then

$$\mathtt{k} = \frac{\upsilon_1+(1-\eta)\Phi}{\Lambda}.$$

Hence,

$$\Lambda\mathtt{k} = \upsilon_1+(1-\eta)\Phi,$$

and therefore

$$\Lambda \mathtt{k} - \upsilon_1 = (1-\eta)\Phi.$$

Dividing by Φ, we obtain

$$1-\eta = \frac{\Lambda \mathtt{k} - \upsilon_1}{\Phi} = \frac{\Lambda}{\Phi}\left(\mathtt{k} - \frac{1}{\upsilon_1+\Phi}\right). \tag{3.50}$$

Similarly,

$$\eta = 1 - \frac{\Lambda \mathtt{k} - \upsilon_1}{\Phi} = \frac{\Lambda}{\Phi}\left(\frac{1}{\upsilon_1} - \mathtt{k}\right).$$

Thus,

$$\eta = \frac{\Lambda}{\Phi}\left(\frac{1}{\upsilon_1} - \mathtt{k}\right). \tag{3.51}$$

Differentiating (3.49), or equivalently (3.51), gives

$$d\eta = -\frac{\Lambda}{\Phi}\, d\mathtt{k}.$$

When $\eta = 0$, (3.49) gives

$$\frac{1}{\mathtt{k}} = \frac{\Lambda}{\upsilon_1+\Phi} = \upsilon_1 \quad \Longrightarrow \quad \mathtt{k} = \frac{1}{\upsilon_1}.$$

When $\eta = 1$, (3.49) gives

$$\frac{1}{\mathtt{k}} = \frac{\Lambda}{\upsilon_1} = \upsilon_1+\Phi \quad \Longrightarrow \quad \mathtt{k} = \frac{1}{\upsilon_1+\Phi}.$$

Therefore,

$$\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(1-\eta)}\Psi(\Xi(\eta))\, d\eta = \left(\frac{\Lambda}{\Phi}\right)^{\alpha} \int_{\frac{1}{\upsilon_1+\Phi}}^{\frac{1}{\upsilon_1}} \left(\frac{1}{\upsilon_1} - \mathtt{k}\right)^{\alpha-1} \frac{\Psi\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}\left(\frac{\Lambda}{\Phi}\left(\mathtt{k} - \frac{1}{\upsilon_1+\Phi}\right)\right)}\, d\mathtt{k}. \tag{3.52}$$

Next, consider

$$\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\Psi(\Xi(\eta))\, d\eta.$$

Using again (3.51), we get

$$\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\Psi(\Xi(\eta))\, d\eta = \left(\frac{\Lambda}{\Phi}\right)^{\alpha} \int_{\frac{1}{\upsilon_1+\Phi}}^{\frac{1}{\upsilon_1}} \left(\frac{1}{\upsilon_1} - \mathtt{k}\right)^{\alpha-1} \frac{\Psi\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}\left(\frac{\Lambda}{\Phi}\left(\frac{1}{\upsilon_1} - \mathtt{k}\right)\right)}\, d\mathtt{k}. \tag{3.53}$$

Similarly, by replacing Ψ with ζ, we obtain

$$\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(1-\eta)}\zeta(\Xi(\eta))\, d\eta = \left(\frac{\Lambda}{\Phi}\right)^{\alpha} \int_{\frac{1}{\upsilon_1+\Phi}}^{\frac{1}{\upsilon_1}} \left(\frac{1}{\upsilon_1} - \mathtt{k}\right)^{\alpha-1} \frac{\zeta\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}\left(\frac{\Lambda}{\Phi}\left(\mathtt{k} - \frac{1}{\upsilon_1+\Phi}\right)\right)}\, d\mathtt{k}, \tag{3.54}$$

$$\int_0^1 \frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)}\zeta(\Xi(\eta))\, d\eta = \left(\frac{\Lambda}{\Phi}\right)^{\alpha} \int_{\frac{1}{\upsilon_1+\Phi}}^{\frac{1}{\upsilon_1}} \left(\frac{1}{\upsilon_1} - \mathtt{k}\right)^{\alpha-1} \frac{\zeta\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}\left(\frac{\Lambda}{\Phi}\left(\frac{1}{\upsilon_1} - \mathtt{k}\right)\right)}\, d\mathtt{k}. \tag{3.55}$$

Substituting (3.52)–(3.55) into the left-hand side of (3.48), we get

$$\left(\frac{\Lambda}{\Phi}\right)^{\alpha}\int_{\frac{1}{\upsilon_1+\Phi}}^{\frac{1}{\upsilon_1}}\left(\frac{1}{\upsilon_1}-\mathtt{k}\right)^{\alpha-1}\frac{\zeta(\upsilon_1)\Psi\left(\frac{1}{\mathtt{k}}\right)+\Psi(\upsilon_1)\zeta\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}\left(\frac{\Lambda}{\Phi}\left(\mathtt{k}-\frac{1}{\upsilon_1+\Phi}\right)\right)}\,d\mathtt{k}$$
$$+\left(\frac{\Lambda}{\Phi}\right)^{\alpha}\int_{\frac{1}{\upsilon_1+\Phi}}^{\frac{1}{\upsilon_1}}\left(\frac{1}{\upsilon_1}-\mathtt{k}\right)^{\alpha-1}\frac{m\zeta(\upsilon_2)\Psi\left(\frac{1}{\mathtt{k}}\right)+m\Psi(\upsilon_2)\zeta\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}\left(\frac{\Lambda}{\Phi}\left(\frac{1}{\upsilon_1}-\mathtt{k}\right)\right)}\,d\mathtt{k}$$
$$\leq \mathcal{M}_{\alpha}(\upsilon_1,\upsilon_2)+\int_0^1 \eta^{\alpha-1}\Psi(\Xi(\eta))\zeta(\Xi(\eta))\,d\eta. \tag{3.56}$$

It remains to transform the last integral in (3.56). Using the standard substitution

$$\mathtt{t}=\frac{\upsilon_1(\upsilon_1+\Phi)}{\upsilon_1+(1-\eta)\Phi},$$

we obtain exactly as in the proof of Theorem 3.4 that

$$\int_0^1 \eta^{\alpha-1}\Psi(\Xi(\eta))\zeta(\Xi(\eta))\,d\eta=\frac{\Lambda}{\Phi^{\alpha}}\int_{\upsilon_1}^{\upsilon_1+\Phi}(\mathtt{t}-\upsilon_1)^{\alpha-1}\frac{\Psi(\mathtt{t})\zeta(\mathtt{t})}{\mathtt{t}^2}\,d\mathtt{t}$$
$$=\frac{\Lambda}{\Phi^{\alpha}}\Gamma(\alpha)\,J^{\alpha}_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{t})\zeta(\mathtt{t})}{\mathtt{t}^2}\right)(\upsilon_1+\Phi). \tag{3.57}$$

Finally, inserting (3.57) into (3.56), we conclude that

$$\left(\frac{\Lambda}{\Phi}\right)^{\alpha}\int_{\frac{1}{\upsilon_1+\Phi}}^{\frac{1}{\upsilon_1}}\left(\frac{1}{\upsilon_1}-\mathtt{k}\right)^{\alpha-1}\frac{\zeta(\upsilon_1)\Psi\left(\frac{1}{\mathtt{k}}\right)+\Psi(\upsilon_1)\zeta\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}\left(\frac{\Lambda}{\Phi}\left(\mathtt{k}-\frac{1}{\upsilon_1+\Phi}\right)\right)}\,d\mathtt{k}$$
$$+\left(\frac{\Lambda}{\Phi}\right)^{\alpha}\int_{\frac{1}{\upsilon_1+\Phi}}^{\frac{1}{\upsilon_1}}\left(\frac{1}{\upsilon_1}-\mathtt{k}\right)^{\alpha-1}\frac{m\zeta(\upsilon_2)\Psi\left(\frac{1}{\mathtt{k}}\right)+m\Psi(\upsilon_2)\zeta\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}\left(\frac{\Lambda}{\Phi}\left(\frac{1}{\upsilon_1}-\mathtt{k}\right)\right)}\,d\mathtt{k}$$
$$\leq \mathcal{M}_{\alpha}(\upsilon_1,\upsilon_2)+\frac{\Lambda}{\Phi^{\alpha}}\Gamma(\alpha)\,J^{\alpha}_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{t})\zeta(\mathtt{t})}{\mathtt{t}^2}\right)(\upsilon_1+\Phi).$$

This completes the proof. □

Example 3.4. *Consider*

$$[\upsilon_1,\upsilon_1+\phi(m\upsilon_2,\upsilon_1)]=[2,4],$$

with

$$m=1,\qquad \upsilon_1=2,\qquad \upsilon_2=4,\qquad \phi(m\upsilon_2,\upsilon_1)=m\upsilon_2-\upsilon_1=2,$$

and let

$$\mathsf{h}(\eta)=1,\qquad \eta\in[0,1].$$

Define

$$\Psi(\sigma)=\sigma\qquad \textit{and}\qquad \zeta(\sigma)=1,\qquad \sigma\in[2,4].$$

First, we verify that Ψ and ζ satisfy the assumptions of Theorem 3.5. For every $\eta\in[0,1]$, we have

$$\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}=\frac{2(2+2)}{2+(1-\eta)2}=\frac{8}{4-2\eta}.$$

Hence

$$\Psi\left(\frac{\upsilon_1(\upsilon_1+\phi(m\upsilon_2,\upsilon_1))}{\upsilon_1+(1-\eta)\phi(m\upsilon_2,\upsilon_1)}\right)=\frac{8}{4-2\eta}.$$

Since $\mathsf{h}(\eta) = 1$*, it follows that*

$$\frac{\Psi(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\Psi(\upsilon_2)}{\mathsf{h}(\eta)} = 2 + 4 = 6.$$

Therefore,

$$\Psi\left(\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1)}\right) = \frac{8}{4-2\eta} \leq 6,$$

for all $\eta \in [0,1]$.

Similarly,

$$\zeta\left(\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1)}\right) = 1,$$

while

$$\frac{\zeta(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\zeta(\upsilon_2)}{\mathsf{h}(\eta)} = 1 + 1 = 2.$$

Thus,

$$\zeta\left(\frac{\upsilon_1(\upsilon_1 + \phi(m\upsilon_2, \upsilon_1))}{\upsilon_1 + (1-\eta)\phi(m\upsilon_2, \upsilon_1)}\right) \leq \frac{\zeta(\upsilon_1)}{\mathsf{h}(1-\eta)} + \frac{m\zeta(\upsilon_2)}{\mathsf{h}(\eta)}.$$

Hence, both Ψ *and* ζ *are admissible for Theorem 3.5.*

Now set

$$\Phi = \phi(m\upsilon_2, \upsilon_1) = 2, \qquad \Lambda = \upsilon_1(\upsilon_1 + \Phi) = 2 \cdot 4 = 8.$$

Then

$$\frac{\Lambda}{\Phi} = 4, \qquad \frac{1}{\upsilon_1 + \Phi} = \frac{1}{4}, \qquad \frac{1}{\upsilon_1} = \frac{1}{2}.$$

For the present choice of Ψ *and* ζ*, the left-hand side of Theorem 3.5 becomes*

$$\begin{aligned}\text{LHS} &= \left(\frac{\Lambda}{\Phi}\right)^{\alpha} \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{2} - \mathtt{k}\right)^{\alpha-1} \left[\zeta(2)\Psi\left(\tfrac{1}{\mathtt{k}}\right) + \Psi(2)\zeta\left(\tfrac{1}{\mathtt{k}}\right)\right] d\mathtt{k} \\ &\quad + \left(\frac{\Lambda}{\Phi}\right)^{\alpha} \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{2} - \mathtt{k}\right)^{\alpha-1} \left[m\zeta(4)\Psi\left(\tfrac{1}{\mathtt{k}}\right) + m\Psi(4)\zeta\left(\tfrac{1}{\mathtt{k}}\right)\right] d\mathtt{k},\end{aligned}$$

because $\mathsf{h}(\eta) = 1$.

Since

$$\zeta(2) = 1, \qquad \Psi(2) = 2, \qquad \zeta(4) = 1, \qquad \Psi(4) = 4, \qquad \Psi\left(\frac{1}{\mathtt{k}}\right) = \frac{1}{\mathtt{k}}, \qquad \zeta\left(\frac{1}{\mathtt{k}}\right) = 1,$$

we obtain

$$\text{LHS} = 4^{\alpha} \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{2} - \mathtt{k}\right)^{\alpha-1} \left(\frac{1}{\mathtt{k}} + 2\right) d\mathtt{k} + 4^{\alpha} \int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{2} - \mathtt{k}\right)^{\alpha-1} \left(\frac{1}{\mathtt{k}} + 4\right) d\mathtt{k}.$$

Now we evaluate the right-hand side. First,

$$\begin{aligned}\mathcal{M}_{\alpha}(\upsilon_1, \upsilon_2) &= \Psi(2)\zeta(2) \int_0^1 \eta^{\alpha-1} d\eta + m^2 \Psi(4)\zeta(4) \int_0^1 \eta^{\alpha-1} d\eta \\ &\quad + \left[m\Psi(2)\zeta(4) + m\Psi(4)\zeta(2)\right] \int_0^1 \eta^{\alpha-1} d\eta \\ &= 2 \cdot \frac{1}{\alpha} + 4 \cdot \frac{1}{\alpha} + (2+4)\frac{1}{\alpha} \\ &= \frac{12}{\alpha}.\end{aligned}$$

Next,

$$\frac{\Psi(\mathtt{t})\zeta(\mathtt{t})}{\mathtt{t}^2} = \frac{\mathtt{t}\cdot 1}{\mathtt{t}^2} = \frac{1}{\mathtt{t}}.$$

Therefore,

$$\begin{aligned}&\frac{\Lambda}{\Phi^\alpha}\Gamma(\alpha)\, J^\alpha_{2^+}\left(\frac{1}{\mathtt{t}}\right)(4)\\ &= \frac{8}{2^\alpha}\int_2^4 (4-\mathtt{t})^{\alpha-1}\frac{1}{\mathtt{t}}\,d\mathtt{t}.\end{aligned}$$

Hence the inequality in Theorem 3.5 reduces to

$$\text{LHS} \le \frac{12}{\alpha} + \frac{8}{2^\alpha}\int_2^4 (4-\mathtt{t})^{\alpha-1}\frac{1}{\mathtt{t}}\,d\mathtt{t}.$$

For some representative values of α, we obtain the following numerical verification:

α	LHS	RHS	Relation
1	11.545177	14.772589	LHS < RHS
1.5	7.943208	9.971604	LHS < RHS
2	6.090355	7.545177	LHS < RHS
3	4.180710	5.090355	LHS < RHS

In particular, for $\alpha = 1$, the left-hand side becomes

$$\begin{aligned}\text{LHS} &= 4\int_{\frac14}^{\frac12}\left(\frac{1}{\mathtt{k}}+2\right)d\mathtt{k} + 4\int_{\frac14}^{\frac12}\left(\frac{1}{\mathtt{k}}+4\right)d\mathtt{k}\\ &= 4\left[\ln\mathtt{k}+2\mathtt{k}\right]_{\frac14}^{\frac12} + 4\left[\ln\mathtt{k}+4\mathtt{k}\right]_{\frac14}^{\frac12}\\ &= (4\ln 2+2) + (4\ln 2+4)\\ &= 8\ln 2 + 6 \approx 11.545177.\end{aligned}$$

Also,

$$\begin{aligned}\text{RHS} &= 12 + \frac{8}{2}\int_2^4 \frac{1}{\mathtt{t}}\,d\mathtt{t}\\ &= 12 + 4\ln 2\\ &\approx 14.772589.\end{aligned}$$

Therefore,

$$11.545177 < 14.772589.$$

Similarly, for $\alpha = 2$, we have

$$\text{LHS} \approx 6.090355, \qquad \text{RHS} \approx 7.545177,$$

and again

$$\text{LHS} < \text{RHS}.$$

Thus, Theorem 3.5 is verified for this example.

Corollary 3.6. *If*

$$\phi(mv_2, v_1) = mv_2 - v_1,$$

then Theorem 3.5 reduces to the following fractional inequality for m-harmonic convex functions:

$$\left(\frac{m\upsilon_1\upsilon_2}{m\upsilon_2-\upsilon_1}\right)^{\alpha}\int_{\frac{1}{m\upsilon_2}}^{\frac{1}{\upsilon_1}}\left(\frac{1}{\upsilon_1}-\mathtt{k}\right)^{\alpha-1}\frac{\zeta(\upsilon_1)\Psi\left(\frac{1}{\mathtt{k}}\right)+\Psi(\upsilon_1)\zeta\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}\left(\frac{m\upsilon_1\upsilon_2}{m\upsilon_2-\upsilon_1}\left(\mathtt{k}-\frac{1}{m\upsilon_2}\right)\right)}\,d\mathtt{k}$$
$$+\left(\frac{m\upsilon_1\upsilon_2}{m\upsilon_2-\upsilon_1}\right)^{\alpha}\int_{\frac{1}{m\upsilon_2}}^{\frac{1}{\upsilon_1}}\left(\frac{1}{\upsilon_1}-\mathtt{k}\right)^{\alpha-1}\frac{m\zeta(\upsilon_2)\Psi\left(\frac{1}{\mathtt{k}}\right)+m\Psi(\upsilon_2)\zeta\left(\frac{1}{\mathtt{k}}\right)}{\mathsf{h}\left(\frac{m\upsilon_1\upsilon_2}{m\upsilon_2-\upsilon_1}\left(\frac{1}{\upsilon_1}-\mathtt{k}\right)\right)}\,d\mathtt{k}$$
$$\leq\mathcal{M}_{\alpha}(\upsilon_1,\upsilon_2)+\frac{m\upsilon_1\upsilon_2}{(m\upsilon_2-\upsilon_1)^{\alpha}}\Gamma(\alpha)\,J^{\alpha}_{\upsilon_1^+}\left(\frac{\Psi(\mathtt{t})\zeta(\mathtt{t})}{\mathtt{t}^2}\right)(m\upsilon_2),$$

where

$$\mathcal{M}_{\alpha}(\upsilon_1,\upsilon_2)=\Psi(\upsilon_1)\zeta(\upsilon_1)\int_0^1\frac{\eta^{\alpha-1}}{[\mathsf{h}(1-\eta)]^2}\,d\eta+m^2\Psi(\upsilon_2)\zeta(\upsilon_2)\int_0^1\frac{\eta^{\alpha-1}}{[\mathsf{h}(\eta)]^2}\,d\eta$$
$$+\left[m\Psi(\upsilon_1)\zeta(\upsilon_2)+m\Psi(\upsilon_2)\zeta(\upsilon_1)\right]\int_0^1\frac{\eta^{\alpha-1}}{\mathsf{h}(\eta)\mathsf{h}(1-\eta)}\,d\eta.$$

4. Conclusions and future remarks

In this article, we introduced a new (h,m)-harmonic Godunova–Levin preinvex class and established new Hermite–Hadamard type inequalities via fractional operators. The proposed framework unifies and extends several existing concepts, supported by non-trivial examples and recoverable special cases. Future work may explore the use of other fractional operators, as well as extensions to interval-valued and fuzzy-valued settings. Additionally, the application of this framework within quantum calculus and related analytical models offers promising research directions.

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